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Numerical Simulation of the Behaviour of Cracks in Axisymmetric Structures by the Dual Boundary Element Method

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Abstract The study of defected axisymmetric structures is among important industrial applications. Detection of such defects, and or the evaluation of intrinsic parameter leads to a better design of those mechanical parts. The first part of the conducting research concerns the evaluation of the stress intensity factors (SIF) in axisymmetric elastic structures with internal or circumferential edge crack using the dual boundary element method (DBEM). Its application to axisymmetric problems requires a stress (hypersingular) boundary integral equation together with the displacement (standard) boundary integral equation, one applied to each side of the crack. This process requires a great algebraic handling due to the complexity of the axisymmetric kernels. Crack surfaces are discretized with discontinuous quadratic boundary elements to satisfy the existence of the finite-part integrals and the continuity of the unit outward normal at corners. SIF evaluation is done using displacements extrapolation at the crack tip. Examples of axisymmetric geometries are analyzed and obtained results are compared to others researchers.

Keywords: Boundary Element Method, Cauchy Principal Value, Hadamard Finite-Part, Fracture Mechanics, Hypersingular Integrals, Linear Elasticity.

1. Introduction

The boundary element method has been successfully applied to axisymmetric elasticity during several years, starting with the works of Kermanidis [1], Mayr [2] and Cruse et al. [3]. Dual integral equations were introduced by Watson [4], in a formulation based on the normal derivative of the displacement equation in plain strain. Hong and Chen [5] presented a general formulation, which incorporates the displacement and the traction boundary integral equations. An effective numerical implementation of the two dimensional DBEM for solving general linear elastic fracture mechanics problems with finite-parts defined was presented by Portela et al. [6]. An axisymmetric hypersingular boundary integral formulation for elasticity problems was developed in De Lacerda and Wrobel [7, 8]. Hypersingular boundary integral equations (HBIE) are derived from a differentiated version of the standard BIE, considering the asymptotic behaviour of their singular and hypersingular

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kernels with evaluation of stress intensity factors in, and propagation of cracks inside cylinder. The strongly singular and hypersingular equations in this formulation are regularized by De Lacerda and Wrobel by employing the singularity subtraction technique. Mukherjee [9] has revisited the same problem and interpreted the HBIE in a finite part sense [10]. The DBEM is a well-established method for the analysis of crack problems [11–14]. Its application to axisymmetric geometries requires the use of the stress boundary integral equation together with the displacement boundary integral equation.

2. Standard and hypersingular boundary integral equations for axisymmetric elastic solid

Axisymmetric geometry is obtained by a 2π rotation of a two-dimensional body about *z*-axis. Under an axisymmetric loading, displacements and stresses are independent of the hoop direction. As consequence of this axisymmetry, a 3-D domain is reduced to a 2-D one, and directions r and z are sufficient to define the problem. In the absence of body forces, the axisymmetric elasticity equation for an internal point of a linear elastic solid Ω with boundary Γ has the following form:

$$u_i(P) = \int_\Gamma \alpha(Q) U_{ij} \cdot (P,Q) t_j(Q) d\Gamma - \int_\Gamma \alpha(Q) T_{ij} \cdot (P,Q) u_j(Q) d\Gamma \tag{1}$$

With i, j= r, z, $\alpha(Q)=2\pi r(Q)$.

P is the internal source point where the ring unit load is applied; *Q* is the integrating field point. r(*Q*): is the radial distance from the field point to the axis of symmetry and u_r, u_z and t_r, t_z are the radial and axial displacements and tractions respectively. U_{ij} and T_{ij} are the axisymmetric displacement and traction kernels, which are functions of the complete elliptic integrals of the first and second kind K and E [15]. Details on the singular behaviour of U and T are presented in reference [7]. When the source point is on Γ and out of the axis of symmetry, U is a combination of regular and weak-singular ($O \ln \bar{r}$) terms, whereas T includes regular, weak, and strong singular terms ($O \bar{r}^{-1}$). The case where P is on the axis of symmetry is still discussed in [7], but not reported here, since we consider an axial hollow geometry. For a source point P on the boundary Γ, the displacement boundary integral equation (Eq. 1), in a limiting process, can be written for a smooth boundary:

$$\frac{1}{2} u_i(P) = \int_\Gamma \alpha(Q) U_{ij} \cdot (P,Q) t_j(Q) d\Gamma - \int_\Gamma \alpha(\overline{Q}) T_{ij} \cdot (P,Q) u_j(Q) d\Gamma \tag{2}$$

The first integral on the right-hand side of Eq. (2) is of Riemman type since the integrand has at most a logarithmic singularity, while the second one is evaluated as a Cauchy Principal Value integral, (the sign on the second integral indicates CPV). An alternative approach to the solution of axisymmetric elasticity problems comes from the Somigliana identity for stresses. Differentiating the displacement equation (Eq. 2) with respect to directions *r* and *z*, substituting into the linear strain-displacement equations and applying Hook's law, an integral equation for stresses at boundary points can be obtained:

$$\frac{1}{2}\sigma_{ij}(P)=\int_{\Gamma}\alpha\left(\overline{Q}\right)D_{ijk}\cdot(P,Q)\,t_k(Q)\,d\Gamma-\int_{\Gamma}\alpha\left(\overline{\overline{Q}}\right)S_{ijk}\cdot(P,Q)\,u_k(Q)\,d\Gamma \tag{3}$$

With i, j, k = r, z. Kernels D_{ijk} and S_{ijk}, are linear combinations of derivatives of U_{ij} and T_{ij}, and consequently, complete elliptic functions K and E. The first integral on the right-hand side of Eq. (3) is evaluated in the CPV sense, while the second one is also improper and must be evaluated in the Hadamard Finite-Part sense, (the sign on the second integral indicates HFP). Multiplying both sides of Eq. (3) by the normal components at the source point $n_j(P)$ leads to the traction equation:

$$\frac{1}{2}t_i(P)=n_j\int_{\Gamma}\overline{\alpha}(Q)\,D_{ijk}\cdot(P,Q)\,t_k(Q)\,d\Gamma-n_j\int_{\Gamma}\overline{\overline{\alpha}}(Q)\,S_{ijk}\cdot(P,Q)\,u_k(Q)\,d\Gamma \tag{4}$$

3. The dual boundary element method

The advantage of the DBEM in solving fracture mechanics problems comes from the fact that only boundaries are discretized, which considerably reduces the size of systems to be solved. Crack propagation analysis is integrated without difficulties since only crack increments are added to the mesh. Discontinuous quadratic elements are employed for the discretization of both geometries (non-crack boundaries, and the crack itself). Discontinuous elements have their edge nodes shifted towards the centre of the element in order to satisfy, smoothness at the boundary nodes, continuity of the displacement derivatives, and boundary curvature at these points. Every crack element has two sides on witch equations (2) and (4) are applied to the elements of each side. A discretized system of equations is obtained where integrals over each element are evaluated.

4. Numerical treatments

The evaluation of different integrals that arise in Eqs. (2) and (4) depends on their singularity. For regular integrals, Gaussian quadrature is straightforward, whereas for singular ones, special treatments are required depending on the singularity type. For weak singular integrals, a local transformation with Gaussian quadrature is employed to improve the accuracy of their evaluation [16]. The singularity subtraction method [6] is used for improper integrals (strong singular and hyper-singular). In the neighborhood of a collocation node, the regular part of the integrand is expressed as a Taylor's expansion of sufficient terms to isolate the singularity. The original improper integral is thus transformed into a sum of regular integral and an integral of a singular function. Standard Gaussian quadrature is then used for numerical evaluation of the regular integral, while the singular function is evaluated analytically. Elliptic functions K and E are approximated by polynomial expressions [15].

4.1. The stress intensity factors evaluation

Near the crack tip, the elastic field is defined by an infinite series expansion that can be decoupled into mode I and II components [17]. Considering only the first term of the expansion, the displacement field on the crack surfaces is identical to the plane strain one. In a polar coordinate system centred at the crack tip, one can write:

$$u_2(\theta = \pi) - u_1(\theta = -\pi) = \frac{4(1-\nu)}{G} K_I \sqrt{\frac{r}{2\pi}} \tag{5}$$

$$u_1(\theta = \pi) - u_1(\theta = -\pi) = \frac{4(1-\nu)}{G} K_{II} \sqrt{\frac{r}{2\pi}} \tag{6}$$

where G is the shear modulus, and ν the Poisson's ratio, K_I and K_{II} are the stress intensity factors for the deformation modes I and II respectively, they can be computed from Eqs. (5) and (6), when the displacements on the crack surfaces are known.

4.2. Crack propagation

The crack growth direction is determined by the maximum principal stress criterion, which stipulates that the crack will grow perpendicularly to the principal stress direction at the crack tip. If K_I and K_{II} are known, this direction is calculated by the following expression:

$$\theta = 2tg^{-1}\left[\frac{1}{4}\left(k \pm \sqrt{k^2+8}\right)\right] \qquad k = K_I / K_{II} \tag{7}$$

where θ is measured from the crack axis in front of the crack tip.

Once the growth direction is known, an increment of length δa is added to the crack at each tip. This procedure constitutes an extremely easy re-meshing approach, since the new increments contributes with a few extra rows and columns to the global system of equations, quickly assembled.

5. Numerical applications

5.1. Evaluation of the convergence of the method

Different cases of axisymmetric cracks in a thick-walled cylinder are considered, using *KSP* code [19], with implementation of a new module for axisymmetric DBEM. The convergence of results using the crack displacement extrapolation is apparent from Fig. 1 as the number n of boundary elements increase. Less than 1% error was achieved with $n = 32$ for the two cases (internal and external circumferential crack) confirming the possibility to carry out good results using smutty meshes.

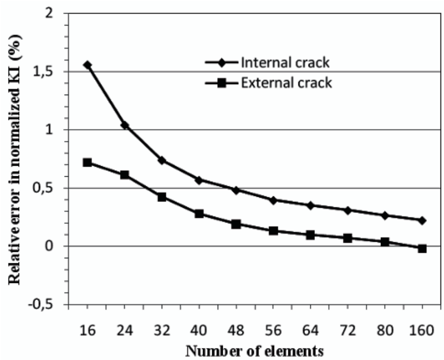

Fig. 1. Evaluation of the error according to the full number of elements used to discretize the geometry

5.2. Axially loaded thick-walled cylinder with circumferential crack

Consider a thick-walled cylinder with internal radius R_i, external radius R_e, height H and material properties E = 70,000 MPa and ν = 0.3, subjected to a tensile axial stress σ = 1.0. The cylinder contains a circumferential crack (internal or external), with radius *a* perpendicular to the axis of symmetry at mid-height position.

Normalized K_I is computed for three values of the ratio R_i/R_e and results are compared to those of the reference [18].

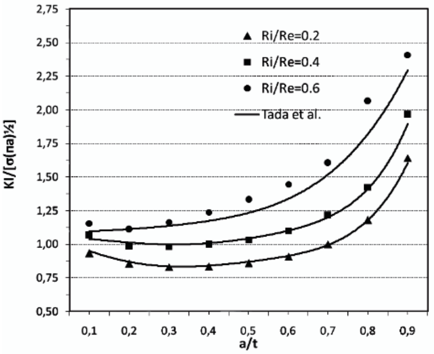

Fig. 2. Comparison of normalized stress intensity factor of an internal circumferential crack in a thick-walled cylinder

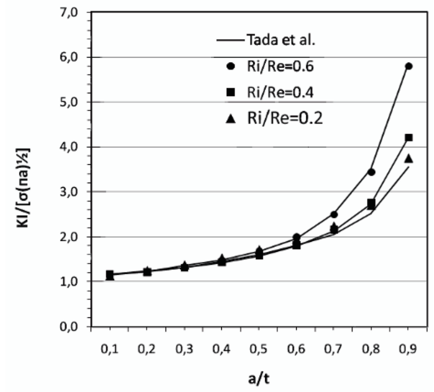

Fig. 3. Comparison of normalized stress intensity factor of an external circumferential crack in a thick-walled cylinder

A very close agreement can be seen for the two smaller values of the ratio, but the difference tends to increase as the thickness of the cylinder decreases for internal crack case (Figs. 2 and 3).

5.3. Cone crack propagation

In this example, simulation of the propagation of a centred internal cone crack, in a thick-walled cylinder is illustrated under different static loading conditions. Data for this example are: $R_i/R_e = 0.4$, $H = 4a$, initial crack length $a/t = 1/12$ and the cone crack is at 45°, $E = 70{,}000$ and $\nu = 0.3$.

The boundary mesh includes 20 discontinuous quadratic elements at each segment. At the crack, 10 equal discontinuous quadratic elements were initially located on each side. At each increment, the crack grows by two elements (one at each tip) of the same type and size of those used for the initial crack. For each load case, a crack trajectory is obtained, for a total of 20 crack increments (Fig. 4a–c). Similar previous results could not be found in literature for comparison but the present results are regular with each applied load.

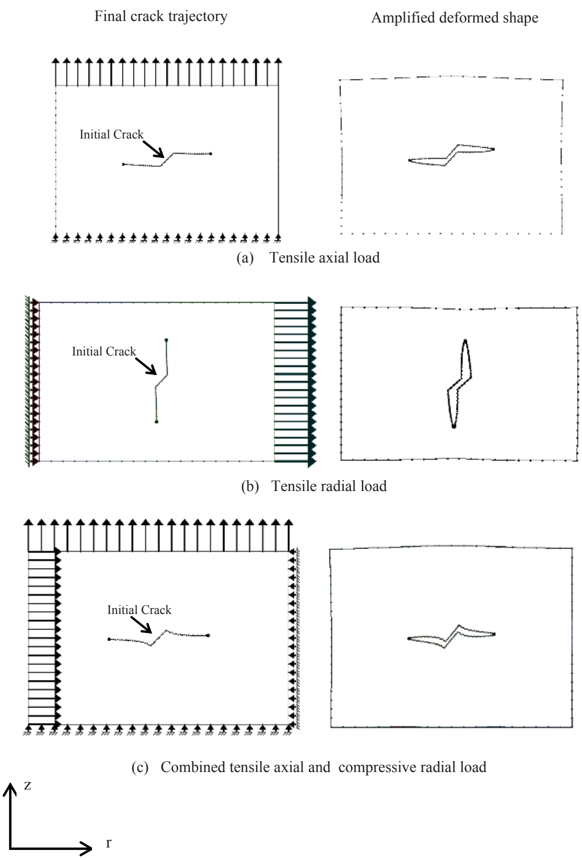

Fig. 4. Crack propagation of a cone crack inside a thick-walled cylinder under three different loading conditions

6. Conclusions

Cracked axisymmetric elastic solids have been analyzed using the dual boundary element method. The DBEM incorporates two independent boundary integral equations with their corresponding fundamental solutions.

Monomial transformation was applied to weak singular integrals, to improve precision with a reduced number of Gauss points. Singularity subtraction technique was applied to strong singular and hypersingular kernels, to allow evaluation of the Hadamard and Cauchy principal Value integrals.The use of discontinuous quadratic boundary element ensure continuity of the strains at the collocation nodes and then, the existence of the finite-part integrals. Stress intensity factors for circumferential or internal cracks were evaluated using the displacement extrapolation technique, and results were compared to analytical solutions. Accurate results were obtained for all treated applications.

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