A new method to solve crack problems based on gradient elasticity

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ABSTRACT. A new constant displacement discontinuity method is presented for the numerical solution of cracks of the three fundamental deformation modes I, II and III lying either in the plane or in the half-plane. This method is based on the strain-gradient elasticity theory in its simplest possible Grade-2 (G2) variant for the solution of classical crack problems. The assumption of the G2 expression for the stresses results to a more accurate average stress value at the mid-point of the straight displacement discontinuity since it takes into account the stress-gradient effect and gives considerably better predictions of the stress intensity factors compared to the compared to the classical elasticity solution. Moreover, the proposed method preserves the simplicity and hence the high speed of computations. For brevity of this paper we present here the specific case of mode-II cracks perpendicular to the free-surface of the half-plane. Validation of the method is made against analytical solution.

RÉSUMÉ. On présente une nouvelle méthode de discontinuité en déplacement pour la solution numérique de fissures en modes I, II ou III dans un plan ou un demi-plan. Cette méthode est basée sur une théorie d'élasticité du second gradient pour la solution de problèmes de fissures classiques. L'hypothèse de second gradient conduit à une contrainte moyenne plus précise au point milieu d'une discontinuité en déplacement droite car l'effet de gradient de contraintes est pris en compte et permet d'obtenir des prédictions bien meilleures des facteurs d'intensité de contraintes si l'on compare à la solution élastique classique. De plus, la simplicité de l'approche permet des calculs numériques très rapides. Dans cet article, nous présentons le cas spécifique d'une fissure en mode II perpendiculaire à la surface libre d'un demi-plan. La méthode est validée par comparaison avec la solution analytique.

KEYWORDS: displacement discontinuity, strain-gradient elasticity, crack problems, stress intensity factors, half-plane.

MOTS-CLÉS: déplacement, discontinuité, élasticité du second gradient, fissures, facteur d'intensité de contraintes, demi-plan.

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1. Introduction

Even though much achievement has been made in Linear Elastic Fracture Mechanics (LEFM) modeling techniques, a simple, albeit fast, and practical crack modeling technique is still needed, in particular for complex multiple crack growth problems (*e.g.* Figure 1) and fatigue problems of brittle materials. Crouch and Starfield (1990) by employing the Neuber-Papkovich potential functions corresponding to point dislocations (*i.e.* displacement discontinuities) has developed such simple and practical technique the so-called Displacement Discontinuity Method (DDM) for solving crack and solid mechanics problems. It is worth mentioning here that DDM may be seen as a special case of the dual Boundary Element Method (BEM) (Chen and Hong, 1999). This method is attractive for researchers and practitioners in the field of LEFM, due to its simplicity as compared to cumbersome complex variables theory of Muskhelishvili, combined with singular integral equations and numerical integration rules.

The assertion in DDM that the stress at the centre of a straight constant displacement discontinuity, as is calculated from first principles of the classical linear elasticity theory, represents the average stress over the element, leads to overestimation of displacements at the crack tips and consequently to large errors for the Stress Intensity Factors (SIFs). This is true if one considers the trapezoidal rule of integration and a constant or linearly varying stress σ in the region of the element. However, for a "quadratically" varying stress this rule must be enhanced, so as to incorporate the effect of curvature, that is to say:

$$\sigma = \langle \sigma \rangle - \frac{1}{24} L^2 \left(\frac{d^2 \sigma}{dx^2} \right)$$
^[1]

Field theories, which are based on averaging rules that include the effect of higher gradients, are called higher gradient theories. In particular above rule [1] represents a 2^{nd} gradient rule, and can be readily generalized in 2 and 3 dimensions by introducing the Laplacian operator instead of the second derivative. This has been done by assuming the constitutive relationship for the stresses:

$$\sigma_{ij} = \left(1 - \ell^2 \nabla^2\right) \left(\lambda \delta_{ij} \varepsilon_{kk} + 2G \varepsilon_{ij}\right), \quad i, j = 1, 2, 3$$
^[2]

where δ_{ij} is the Kronecker delta, ∇^2 is the Laplacian operator, λ and G are the usual Lamè constants, Einstein's summation rule is assumed, ℓ^2 is the straingradient term that has dimensions of length squared, and ε_{ij} denotes the symmetric strain tensor.

Based on the above approach the stress at the centre of a straight dislocation is derived from the strain-gradient elasticity theory in its simplest possible Grade-2 (second gradient of strain or G2 theory) variant (Vardoulakis *et al.*, 1996; Exadaktylos *et al.*, 1996; Exadaktylos and Aifantis, 1996; Exadaktylos, 1998;

Exadaktylos, 1999; Exadaktylos and Vardoulakis, 2001). The value of the strain gradient coefficient or length-scale ℓ that gives the exact agreement of the mid-point displacement of the uniformly pressurized straight displacement discontinuity with the analytical solution for the uniformly pressurized crack, assuming that the latter is discretized with only one element, may be easily found. Since the errors of the classical DDM are larger in the regions close to the tips where also the stress distribution along the crack elements display higher gradients, a simple parabolic dependence of the strain gradient coefficient on the x-coordinate of the centre of the i-th element lying along Ox-axis was assumed. This G2 formulation applies only on the crack elements and not outside them, where it is assumed that the classical stresses are valid; furthermore, it does not alter the nature of the classical elasticity problem of a cracked body we are aiming to solve. That is to say, no extra boundary conditions along the crack are imposed that are necessary for the solution of a strain-gradient elasticity problem, apart from those prescribed by classical elasticity theory.

In two previous papers (Exadaktylos and Xiroudakis, 2010a, 2010b), the G2 constant displacement discontinuity (G2CDD) method was presented for the solution of mode I, II and III crack problems in an infinite plane isotropic elastic body as is illustrated in Figure 2. In this paper, the previous work is further elaborated to incorporate automatically traction-free boundary conditions for a semi-infinite region. Herein, a specific problem is solved in the next Section that is a special case of the general problem of a Displacement Discontinuity (DD) over an arbitrarily oriented, finite line segment in a semi-infinite body. It is instructive, however, to treat this special case separately before considering the more complicated general problem. Hereafter, tensile stresses are considered as positive quantities and the unit length is chosen to be the half-width of the DD element since the sizes of the elements are taken to be equal.

2. The half-plane G2 solution for a displacement discontinuity over a finite line segment normal to the free surface

For the extension of the G2CDD method to situations in which the region to be analyzed is affected by the proximity of a traction-free plane surface, it is necessary to obtain the solution for a constant displacement discontinuity over an arbitrarily oriented, finite line segment in a semi-infinite body. This solution is constructed by superposition from the infinite body results using the classical method of images (Hirth and Lothe, 1982). It consists of two parts, namely, the actual solution already found for an infinite body with a constant displacement discontinuity over an arbitrarily oriented, finite line segment in y < 0, an "image discontinuity" in y > 0that cancels out the shear stresses on y = 0, and a continuous distribution of normal stress on y = 0 to make the cancel out the normal tractions on the surface of the semi-infinite region $y \le 0$. Hence, the complete solution is given by the sum of the three separate solutions. The stress tensor due to the actual DD will be denoted by $\sigma_{ii(A)}$, those due to its image by $\sigma_{ii(I)}$, and those resulting from the supplemental

solution by $\sigma_{ij(S)}$, wherein indices i, j denote the Cartesian coordinates x, y. The complete solution for the half-plane $y \le 0$ then may be represented as follows:



Figure 1. A plane isotropic elastic body containing isolated or mutually intersecting cracks $L_1, L_2, ..., L_k$ and subjected to normal stresses σ_x, σ_y and shear stress τ_ℓ at infinity



Figure 2. Distribution of horizontal stress in a plane isotropic elastic body containing isolated or mutually intersecting cracks and subjected to far-field horizontal stress

As is illustrated in Figure 3, local coordinate systems attached on the actual and image finite linear segments are adopted in the following fashion, respectively,

$$\bar{x}_A = (y+h); \quad \bar{y}_A = -x, \tag{4}$$

$$\bar{x}_I = (y - h); \quad \bar{y}_I = -x \tag{5}$$

where *h* is the depth of the dislocation centre. We seek the solution referring to the global coordinate system Oxy for the stresses in plane strain conditions, produced by an arbitrarily oriented finite straight Mode II dislocation lying in the lower halfplane at a depth h and occupying the line segment $-1 < \bar{x}_A < 1$, $\bar{y}_A = 0$, *i.e.* for a = 1, with no loading at infinity and displaying a constant DD of magnitude $D_{\bar{x}}$ in local coordinates (Figure 1).



Figure 3. Half plane solution for a vertical crack with the method of images

The local stresses in global coordinates read as follows

$$\sigma_{xx} = \sigma_{\overline{yy}}(x_A, y_A), \sigma_{xy}(x_A, y_A) = -\sigma_{\overline{xy}}(x_A, y_A), \sigma_{yy} = \sigma_{\overline{xx}}(x_A, y_A)$$
[6]

The solution for Mode II stresses by employing the Fourier transform technique are (Exadaktylos and Xiroudakis, 2010b):

$$\sigma_{xx(A)}^{II}(x,y) = \frac{GD_{\bar{x}}}{2(1-\nu)} \sqrt{\frac{2}{\pi}} x \int_{0}^{\infty} \xi^{3/2} \left(1 + \ell^{2} \xi^{2}\right) J_{1/2}(\xi) e^{-|x|\xi} \sin((y+h)\xi) d\xi$$
^[7]

$$\sigma_{xy(A)}^{II}(x,y) = \frac{GD_{\overline{x}}}{2(1-\nu)} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^{1/2} \left(1 + \ell^{2} \xi^{2}\right) (1 - |x|\xi) J_{1/2}(\xi) e^{-|x|\xi} \cos((y+h)\xi) d\xi$$
[8]

$$\sigma_{yy(A)}^{II}(x,y) = \frac{GD_{\bar{x}}}{2(1-\nu)} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^{1/2} \left(1 + \ell^{2} \xi^{2}\right) (2-x\xi) J_{1/2}(\xi) e^{-|x|\xi} \sin((y+h)\xi) d\xi \qquad [9]$$

On the other hand the image element solution may be easily found as follows:

$$\sigma_{xx(I)}^{II}(x,y) = -\frac{GD_{\bar{x}}}{2(1-\nu)} \sqrt{\frac{2}{\pi}} x_{0}^{\tilde{y}} \xi^{3/2} (1+\ell^{2}\xi^{2}) J_{1/2}(\xi) e^{-|x|\xi} \sin((y-h)\xi) d\xi$$
[10]

$$\sigma_{xy(l)}^{II}(x,y) = -\frac{GD_{\bar{x}}}{2(1-\nu)} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^{1/2} \left(1 + \ell^{2} \xi^{2}\right) \left(1 - |x|\xi\right) J_{1/2}(\xi) e^{-|x|\xi} \cos((y-h)\xi) d\xi \qquad [11]$$

$$\sigma_{yy(l)}^{II}(x,y) = -\frac{GD_{\bar{x}}}{2(1-\nu)} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^{1/2} (1+\ell^{2}\xi^{2}) (2-x\xi) J_{1/2}(\xi) e^{-|x|\xi} \sin((y-h)\xi) d\xi \qquad [12]$$

The analytical expressions for the above semi-infinite integrals may be found analytically through formal integration rules (Exadaktylos and Xiroudakis, 2010b). Furthermore, for this problem the supplementary stresses may be found after certain mathematical manipulations as follows:

$$\sigma_{xx(S)}^{II}(x,y) = \frac{GD_{\overline{x}}}{2\pi(1-\nu)} x \left[\left(\frac{2}{x^{2} + (h+1-y)^{2}} - \frac{2}{x^{2} + (h-1-y)^{2}} \right) - \left(\frac{4(y(y+h+1)-(h+1)^{2})}{(x^{2} + (h+1-y)^{2})^{2}} - \frac{4(y(y+h-1)-(h-1)^{2})}{(x^{2} + (h-1-y)^{2})^{2}} \right) + \left(\frac{16y(h+1)(h+1-y)^{2}}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{16y(h-1)(h-1-y)^{2}}{(x^{2} + (h-1-y)^{2})^{3}} \right) + 4\ell^{2} \left\{ \left(\frac{1}{(x^{2} + (h+1-y)^{2})^{2}} - \frac{1}{(x^{2} + (h-1-y)^{2})^{2}} \right) - \left(\frac{y(4y-11(h+1)) + 4(h+1)^{2}}{(x^{2} + (h-1-y)^{2})^{3}} - \frac{(y(4y-11(h-1))) + 4(h-1)^{2}}{(x^{2} + (h-1-y)^{2})^{3}} \right) + 24 \left(\frac{(h+1-y)^{2}(y(y-8(h+1))) + (h+1)^{2}}{(x^{2} + (h+1-y)^{2})^{4}} - \frac{(h-1-y)^{2}(y(y-8(h-1))) + (h-1)^{2}}{(x^{2} + (h-1-y)^{2})^{4}} \right) + 192 \left(\frac{y(h+1)(h+1-y)^{4}}{(x^{2} + (h+1-y)^{2})^{5}} - \frac{y(h-1)(h-1-y)^{4}}{(x^{2} + (h-1-y)^{2})^{5}} \right) \right\} \right]$$

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$$\begin{aligned} \sigma_{xy(s)}^{H}(x,y) &= \frac{GD_{\overline{x}}}{2\pi(1-y)} y \left[\left(\frac{2}{x^{2} + (h+1-y)^{2}} - \frac{2}{x^{2} + (h-1-y)^{2}} \right) + \\ &+ \left(\frac{4(y+2(h+1))(h+1-y)}{(x^{2} + (h+1-y)^{2})^{2}} - \frac{4(y+2(h-1))(h-1-y)}{(x^{2} + (h-1-y)^{2})^{2}} \right) - \\ &- \left[\frac{16(h+1)(h+1-y)^{3}}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{16(h-1)(h-1-y)^{3}}{(x^{2} + (h-1-y)^{2})^{3}} \right] + \\ &+ 12\ell^{2} \left\{ \left[\frac{1}{(x^{2} + (h+1-y)^{2})^{2}} - \frac{(h-1-y)(2y-7(h-1))}{(x^{2} + (h-1-y)^{2})^{3}} \right] - \\ &- 8 \left\{ \frac{(y-1)(h+1)(h+1-y)^{3}}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{(y-1)(h-1)(h-1-y)^{3}}{(x^{2} + (h-1-y)^{2})^{3}} \right] - \\ &- 64 \left[\frac{(h+1)(h+1-y)^{5}}{(x^{2} + (h+1-y)^{2})^{5}} - \frac{(h-1)(h-1-y)^{5}}{(x^{2} + (h-1-y)^{2})^{5}} \right] \\ &- 64 \left[\frac{(h+1)(h+1-y)^{2}}{(x^{2} + (h+1-y)^{2})^{5}} - \frac{(h-1)(h-1-y)^{5}}{(x^{2} + (h-1-y)^{2})^{5}} \right] \\ &- \left[\frac{4(y(y-(h+1))(h+1-y)^{2}}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{16y(h-1)(h-1-y)^{2}}{(x^{2} + (h-1-y)^{2})^{2}} \right] - \\ &- \left[\frac{16y(h+1)(h+1-y)^{2}}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{16y(h-1)(h-1-y)^{2}}{(x^{2} + (h-1-y)^{2})^{3}} \right] + \\ &+ 4\ell^{2} \left[\left(\frac{1}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{16y(h-1)(h-1-y)^{2}}{(x^{2} + (h-1-y)^{2})^{3}} \right] + \\ &+ 4\ell^{2} \left[\left(\frac{1}{(x^{2} + (h+1-y)^{2})^{3}} - \frac{(y(2y-(h-1))-(h-1)^{2}}{(x^{2} + (h-1-y)^{2})^{3}} \right] - \\ &- \left[- \frac{24}{((h+1-y)^{2}(y(y-6(h+1))-(h+1)^{2})} - \frac{(h-1-y)^{2}(y(y-6(h-1))-(h-1)^{2})}{(x^{2} + (h-1-y)^{2})^{3}} - \frac{(15)}{(x^{2} + (h-1-y)^{2}} \right] - \\ &- 192 \left(\frac{y(h+1)(h+1-y)^{4}}{(x^{2} + (h+1-y)^{2}} - \frac{y(h-1)(h-1-y)^{4}}{(x^{2} + (h-1-y)^{2}} \right) \right] \right\} \right] \\ \end{array}$$

3. Validation example

Problems involving an infinite body with cracks – even intersecting cracks – subjected to complex loading conditions, are solved almost trivially by the DD method. Each crack is divided into a series of segments over which stress boundary conditions are known, and the appropriate equations are solved for the discontinuity components at each segment. Stresses elsewhere in the body then can be computed by summing the contributions of all the individual discontinuities. One of the main objectives of many analyses of crack problems in the context of LEFM is to obtain values of the SIFs K_I , K_{II} at the crack tips. A simple way of accomplishing this is to use the known LEFM relationships:

$$K_{I} = -\frac{G}{4(1-\nu)} \lim_{r \to 0} \left\{ \sqrt{\frac{2\pi}{r}} D_{y}(r) \right\}, \quad K_{II} = -\frac{G}{4(1-\nu)} \lim_{r \to 0} \left\{ \sqrt{\frac{2\pi}{r}} D_{x}(r) \right\}$$
[16]

where the negative sign is due to the adopted sign convention for the DD's and stresses, $D_y(r)$, $D_x(r)$ are the normal, shear and anti-plane shear components of displacement discontinuity a distance r from the crack tip(s). For practical purposes, the limits in Equations [16] can be approximated by evaluating the expressions for a fixed value of r, small in relation to the size of the crack. The SIFs are simply calculated numerically by using the displacement discontinuity at the midpoints of the crack-tip cracks. Thus, accurate values of SIFs may be obtained if the DD distributions in the vicinity of the crack tip(s) are known accurately.



Figure 4. Half plane with a vertical crack subjected to far-field uniform shear tractions

Next, the proposed method is validated against the vertical crack problem illustrated in Figure 4. This case refers to the straight crack normal to the freesurface of the half-plane subjected to uniform in-plane shear stress at infinity. The closed form expressions for the mode II SIF at crack tips A and B are given by Tada *et al.* (1973). Figure 5 shows the variation of the dimensionless mode II SIF at the two crack tips A, B as is predicted by the two numerical methods, namely CDD and G2CDD, and by the semi-analytical solution, as the relative crack distance from the free surface increases and for a fixed number of discretization elements of the crack (N = 10). From this figure the very nice agreement of the G2CDD method and its superiority compared with the CDD method may be noticed.



Figure 5. Comparison of the dependence of mode II SIF at the two tips A, B of the vertical crack discretized with N = 10 elements, on its relative distance from the free surface predicted by the semi-analytical solution (continuous and dashed lines) and the two numerical methods CDD and G2CDD

4. Conclusions

The innovation of the proposed method lies in the fact that this method is based on the strain-gradient elasticity theory in its simplest possible G2 variant for the solution of classical crack problems, *i.e.* cracks exhibiting inverse square root singularity. The assumption of the G2 expression [2] for the stresses results to a more accurate average stress value at the mid-point of the straight displacement discontinuity compared to the classical elasticity solution since it takes into account the stress-gradient effect. This new element with only one collocation point at element centre gives considerably better predictions of the stress intensity factors

compared to the constant displacement discontinuity element. Moreover, the new method preserves the simplicity and hence the high speed of computations. For brevity of this paper we have presented the solution for the specific case of mode-II cracks perpendicular to the free-surface of the half-plane. Validation of the method was successfully performed against analytical solution.

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