Engineering Analysis with Boundary Elements ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation (DRBIE) method

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ARTICLE INFO

Article history:
Received 18 May 2009
Accepted 2 July 2009

Keywords:
Boundary integral equation (BIE) method
Dual reciprocity method (DRM)
Second-order hyperbolic telegraph equation
Radial basis functions (RBFs)

ABSTRACT

In this paper, we use a numerical method based on the boundary integral equation (BIE) and an application of the dual reciprocity method (DRM) to solve the second-order one space-dimensional hyperbolic telegraph equation. Also the time stepping scheme is employed to deal with the time derivative. In this study, we have used three different types of radial basis functions (cubic, thin plate spline and linear RBFs), to approximate functions in the dual reciprocity method (DRM). To confirm the accuracy of the new approach and to show the performance of each of the RBFs, several examples are presented. The convergence of the DRBIE method is studied numerically by comparison with the exact solutions of the problems.

1. Introduction

We consider the second-order linear hyperbolic telegraph equation in one-space dimension, given by

$$u_{tt}(x,t)+2\alpha u_t(x,t)+\beta^2 u(x,t)=u_{xx}(x,t)+f(x,t),\quad a\le x\le b,\ \ t\ge 0, \tag{1.1}$$

both the electric voltage and the current in a double conductor, satisfy the telegraph equation, where x is distance and t is time. For $\alpha>0$, $\beta=0$ Eq. (1.1) represents a damped wave equation and for $\alpha>\beta>0$, it is called telegraph equation.

1.1. A brief introduction to applications

Equations of the form Eq. (1.1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. Interaction between convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physical, chemical and biological process [1–4]. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction–diffusion for such branches of sciences. For example biologists encounter these equations in the study of pulsate blood flow in arteries and in one-dimensional random motion of bugs along a hedge [5]. Also the propagation of acoustic waves in Darcy-type porous media [6], and parallel flows of viscous Maxwell fluids [7] are just some of the phenomena governed [8,9] by Eq. (1.1). Some discussions about derivation of the telegraph equation is described in [2]. In Section 4, we present a specific application of the telegraph equation.

1.2. Some theoretical aspects

Bereanu [10] discussed about existence, nonexistence and multiplicity for the periodic solutions of the nonlinear telegraph equation with bounded nonlinearities. A maximum principle for bounded and periodic solutions of the telegraph equation, presented in [11] and [12], respectively. Authors of [13] discussed about the existence of time-bounded solutions of nonlinear bounded perturbation of the telegraph equation with Neumann boundary condition. Wang and An [14], discussed the existence of positive doubly periodic solutions for nonlinear telegraph system, using the method of upper and lower solutions. Also the nonnegative doubly periodic solutions for nonlinear telegraph system have been obtained in [15]. The existence of time-periodic solutions of the telegraph equation can be found in [16–18] and the references therein.

1.3. Literature review

Abdusalam [19], presented the technique of asymptotic solution to find traveling wave solution for the telegraph model of dispersive variability which is a generalization of the Julian Cook model. In [20] a new spline different scheme based on quadratic spline interpolations is presented for solving the one space-dimensional linear hyperbolic equation of the form

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doi:10.1016/j.enganabound.2009.07.002

Eq. (1.1). This method is second and fourth-order accurate in time and space directions, respectively. An enormous amount of research by conditionally stable finite difference schemes has been done [21–23] for numerical solution of the second-order linear hyperbolic Eq. (1.1). In [24] a three-level implicit unconditionally stable alternating direction implicit (ADI) method for the two-dimensional linear hyperbolic equations is presented, which its accuracy is second order for both time and space components. Also Mohanty et al. in [25] used the same scheme for three-space-dimensional linear hyperbolic equation. An implicit three-level unconditionally stable difference scheme of $O(k^2+h^2)$, where k, h are the time and space step length, respectively, is proposed in [26] for the one-space-dimensional linear hyperbolic Eq. (1.1). Mohanty [5] developed an unconditionally stable difference scheme of second and fourth order in time and space components, respectively, to solve the one-space-dimensional linear hyperbolic equation in which two free parameters are introduced. An application of the Rothe–Wavelet method to the solution of telegraph equation is proposed in [27]. Authors of [27] described how wavelets may be used in slightly different manner, for the discretization of the telegraph equation. In [28] a numerical scheme for solving the second-order one-space-dimensional linear hyperbolic equation, by using the shifted Chebyshev cardinal functions is presented. Dehghan and Shokri [29,30] studied the numerical scheme to solve one and two-dimensional hyperbolic equations using collocation points and approximating directly the solution using the thin-plate-spline radial basis functions. In [2] a numerical method, based on the combination of a high-order compact finite-difference scheme to approximate the spatial derivative and the collocation technique for the time component was proposed to solve the one-space-dimensional linear hyperbolic equation. The authors of [31] developed an efficient approach for solving the two-dimensional linear hyperbolic telegraph equation, by the compact finite difference approximation of fourth order and collocation method. A numerical scheme, based on the shifted Chebyshev tau method is proposed in [32] to solve this equation. In [56], an explicit difference scheme is discussed for the numerical solution of the linear hyperbolic equation of the form Eq. (1.1).

1.4. Organization of the current paper

The outline of this article is as follows. In Section 2, the boundary integral equation (BIE) method and the dual reciprocity (DR) technique for Eq. (1.1) are described. In Section 3, the results of numerical experiments are presented. An application of the model described in this work is given in Section 4. The last concluding section summarizes the major finding of this study.

2. Integral equation formulation and application of DRM

We consider Eq. (1.1), with the following initial and boundary conditions: The initial conditions will be assumed to have the form:

$$u(x,0)=v_1(x), \quad a\le x\le b, \tag{2.1}$$

$$u_t(x,0)=v_2(x), \quad a\le x\le b, \tag{2.2}$$

while the boundary conditions are given by Dirichlet form:

$$u(a,t)=h_1(t), \quad u(b,t)=h_2(t), \quad t\ge 0, \tag{2.3}$$

Neumann form:

$$u_x(a,t)=k_1(t), \quad u_x(b,t)=k_2(t), \quad t\ge 0, \tag{2.4}$$

or mixed boundary conditions. Our aim is to present a different numerical approach for solving the telegraph equation. In order to derive an integral equation formulation for Eq. (1.1), we employ the following integral identity:

$$\int_a^b [u_{tt}+2\alpha u_t+\beta^2 u-u_{xx}-f(x,t)]\omega\, dx=0, \tag{2.5}$$

where ω is the weight function. The main idea behind this integral identity, follows from the well-known weighted residual method, that is a useful concept to give a common interpretation to various numerical techniques for the solution of differential equations. The goal of these numerical methods in general, is to make the weighted residual equal to zero. In this paper, we use a special weight function, that is the fundamental solution for the one-dimensional Laplacian operator, defined by the equation:

$$\frac{\partial^2\omega}{\partial x^2}(x,\xi)=\delta(x,\xi), \quad a\le x,\xi\le b, \tag{2.6}$$

where δ is the dirac delta function and x, ξ are a field point and a source point, respectively. The fundamental solution and its derivative are given in the following:

$$\omega(x,\xi)=\tfrac{1}{2}|x-\xi|, \tag{2.7}$$

$$\omega_{,x}(x,\xi)=\omega'(x,\xi)=\tfrac{1}{2}\,\mathrm{sgn}(x-\xi), \tag{2.8}$$

in which sgn denotes the signum function.

2.1. The idea of using the dual reciprocity method in the integral equation formulation

The boundary element method (BEM) is a very nice established numerical calculation technique to solve many kinds of engineering problems. We refer the interested readers to [33,34]. This method is applied for nonlinear, inhomogeneous and time-dependent problems of partial differential equations. However in the recent cases in that for nonlinear, inhomogeneous and time-dependent problems of PDEs the integral equation of the problem generally includes the domain integrals, so we have to discretize the domain with not only boundary elements, but also internal domain elements, therefore in the boundary element analysis for this case, one of the advantages which is reduction in the dimension of analysis domain has disappeared [35]. Because of keeping this advantage, some developments have helped to handle the domain integral in a more elegant manner, such as the multiple reciprocity method (MRM) [36] and the dual reciprocity method (DRM). The DRM as a very powerful technique for solving general elliptic equations of the type $\nabla^2 u=b$, was first introduced by Nardini and Brebbia [37] and it has been described in details in [38] (the interested readers to application of the DRM are referred to [39–41]). One of the strong points of DRM is that various important engineering problems can be solved with a knowledge of only a few basic fundamental solutions. In [42] Chen and Wong have used dual boundary element and multiple reciprocity method for solving the one-dimensional eigenproblems. Authors in [43–45] used the dual reciprocity method and boundary integral equation approach for solving time independent Burgers' equation, one-dimensional S-G equation and Cahn–Hillard equation, respectively. Ang [46], employed a similar approach to solve an initial-boundary value problem that combines Neumann and integral condition for the wave equation [47]. Also Ang in [48] presented a time-stepping dual reciprocity boundary element method for numerical solution of a generalized nonlinear Schrödinger equation [49]. In [50] a new weakly nonlinear numerical model, called the perturbation DRBEM model is presented for wave diffraction and refraction governed by the Boussinesq equations.

In this paper we are going to present a simple and efficient approach based on the boundary integral equation (BIE) technique and the dual reciprocity method (DRM), for solving the telegraph equation. The time derivative and the inhomogeneous terms are interpolated by $\phi(r)$, that called radial basis function (RBF), where r denotes the distance between a source point x_k and field point x. It is remarkable that the RBFs are known as a powerful tool for scattered data interpolation. The function $\phi(r)$ can be of various types, for example: polynomials of any chosen degree such as linear, cubic, etc., thin plate spline (TPS), multi quadrics (MQ), inverse multiquadrics (IMQ) and Gaussian forms, etc. In [37], Nardini and Brebbia used the function $\phi(r)=1+r$ as the approximation function in DRM. Some researchers [51,52] found that r^3 can be used as an improvement on r. The thin-plate-spline or (TPS), $r^2\log r$ is another RBF, that has been employed for problems involving body forces of high order [53]. Partridge and his co-workers have employed some of these radial basis functions, for solving some different problems, to obtain the utilization of different types of RBFs as approximation functions used in the DRM. Also in their investigation to bring out the advantage and disadvantages of each, the radial basis functions as "local functions" have been combined with the "global" ones (augmentation terms), such as polynomials (the terms from Pascal triangle $1, x, y, xy$, etc.), Sine expansions and other known functions [54,55]. Therefore, these functions and their efficiency are presented for solving some different problems. In an attempt to bring out the wealth of another types of RBFs, except for the linear form of radial functions as approximation functions used in the dual reciprocity method, and to give a comparison between the results of each, we have considered three types of them: the cubic, linear and thin plate splines. To demonstrate accuracy and usefulness of the method proposed in the current paper and also the performance of using these three types of RBFs, the numerical results are presented and the comparisons are taken with the exact solutions.

2.2. Implementation of the DRM

For implementation of the dual reciprocity method we meshed the interval $[a,b]$ to $N-1$ parts and chose N source points $x_j, j=1,2,\ldots,N$ in $[a,b]$, where $a=x_1<x_2<\cdots<x_N-1<x_N=b$. In this step we made an interpolation for approximating the time derivatives, and the inhomogeneous terms in the following [35]:

$$u_{tt}+2\alpha u_t+\beta^2 u-f(x,t)=b(x,t)=\sum_{k=1}^{N}\phi_k(x)\eta_k(t), \tag{2.9}$$

where ϕ_k is a radial basis function. In order to study the effect of using different types of RBFs as interpolation functions used in the dual reciprocity method, we choose three type of them: the cubic, thin plate spline and the linear RBFs. In Table 1, we have presented these RBFs that are employed in the current work:

In this table note that $r_k=|x-x_k|$ for $k=1,2,\ldots,N$. From (2.5), (2.9) we have

$$\int_a^b \frac{\partial^2 u}{\partial x^2}\omega\, dx=\sum_{k=1}^{N}\left[\int_a^b \phi_k^l(x)\omega\, dx\right]\eta_k(t), \quad l=1,2,3. \tag{2.10}$$

Table 1
Three types of RBFs, that are used as interpolation functions in DRM.

Cubic	Thin-plate-spline	Linear
$\phi_k^1=1+r_k^3$	$\phi_k^2=r_k^4\log(r_k)$	$\phi_k^3=1+r_k$

Table 2
The particular solutions f_k^l and their derivatives $f'^{(l)}_k$, $(l=1,2,3)$, for the mentioned RBFs.

Cubic	Thin-plate-spline	Linear
$f_k^1=\frac{1}{2}r_k^2+\frac{1}{20}r_k^5$	$f_k^2=\frac{1}{30}r^6\log(r)-\frac{11}{900}r^6$	$f_k^3=\frac{1}{2}r_k^2+\frac{1}{6}r_k^3$
$f'^{(1)}_k=\frac{\partial f_k^1}{\partial x}=r_k+\frac{1}{4}r_k^4$	$f'^{(2)}_k=\frac{\partial f_k^2}{\partial x}=\frac{1}{5}r^5\log(r)-\frac{1}{25}r^5$	$f'^{(3)}_k=\frac{\partial f_k^3}{\partial x}=r_k+\frac{1}{2}r_k^2$

We put the value of the function ϕ_k at source point x_i by ϕ_{ik} for $i=1,2,\ldots,N$, and set F as an $N\times N$ matrix that $F(i,k)=\phi_{ik}$ and $G=F^{-1}$, then we have

$$\eta_k(t)=\sum_{j=1}^{N}G_{kj}b_j(t), \tag{2.11}$$

where $b_j(t)=b(x_j,t)$. Now we define a particular solution f_k^l associated with each ϕ_k^l, for $l=1,2,3$ satisfying the following equation:

$$\frac{\partial^2 f_k^l}{\partial x^2}=\phi_k^l, \quad l=1,2,3. \tag{2.12}$$

The particular solutions f_k^l and their derivatives $f'^{(l)}_k$ for $l=1,2,3$, are shown in Table 2.

According to linearization and approximation of the nonlinear integral equation (2.5) by RBFs and substituting (2.12) into (2.10) and applying the integration by parts [44], we can get the following expression:

$$[\omega_i(x)q(x,t)]_a^b-[u(x,t)\omega_i'(x)]_a^b+u_i(t)$$
$$=\sum_{k=1}^{N}[[\omega_i f'^{(l)}_k]_a^b-[f_k^l\omega_i']_a^b+f_{ik}^l]\eta_k(t), \quad l=1,2,3, \tag{2.13}$$

for $i=1,2,\ldots,N$, where $q(x,t)=\partial u(x,t)/\partial x$ and $\omega_i(x)=\omega(x,\xi)|_{\xi=x_i}$ are weight functions at the points $i=1,2,\ldots,N$ which satisfy in Eq. (2.6). $u_i(t)=u(\xi,t)|_{\xi=x_i}$, $f_{ik}^l=f_k^l(x_i)$, $l=1,2,3$. If we set

$$S_{ik}=[\omega_i f'^{(l)}_k]_a^b-[f_k^l\omega_i']_a^b+f_{ik}^l, \quad l=1,2,3, \tag{2.14}$$

then Eq. (2.13) takes the following form:

$$[\omega_i(x)q(x,t)]_a^b-[u(x,t)\omega_i'(x)]_a^b+u_i(t)=\sum_{k=1}^{N}S_{ik}\eta_k(t). \tag{2.15}$$

From (2.11), the right-hand side of Eq. (2.15) is

$$\sum_{k=1}^{N}S_{ik}\eta_k(t)=\sum_{k=1}^{N}S_{ik}\sum_{j=1}^{N}G_{kj}b_j(t)=\sum_{j=1}^{N}M_{ij}b_j(t), \tag{2.16}$$

where

$$M_{ij}=\sum_{k=1}^{N}S_{ik}G_{kj}. \tag{2.17}$$

If we set

$$E=\begin{bmatrix}-\omega_1(a) & \omega_1(b)\\ -\omega_2(a) & \omega_2(b)\\ \vdots & \vdots\\ -\omega_N(a) & \omega_N(b)\end{bmatrix}, \quad H=\begin{bmatrix}-\omega_1'(a) & \omega_1'(b)\\ -\omega_2'(a) & \omega_2'(b)\\ \vdots & \vdots\\ -\omega_N'(a) & \omega_N'(b)\end{bmatrix}, \tag{2.18}$$

then Eqs. (2.15)–(2.17) yield

$$[E]\begin{bmatrix}q_1(t)\\ q_N(t)\end{bmatrix}-[H]\begin{bmatrix}u_1(t)\\ u_N(t)\end{bmatrix}+[I]\begin{bmatrix}u_1(t)\\ u_2(t)\\ \vdots\\ u_N(t)\end{bmatrix}=[M]\begin{bmatrix}b_1(t)\\ b_2(t)\\ \vdots\\ b_N(t)\end{bmatrix}, \tag{2.19}$$

Please cite this article as: Dehghan M, Ghesmati A. Solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation.... Eng Anal Bound Elem (2009), doi:10.1016/j.enganabound.2009.07.002

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where I is the $N \times N$ identity matrix, $q_1(t) = q(a,t)$, $q_N(t) = q(b,t)$ and $u_j(t) = u(x_j, t)$. We note that

$$b_j(t) = \frac{\partial^2 u}{\partial t^2}(x_j, t) + 2\alpha \frac{\partial u}{\partial t}(x_j, t) + \beta^2 u(x_j, t) - f(x_j, t). \tag{2.20}$$

Now (2.19) constitutes a system of N equations in N unknowns which are functions of t. In the numerical solutions, we can use Dirichlet, Neumann and mixed boundary conditions. If the Dirichlet boundary conditions are applied, the unknown variables are given by $q_1(t)$, $q_N(t)$ and $u_j(t)$, for $j = 2, 3, \ldots, N-1$. For Neumann boundary conditions the unknowns are given by $u_j(t)$, for $j = 1, 2, \ldots, N$. Finally for mixed boundary conditions we can derive the unknown variables similarly. This system is solved approximately using the iterative scheme described in the following. Also we use the following finite difference approximations for time derivative operator:

$$u_{tt}|_j \simeq \frac{u_j^{(n+1)} - 2u_j^{(n)} + u_j^{(n-1)}}{(\tau)^2}, \quad j = 1, 2, \ldots, N, \tag{2.21}$$

$$u_t|_j \simeq \frac{u_j^{(n)} - u_j^{(n-1)}}{(\tau)}, \quad j = 1, 2, \ldots, N, \tag{2.22}$$

and make the following approximation by using the Crank–Nicolson technique:

$$q_j(t) \simeq \tfrac{1}{2}[q_j^{(n)} + q_j^{(n+1)}], \quad j = 1, N. \tag{2.23}$$

For u we employ the subsequent approximation:

$$u_j(t) \simeq \tfrac{1}{3}[u_j^{(n-1)} + u_j^{(n)} + u_j^{(n+1)}], \quad j = 1, 2, \ldots, N, \tag{2.24}$$

where $u_j^{(n)} = u(x_j, n\tau)$ and $q_j^{(n)} = q(x_j, n\tau)$, f is the inhomogeneous term. We set $\lambda = 1/(\tau)^2$, $\gamma = 1/\tau$, $\mathbf{u} = [u_1, u_2, \ldots, u_N]^T$ and $\Psi(x_j, t) = f(x_j, t)$. Now we put Eqs. (2.20)–(2.24) in Eq. (2.19), so we get

$$\frac{1}{2}[E]\left(\begin{bmatrix} q_1 \\ q_N \end{bmatrix}^{(n)} + \begin{bmatrix} q_1 \\ q_N \end{bmatrix}^{(n+1)}\right) - \frac{1}{3}[H]\left(\begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n-1)} + \begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n)} + \begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n+1)}\right)$$
$$+\frac{1}{3}[I]([\mathbf{u}]^{(n-1)} + [\mathbf{u}]^{(n)} + [\mathbf{u}]^{(n+1)}) = \lambda[M]([\mathbf{u}]^{(n-1)} - 2[\mathbf{u}]^{(n)} + [\mathbf{u}]^{(n+1)})$$
$$+2\alpha\gamma[M]([\mathbf{u}]^{(n)} - [\mathbf{u}]^{(n-1)}) + \frac{\beta^2}{3}[M]([\mathbf{u}]^{(n-1)} + [\mathbf{u}]^{(n)} + [\mathbf{u}]^{(n+1)}) - [M][\Psi], \tag{2.25}$$

where $\Psi = [f(x_1, t), f(x_2, t), \ldots, f(x_N, t)]^T$. At the first time level, when $n = 0$, according to the initial conditions that were introduced in (2.1) and (2.2), we apply the following assumptions:

$$u_j^{(0)} = v_1(x_j)$$

and

$$u_j^{(-1)} = u_j^{(0)} - (\tau)v_2(x_j) = v_1(x_j) - (\tau)v_2(x_j).$$

For Dirichlet boundary conditions, Eq. (2.25) is solved as a system of linear algebraic equations, for unknowns $u_j^{(n+1)}$, $(j = 2, 3, \ldots, N-1)$ and $q_j^{(n+1)}$, $(j = 1, N)$. We assume that $u_j^{(n-1)}, u_j^{(n)}$ for $j = 2, 3, \ldots, N-1$, are known from the two previous time levels, $q_j^{(n)}$ from the previous time level and $u_j^{(n-1)}$, $u_j^{(n)} (j = 1, N)$ from the boundary conditions (2.3). Similarly, for Neumann and Robbin boundary conditions, the known and unknown variables are characterized and the above system can be solved at each time level.

3. Numerical results

The following models, have been solved by DRBIE method. To obtain the accuracy and applicability of different types of RBFs for approximation in the dual reciprocity method, the root-mean-square (RMS) errors have been reported

$$\|\mathrm{e}\|_{L_2} = \sqrt{\sum_{j=1}^{N} |\mathrm{e}_j|^2}, \tag{3.1}$$

$$\mathrm{RMS} = \frac{\|\mathrm{e}\|_{L_2}}{\sqrt{N}}, \tag{3.2}$$

where

$\mathrm{e} = u_{\mathrm{exact}} - u_{\mathrm{app}}$,

$\mathrm{e}_j = (u_{\mathrm{exact}})_j - (u_{\mathrm{app}})_j$

and

$N = \mathrm{length}(\mathrm{e})$.

3.1. Example 1

We consider the second-order hyperbolic telegraph equation of the form Eq. (1.1), with $\alpha = 4$, $\beta = 2$, $\tau = t_i - t_{i-1} = 0.01$, $\chi = x_i - x_{i-1} = 0.05, 0.02, 0.01$, in the interval $[0, 2\pi]$ and $t \in [0, 3]$. In this case we have $f(t) = (2 - 2\alpha + \beta^2)\exp(-t)\sin(x)$. The exact solution by [29,32] is

$$u(x,t) = \exp(-t)\sin(x). \tag{3.3}$$

The Neumann boundary conditions obtained from the exact solution. The initial conditions are given by

$$v_1(x) = u(x,0) = \sin(x), \quad v_2(x) = u_t(x,0)$$
$$= -\sin(x), \quad 0 \leq x \leq 2\pi. \tag{3.4}$$

The root-mean-square error is presented in Table 3. The space–time graph of the estimated solution up to $t = 3$ is shown in Fig. 1, and the estimated–analytical solution graph, for some different times and $x \in [0, 2\pi]$ is presented in Fig. 2. Note that the following assumptions are considered to acquire the results for plotting the figures of numerical solutions ($\tau = 0.01$, $\alpha = 4$, $\beta = 2$, $\chi = 0.05$).

3.2. Example 2

In this problem, we consider the hyperbolic telegraph equation (1.1). The analytical solution of this example is [28]

$$u(x,t) = \tan\left(\frac{x+t}{2}\right), \tag{3.5}$$

with the following initial conditions:

$$v_1(x) = u(x,0) = \tan\left(\frac{x}{2}\right), \quad v_2(x) = u_t(x,0)$$
$$= \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right), \quad 0 \leq x \leq 2. \tag{3.6}$$

Table 3
RMS error between the exact and the (DRBIE) solutions of the Example 1 at final time $t = 3$ with $\tau = 0.01$, $\alpha = 4$, $\beta = 2$.

RMS error			
$\chi = (x_i - x_{i-1})$	Cubic RBF: $1 + r^3$	TPS RBF: $r^4\log(r)$	Linear RBF: $1 + r$
0.05	7.125E−005	9.017E−005	3.010E−004
0.02	1.712E−005	2.943E−005	7.128E−005
0.01	8.218E−006	8.991E−006	4.320E−005

The Neumann boundary condition are obtained from the exact solution. In solving the example with DRBIE method, the following assumptions are considered: $\alpha = 10$, $\beta = 5$, $\tau = t_i - t_{i-1} = 0.01$, $\chi = x_i - x_{i-1} = 0.05, 0.01, 0.005$, $x \in [0, 2]$ and $t \in [0, 1]$. In this case we have $f(t) = \alpha(1 + \tan^2(x + t/2)) + \beta^2 \tan(x + t/2)$.

The space–time graph of the numerical solution up to $t = 1$ is presented in Fig. 3. The graph of analytical and estimated solutions for some different times and $x \in [0, 2]$ is presented in Fig. 4. In order to obtain the numerical solutions to plot the figures, the following assumptions are considered ($\tau = 0.01$, $\alpha = 10$, $\beta = 5$, $\chi = 0.05$).

The accuracy of the DRBIE method is measured by using the RMS error. The errors are reported in Table 4.

3.3. Example 3

Consider the hyperbolic telegraph equation (1.1), with $\alpha = 6$, $\beta = 2$, $\tau = t_i - t_{i-1} = 0.01$, $\chi = x_i - x_{i-1} = 0.05, 0.01, 0.005$, in the interval $[0, 4]$ and $t \in [0, 2]$. In this case we have $f(t) = -2\alpha \sin(t)\sin(x) + \beta^2 \cos(t)\sin(x)$. The exact solution of the above differential equation is [28,2]

$$u(x, t) = \cos(t)\sin(x), \tag{3.7}$$

subject to the following initial conditions:

$$v_1(x) = u(x, 0) = \sin(x), \quad v_2(x) = u_t(x, 0) = 0, \quad 0 \le x \le 4. \tag{3.8}$$

Also the Neumann boundary condition is obtained from the exact solution.

The accuracy of the scheme is measured by using the RMS error. The errors are reported in Table 5. The space–time graph of the numerical solution up to $t = 2$ is presented in Fig. 5. The graph of analytical and estimated solutions for several different times and $x \in [0, 4]$ is presented in Fig. 6. Also note that we consider these suppositions to plot the figures of the obtained numerical solutions ($\tau = 0.01$, $\alpha = 6$, $\beta = 2$, $\chi = 0.05$).

3.4. Example 4

We finally close our numerical simulation by studying the telegraph equation of the form Eq. (1.1), with the following assumptions: $\alpha = \frac{1}{2}$, $\beta = 1$, $\tau = t_i - t_{i-1} = 0.01$, $\chi = x_i - x_{i-1} = 0.05, 0.01, 0.005$, $x \in [1, 4]$ and $t \in [0, 5]$. In this case we have $f(t) = (2 - 2t + t^2)(x - x^2)\exp(-t) + 2t^2 \exp(-t)$. The exact solution of the above differential equation is given by [29,32]

$$u(x, t) = (x - x^2)t^2 \exp(-t). \tag{3.9}$$

The initial conditions for this example are

$$v_1(x) = u(x, 0) = 0, \quad v_2(x) = u_t(x, 0) = 0, \quad 1 \le x \le 4. \tag{3.10}$$

The Neumann boundary condition is obtained from the exact solution.

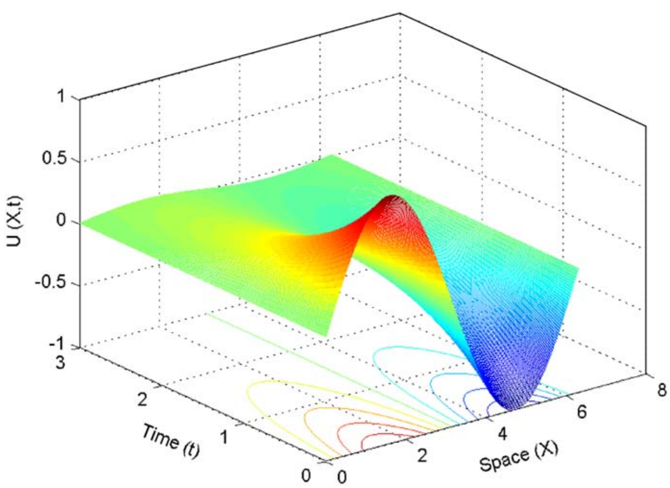

Fig. 1. Space–time graph of the estimated solution up to $t = 3$, with $\tau = 0.01$, $\alpha = 4$, $\beta = 2$, $\chi = 0.05$ by using the linear RBFs as approximation functions in DRM.

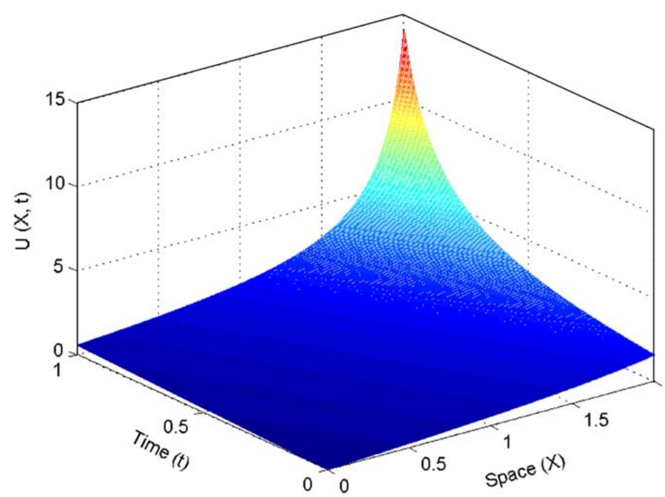

Fig. 3. Space–time graph of the estimated solution up to $t = 1$, with $\tau = 0.01$, $\alpha = 10$, $\beta = 5$, $\chi = 0.05$ by using the linear RBFs as approximation functions in DRM.

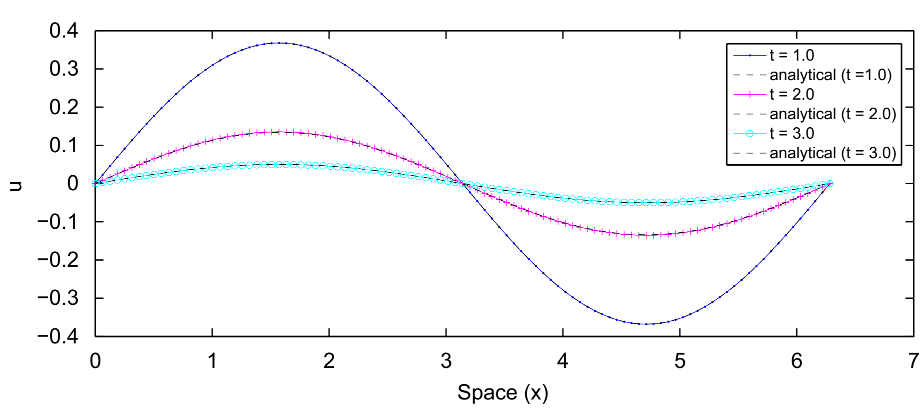

Fig. 2. Analytical–estimated graph of Example 1, for $x \in [0, 2\pi]$ and $t = 1, 2, 3$.

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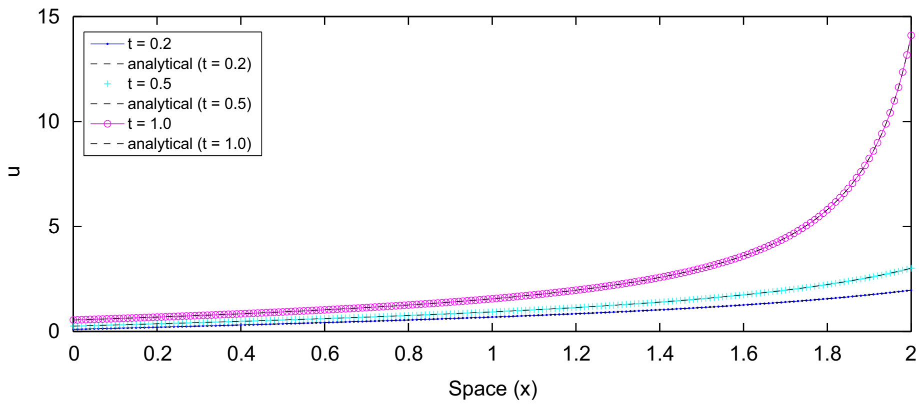

Fig. 4. Analytical–estimated graph Example 2, for $x \in [0, 2]$ and $t = 0.2, 0.5, 1$.

Table 4
RMS error between the exact and the (DRBIE) solutions of the Example 2 at final time $t = 1$, with $\tau = 0.01$, $\alpha = 10$, $\beta = 5$.

RMS error			
$\chi = (x_i - x_{i-1})$	Cubic RBF: $1 + r^3$	TPS RBF: $r^4 \log(r)$	Linear RBF: $1 + r$
0.05	3.712E−005	5.094E−005	9.375E−005
0.01	7.375E−006	9.527E−006	4.001E−005
0.005	3.978E−006	4.101E−006	1.903E−005

Table 5
RMS error between the exact and the (DRBIE) solutions of the Example 3 at final time $t = 2$, with $\tau = 0.01$, $\alpha = 6$, $\beta = 2$.

RMS error			
$\chi = (x_i - x_{i-1})$	Cubic RBF: $1 + r^3$	TPS RBF: $r^4 \log(r)$	Linear RBF: $1 + r$
0.05	5.905E−005	6.287E−005	2.153E−004
0.01	6.921E−006	7.997E−006	7.012E−005
0.005	4.325E−006	5.305E−006	4.003E−005

The root-mean-square error is presented in Table 6, also the space–time graph of the estimated solution up to $t = 5$ is presented in Fig. 7. The graph of analytical and estimated solutions for some different times and $x \in [1, 4]$ is presented in Fig. 8. The following assumptions are deliberated to plot the obtained numerical results ($\tau = 0.01$, $\alpha = \frac{1}{2}$, $\beta = 1$, $\chi = 0.05$).

One main advantage in the application of the dual reciprocity in our study based on the boundary integral equation, is that: there is no integration operator during our numerical process. In fact the integration is reduced to calculating the function in two boundary points such as a, b, only, and it shows the efficiency of the new numerical technique (see Eq. (2.13)). In all problems, when the total number of collocation nodes increased, the numerical errors declined, therefore the convergence of the method can be seen obviously from the numerical reports in the columns of the tables from up to down. Another point about using three types of RBFs for approximation functions in DRM: as it has been presented in all tables of RMS error, the accuracy of the method is decreased by using the cubic, TPS and the linear-RBFs, respectively. In this paper, our intention of using two different types of RBFs, except for the linear one, is to employ new types of radial functions as approximation functions used in the DRM in our approach, these are more accurate than the linear types. From all tables it concluded that the difference of errors between the cubic and TPS forms is little, and the results are almost the same, also the results in both of them are more accurate than the results obtained from the linear ones. All of the above conclusions are shown in the table of errors clearly, and can be seen from left to right in each row of the tables. To show the numerical results, the graphs of the examples are plotted. A remarkable point about these graphs is that all of the figures are plotted for the results obtained by using the linear radial basis functions (LRBFs), as approximation functions in the DRM. It is obvious that the figures of the corresponding results obtained from application of other kinds of RBFs (e.g. cubic and TPS) are more accurate than the linear one. Therefore we ignore of presenting these graphs.

In next section we follow [3,4] to present a specific application of the model investigated in this paper.

4. Dispersive wave propagation as an application of the telegraph equation

Some discussions about derivation of the telegraph equation is described in [2]. This section demonstrates how linear equations that describe wave propagation can distort a propagating disturbance because of an effect called dispersion [3]. The telegraph equation $u_{tt} - c^2 u_{xx} + a u_t + bu = 0$, with c, a and b positive constants describes bidirectional wave propagation, and it was first derived to model telephonic communication along land lines. To see how a harmonic plane wave (a sinusoid) moving along the x-axis and governed by this equation is propagated, we consider the function $u(x, t)$ that is the real part of

$$\hat{u}(x, t) = A\exp[im(x - ct)], \quad (A \ real),$$

and start by substituting $\hat{u}(x, t)$ into the telegraph equation [3] (this is equivalent to substituting $u(x, t) = A\cos[m(x - ct)]$ into the equation). Defining the wavelength $\lambda = 2\pi/m$, the wave number $k = 2\pi/\lambda$ and the frequency $\omega = 2\pi c/\lambda$ of the harmonic wave allows $\hat{u}(x, t)$ to be written

$$\hat{u}(x, t) = A\exp[i(kx - \omega t)].$$

When this expression is substituted into the telegraph equation, the following compatibility condition is found between k and ω in order that the harmonic wave is a solution of the equation:

$$\omega^2 + ia\omega - (b + c^2k^2) = 0.$$

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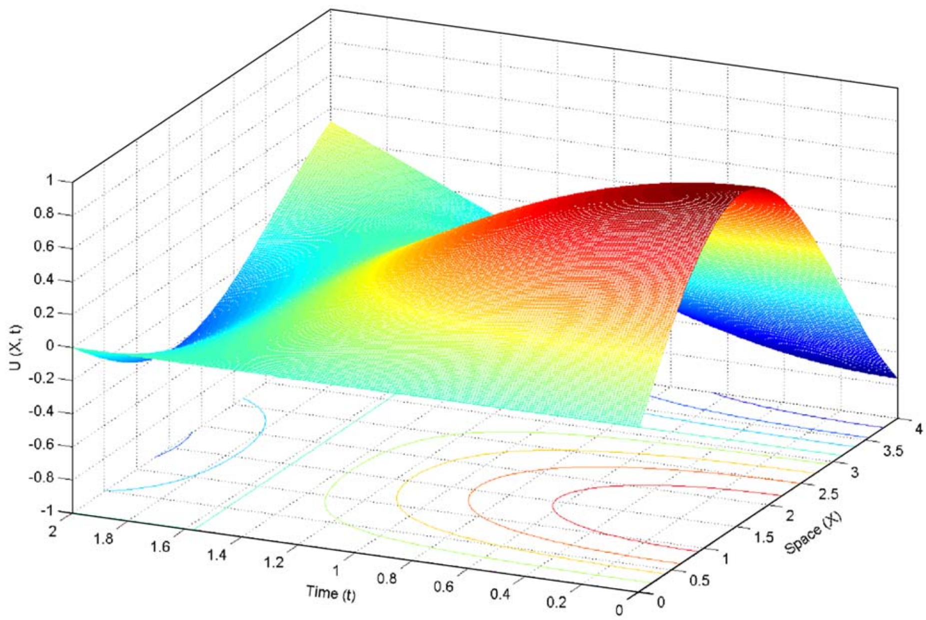

Fig. 5. Space–time graph of the estimated solution up to $t = 2$, with $\tau = 0.01$, $\alpha = 6$, $\beta = 2$, $\chi = 0.05$ by using the linear RBFs as approximation functions in DRM.

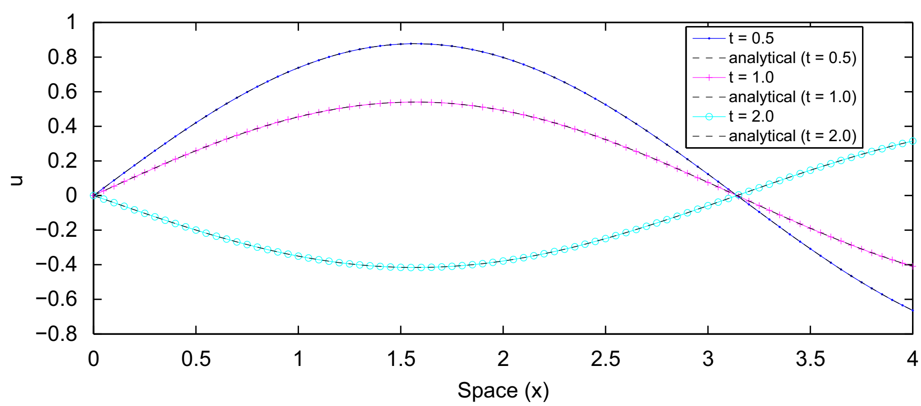

Fig. 6. Analytical–estimated graph Example 3, for $x \in [0, 4]$ and $t = 0.5, 1, 2$.

Table 6
RMS error between the exact and the (DRBIE) solutions of the Example 4 at final time $t = 5$, with $\tau = 0.01$, $\alpha = \frac{1}{2}$, $\beta = 1$.

RMS error			
$\chi = (x_i - x_{i-1})$	Cubic RBF: $1 + r^3$	TPS RBF: $r^4\log(r)$	Linear RBF: $1 + r$
0.05	7.012E–005	1.005E–004	5.903E–004
0.01	1.301E–005	6.651E–005	9.979E–005
0.005	9.105E–006	1.014E–005	5.004E–005

This result is called the dispersion relation for the telegraph equation, and for real k it shows that ω is complex, with

$$\frac{\omega}{k} = -i\frac{a}{2k} \pm \frac{1}{2k}(4c^2k^2 + 4b - a^2)^{1/2}.$$

The quantity $kx - \omega t$ determines the phase of the wave, so that a wave of constant phase propagates with $kx - \omega t =$ constant, showing that the phase velocity of the wave is $v_p = \omega/k$. However,

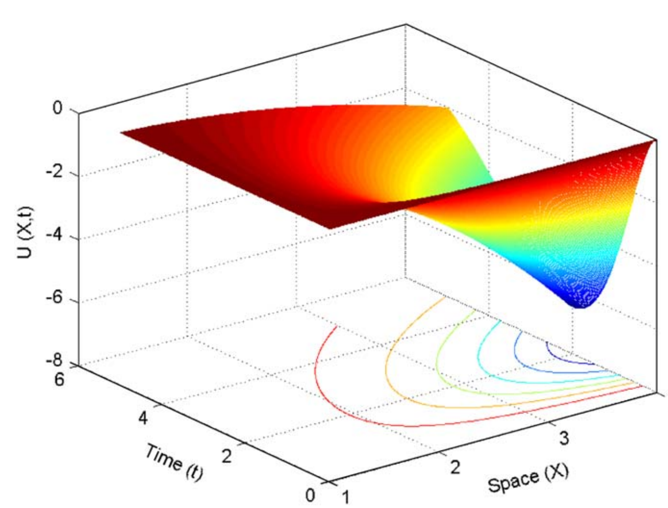

Fig. 7. Space–time graph of the estimated solution up to $t = 5$, with $\tau = 0.01$, $\alpha = \frac{1}{2}$, $\beta = 1$, $\chi = 0.05$ by using the linear RBFs as approximation functions in DRM.

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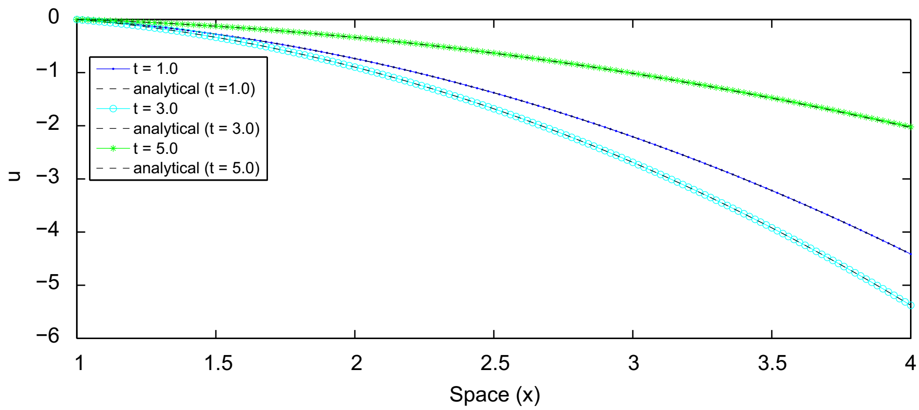

Fig. 8. Analytical–estimated graph Example 4, for $x \in [1,4]$ and $t = 1,3,5$.

the dispersion relation shows that ω/k is a function of ω, so it follows that waves with different frequencies ω will propagate with different phase speeds v_p [4]. Consequently, with the use of Fourier series, any periodic initial disturbance at time $t = 0$ can be decomposed into a sum of harmonic components. So because each component propagates with a different phase speed, when they are recombined to form the solution at later times $t_1, t_2, \ldots$, it follows that the wave shape will have changed with time. This change of shape of the wave is said to be due to dispersion. When the dispersion relation is used in $\hat{u}(x,t)$, it turns out that [3]

$$u(x,t) = Re\left\{A\exp\left(\frac{-at}{2}\right) \times \exp\left(ik\left[x \mp \frac{t}{2k}(4c^2k^2 + 4b - a^2)^{1/2}\right]\right)\right\}.$$

This confirms the dispersive nature of the telegraph equation, and when $a > 0$ it shows that the magnitude of the wave decays exponentially with time. If, however, $4b = a^2$ the dispersive effect vanishes and the wave propagates without change of shape, but with an exponential decay called dissipation. Such waves are said to be relatively undistorted. It was this condition that was first used to adjust the parameters in a telephone land line to remove distortion of the transmitted message due to dispersion. The decay, or dissipation, was corrected by the insertion of amplifiers at regular points along the line [4].

5. Conclusion

In this study, the boundary integral equation method and the dual reciprocity technique (BIE-DR) were used to obtain approximate analytical solutions of the second-order one-dimensional hyperbolic telegraph equation. In formulation of the boundary integral equation method, we employed the fundamental solution of the Laplacian operator as the weight function. To implement the dual reciprocity (DR) as a technique of the boundary element method (BEM), in that for interpolation of the inhomogeneous and time derivative terms, three types of radial basis functions (cubic, thin plate spline and linear RBFs) were utilized. To demonstrate the accuracy and usefulness of this method, some numerical examples were presented. For all test problems, the RMS error was reported, also the graphs of the numerical solutions were shown. A good agreement between the results for the DRBIE technique, and the exact solutions was observed clearly. The results of our numerical experiments, confirm the validity of the new technique. From using other types of RBFs except for the linear ones ($\phi(r) = 1 + r$) as approximation functions used in DRM we found that the accuracy in the cubic form ($\phi(r) = 1 + r^3$) was more than TPS ($\phi(r) = r^4 \log r$), and in both of them was more than the linear form. Therefore the cubic and TPS forms of RBFs can be good replacements for the traditional ones, i.e. linear forms.

Acknowledgments

The authors are very grateful to both reviewers of this paper for their comments and suggestions which have improved the paper.

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