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The use of He's variational iteration method for obtaining the free vibration of an Euler–Bernoulli beam

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ABSTRACT

This paper presents a way of using He's variational iteration method to solve free vibration problems for an Euler–Bernoulli beam under various supporting conditions. By employing this technique, the beam's natural frequencies and mode shapes can be obtained and a rapidly convergent sequence is obtained during the solution. The results obtained are the same as the results obtained by the Adomian decomposition method. It is verified that the present method is accurate and it provides a simple and efficient approach for solving vibration problems for uniform Euler–Bernoulli beams. A robust and efficient algorithm is also programmed using Matlab based on the present method, which can be easily used to solve Euler Bernoulli beam problems.

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1. Introduction

Vibration problems for uniform Euler–Bernoulli beams have been solved by different approaches. Smith et al. [1] presented a fully Sinc–Galerkin method in both space and time for solving the Euler–Bernoulli beam problems with fixed and cantilever boundary conditions. The authors proved that the Sinc discretizations were especially suited for such problems and good numerical results were yielded from this method. Yeih et al. [2] obtained the natural frequencies and natural modes for an Euler–Bernoulli beam using a dual multiple reciprocity method (MRM) and the singular value decomposition method. Yeih's method was able to avoid the spurious eigenvalue problem and modes resulted from applying the conventional MRM. Naguleswaran [3–6] obtained an approximate solution for the transverse vibration of the uniform Euler–Bernoulli beam under linearly varying axial force. The author also extended this approach to find the natural frequencies, sensitivity and mode shapes of an Euler–Bernoulli beam with up to three step changes in cross-section.

A recent innovative method in solving these problems is presented by Lai et al. [7]. In this method, the Adomian decomposition method (ADM) is applied to solve the Euler–Bernoulli beam vibration problem. On employing the ADM, the governing differential equation becomes a recursive algebraic equation and boundary conditions become simple algebraic frequency equations. It was shown by the researchers that after some simple mathematical operations, any i th natural frequency and any i th mode shape can be obtained. Compared to the aforementioned approaches, Lai's method is simpler and more straightforward.

In this paper, He's variational iteration method is employed to solve the Euler–Bernoulli beam problem. This method was developed by the Chinese mathematician Ji-Huan He as a modification of a general Lagrange multiplier method [8–12]. This method has been extensively applied as a powerful tool for solving various kinds of problems, such as the Fokker–Planck

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equation [13], the quadratic Riccati differential equation [14], nonlinear equations in heat transfer problems [15,16], nonlinear ordinary differential equations [17], nonlinear Jaulent–Miodek equations [18], nonlinear equations with Riemann–Liouville fractional derivatives [19], the Cauchy reaction–diffusion problem [20], the semi-linear inverse parabolic equation [21], and the differential-difference equation [22], etc.

2. He's variational iteration method (HVIM)

In this section, the concept of He's variational iteration method is briefly introduced. Consider the general nonlinear differential equation given in the form

$$Lu(t) + Nu(t) = g(t) \quad (1)$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known function. By using the variational iteration method, a correction functional can be constructed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda (Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)) d\xi \quad (2)$$

where λ is a general Lagrange multiplier, which can be determined optimally via variational theory; the subscript n means the n th approximation; \tilde{u}_n is a restricted variation and $\delta\tilde{u}_n = 0$.

3. Using the HVIM to analyze the free vibration problem for a uniform Euler–Bernoulli beam

Ignoring shear deformation and rotary inertia effect, the equation of motion for lateral vibrations of a uniform Euler–Bernoulli beam can be written as

$$EI \frac{\partial^4 y(x, t)}{\partial x^4} + \rho A \frac{\partial^2 y(x, t)}{\partial t^2} = 0, \quad 0 < x < l \quad (3)$$

and the boundary conditions are given as

$$\left[c_{r3} \frac{\partial^3 y(x, t)}{\partial x^3} + c_{r2} \frac{\partial^2 y(x, t)}{\partial x^2} + c_{r1} \frac{\partial y(x, t)}{\partial x} + c_{r0} y(x, t) \right] \Big|_{x=0} = 0, \quad r = 1, 2 \quad (4)$$

$$\left[d_{r3} \frac{\partial^3 y(x, t)}{\partial x^3} + d_{r2} \frac{\partial^2 y(x, t)}{\partial x^2} + d_{r1} \frac{\partial y(x, t)}{\partial x} + d_{r0} y(x, t) \right] \Big|_{x=l} = 0, \quad r = 1, 2 \quad (5)$$

where $y(x, t)$ is the lateral deflection at distance x along the length of the beam and time t , EI , ρ , and A are the flexural rigidity, the density, and the cross-sectional area of the beam, respectively. c_{r0} , c_{r1} , c_{r2} , c_{r3} , d_{r0} , d_{r1} , d_{r2} , d_{r3} are constants coming from different boundary conditions for Euler–Bernoulli beams, where $r = 1$ and 2 . For any mode of vibration, the lateral deflection $y(x, t)$ can be written in the form

$$y(x, t) = Y(x)h(t) \quad (6)$$

where $Y(x)$ is the model deflection and $h(t)$ is a harmonic function of time. If ω denotes the frequency of $h(t)$, then

$$\frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 Y(x)h(t). \quad (7)$$

Eq. (3) is thus reduced to a differential equation

$$EI \frac{d^4 Y(x)}{dx^4} - \rho A \omega^2 Y(x) = 0, \quad 0 < x < l. \quad (8)$$

Without loss of generality the following dimensionless quantities can be introduced:

$$X = \frac{x}{l}, \quad Y(X) = \frac{Y(x)}{l}, \quad P = \frac{\rho A \omega^2 l^4}{EI} \quad (9)$$

where P is the eigenvalue for this problem. Then Eq. (8) can be written in a dimensionless form as

$$\frac{d^4 Y(X)}{dX^4} - PY(X) = 0, \quad 0 < X < 1. \quad (10)$$

Similarly, the boundary conditions (Eqs. (4) and (5)) can also be expressed in the dimensionless form

$$\left[\alpha_{r3} \frac{d^3 Y(X)}{dX^3} + \alpha_{r2} \frac{d^2 Y(X)}{dX^2} + \alpha_{r1} \frac{dY(X)}{dX} + \alpha_{r0} Y(X) \right] \Big|_{X=0} = 0, \quad r = 1, 2 \quad (11)$$

$$\left[\beta_{r3} \frac{d^3 Y(X)}{dX^3} + \beta_{r2} \frac{d^2 Y(X)}{dX^2} + \beta_{r1} \frac{dY(X)}{dX} + \beta_{r0} Y(X) \right] \Big|_{X=1} = 0, \quad r = 1, 2. \quad (12)$$

By comparing Eq. (10) with Eq. (1) the correctional function can be obtained as

$$Y_{n+1}(X) = Y_n(X) + \int_0^X \lambda \left(\frac{d^4 Y_n(t)}{dt^4} - PY_n(t) \right) dt. \tag{13}$$

Integrating the integral by parts, Eq. (13) becomes

$$Y_{n+1}(X) = Y_n(X) + \lambda Y_n^{(3)}(X) - \lambda^{(1)} Y_n^{(2)}(X) + \lambda^{(2)} Y_n^{(1)}(X) - \lambda^{(3)} Y_n(X) + \int_0^X [\lambda^{(4)} - P\lambda] Y_n(t) dt. \tag{14}$$

By taking the variation on both sides of Eq. (14) with respect to Y_n , one can obtain

$$\delta Y_{n+1}(X) = \delta Y_n(X) + \lambda \delta Y_n^{(3)}(X) - \lambda^{(1)} \delta Y_n^{(2)}(X) + \lambda^{(2)} \delta Y_n^{(1)}(X) - \lambda^{(3)} \delta Y_n(X) + \int_0^X [\lambda^{(4)} - P\lambda] \delta Y_n(t) dt. \tag{15}$$

The stationary conditions obtained from Eq. (15) are as follows:

$$\lambda|_{t=X} = 0. \tag{16}$$

$$\lambda^{(1)}|_{t=X} = 0. \tag{17}$$

$$\lambda^{(2)}|_{t=X} = 0. \tag{18}$$

$$1 - \lambda^{(3)}|_{t=X} = 0. \tag{19}$$

$$\lambda^{(4)} - P\lambda = 0. \tag{20}$$

From Eqs. (16)–(20), the Lagrange multiplier λ can be derived as

$$\lambda = \frac{(t - X)^3}{6}. \tag{21}$$

Next, by substituting Eq. (21) into Eq. (13) the correctional function can be represented as

$$Y_{n+1}(X) = Y_n(X) + \int_0^X \frac{(t - X)^3}{6} \left(\frac{d^4 Y_n(t)}{dt^4} - PY_n(t) \right) dt. \tag{22}$$

In order to start the iterations using Eq. (22), $Y_0(X)$ is needed, which is represented as a Maclaurin series with the first four terms

$$Y_0(X) = \sum_{m=0}^3 \frac{X^m}{m!} Y^{(m)}(0) = Y(0) + Y^{(1)}(0)X + \frac{Y^{(2)}(0)}{2!}X^2 + \frac{Y^{(3)}(0)}{3!}X^3. \tag{23}$$

From Eq. (22) one can obtain

$$\begin{aligned} Y_1(X) &= Y_0(X) + \int_0^X \frac{(t - X)^3}{6} \left(\frac{d^4 Y_0(t)}{dt^4} - PY_0(t) \right) dt \\ Y_2(X) &= Y_1(X) + \int_0^X \frac{(t - X)^3}{6} \left(\frac{d^4 Y_1(t)}{dt^4} - PY_1(t) \right) dt \\ \dots \\ Y_k(X) &= Y_{k-1}(X) + \int_0^X \frac{(t - X)^3}{6} \left(\frac{d^4 Y_{k-1}(t)}{dt^4} - PY_{k-1}(t) \right) dt. \end{aligned} \tag{24}$$

After obtaining $Y_k(X)$, the solution for Eq. (10) then can be expressed as

$$Y(X) = \lim_{k \rightarrow \infty} Y_k. \tag{25}$$

Since it is not possible to set k as infinity, a large number n is therefore selected and substituted into Eq. (25) instead of ∞ , where n is decided on the basis of the accuracy required;

$$Y(X) = Y_n(X). \tag{26}$$

From Eqs. (23) and (24) it is known that $Y_0(X)$ is a function of $Y(0)$ and X , and $Y_k(X)$ is a function of $Y(0)$, P , t , and X . Similarly, $Y_k(X)^{(1)}$, $Y_k(X)^{(2)}$, $Y_k(X)^{(3)}$ are also the functions of $Y(0)$, P , t , and X . Hence Eq. (12), one of the boundary conditions, can be rewritten as

$$\sum_{j=0}^3 f_{rj}^{[k]}(P) Y^{(j)}(0) = 0, \quad r = 1, 2 \tag{27}$$

where $f_{rj}^{[k]}$ are polynomials of P with respect to k . By solving Eqs. (11) and (27) simultaneously for the nontrivial solutions $Y^{(j)}(0)$ ($j = 0, 1, 2, 3$), the i th estimated eigenvalue P_i^k corresponding to k can be obtained, and the number of iterations n is decided by the following equation:

$$|P_i^{[n]} - P_i^{[n-1]}| \leq \epsilon \tag{28}$$

where ϵ is a small value preset according to the accuracy required. If Eq. (28) is satisfied, then $P_i^{[n]}$ will become the i th eigenvalue P_i . By substituting P_i into Eq. (24), $Y_n(X)$ can be found. However, finding $P_i^{[n]}$ from Eqs. (11) and (27) is very complicated and tedious; therefore a simplified method [7] is applied here to find $P_i^{[n]}$. This method reduces the number of terms in Eqs. (11) and (27) by applying the given boundary conditions of the beam at its left end $X = 0$. One can choose any two values of $Y^{(j)}(0)$ ($j = 0, 1, 2, 3$), as the arbitrary constants, and the remaining two quantities are functions of these two arbitrary constants. Then the two equations can be easily solved and the nontrivial solutions as well as the eigenvalue can be found. The following sections discuss different boundary conditions at the left end $X = 0$ for this problem.

Case 1: Free end at left end $X = 0$ ($x = 0$; see Eq. (9)).

In this case, the shear force and bending moment are zero at $X = 0$, that is $Y^{(2)}(0) = 0$, $Y^{(3)}(0) = 0$, and both $Y^{(1)}(0)$ and $Y(0)$ are considered as arbitrary constants which depend on the given boundary conditions at the right end $X = 1$ ($x = l$ according to Eq. (9)). Hence Eq. (27) can be simplified as

$$f_{r0}^{[k]}(P)Y^{(0)}(0) + f_{r1}^{[k]}(P)Y^{(1)}(0) = 0, \quad r = 1, 2. \tag{29}$$

For nontrivial solutions ($x, y \neq 0$), the equation is given as

$$\begin{vmatrix} f_{10}^{[k]}(P) & f_{11}^{[k]}(P) \\ f_{20}^{[k]}(P) & f_{21}^{[k]}(P) \end{vmatrix} = 0. \tag{30}$$

The i th estimated eigenvalue $P_i^{[k]}$ corresponding to k can then be obtained from Eq. (30). Without loss of any generality, the beam can be considered as connected to a translational spring and a rotational spring at its right end $X = 1$ ($x = l$). The boundary conditions at $X = 1$ can then be presented as

$$\left[EI \frac{d^2 Y(x)}{dx^2} + k_{RR} \frac{dY}{dx} \right]_{x=l} = 0 \tag{31}$$

$$\left[EI \frac{d^3 Y(x)}{dx^3} + k_{TR} Y(x) \right]_{x=l} = 0 \tag{32}$$

where k_{RR} is the rotational spring constant and k_{TR} is the translational spring constant. The above equations can also be expressed in a dimensionless form as

$$Y^{(2)}(1) + \beta_{RR} Y^{(1)}(1) = 0 \tag{33}$$

$$Y^{(3)}(1) + \beta_{TR} Y(1) = 0 \tag{34}$$

where $\beta_{RR} = k_{RR}/EI$ and $\beta_{TR} = k_{TR}/EI$. Starting from $k = 1$, assuming $Y_k(X)$ as $Y(X)$ and substituting $Y_k(1)$ from Eq. (24) into Eqs. (31) and (32), one can obtain

$$Y_k^{(2)}(1) + \beta_{RR} Y_k^{(1)}(1) = 0 \tag{35}$$

$$Y_k^{(3)}(1) - \beta_{TR} Y_k(1) = 0. \tag{36}$$

Comparing Eqs. (33) and (34) to Eq. (29), the values of $f_{10}(P), f_{11}(P), f_{20}(P), f_{21}(P)$ can be determined. Next, one substitutes these values into Eq. (30) and the i th estimated eigenvalue $P_i^{[k]}$ can be obtained with respect to k . If this $P_i^{[k]}$ satisfies Eq. (28) then k becomes n and $P_i^{[k]}$ is P_i . Otherwise if Eq. (28) is not satisfied with the current $P_i^{[k]}$, the same procedure is to be repeated with increasing k until Eq. (28) is satisfied. Finally, $Y(X)$ is determined from Eq. (26) on the basis of the n found.

Case 2: Clamped at the left end $X = 0$ ($x = 0$).

In this case, the deflection and slope are zero at $X = 0$, that is $Y(0) = 0$ and $Y^{(1)}(0) = 0$. $Y^{(2)}(0)$ and $Y^{(3)}(0)$ are arbitrary constants which depend on the given boundary conditions at the right end $X = 1$ ($x = l$). Eq. (27) then can be simplified as

$$f_{r2}^{[k]}(P)Y^{(2)}(0) + f_{r3}^{[k]}(P)Y^{(3)}(0) = 0, \quad r = 1, 2. \tag{37}$$

For nontrivial solutions, the equation is given as

$$\begin{vmatrix} f_{12}^{[k]}(P) & f_{13}^{[k]}(P) \\ f_{22}^{[k]}(P) & f_{23}^{[k]}(P) \end{vmatrix} = 0. \tag{38}$$

The i th estimated eigenvalue $P_i^{[k]}$ corresponding to k can be obtained from Eq. (38). Without loss of any generality, the beam is considered as connected to a translational spring and a rotational spring at its right end $X = 1$ ($x = l$). The boundary conditions at $X = 1$ are given as Eqs. (33) and (34).

Starting from $k = 1$, assuming $Y_k(X)$ as $Y(X)$ and substituting $Y_k(1)$ from Eq. (24) into Eqs. (33) and (34), we can obtain

$$Y_k^{(2)}(1) + \beta_{RR}Y_k^{(1)}(1) = 0 \tag{39}$$

$$Y_k^{(3)}(1) - \beta_{TR}Y_k(1) = 0. \tag{40}$$

$f_{12}(P), f_{13}(P), f_{22}(P), f_{23}(P)$ in Eq. (37) can be determined for this case by comparing Eqs. (39) and (40) to that equation. The i th estimated eigenvalue $P_i^{[k]}$ with respect to k is obtained by substituting these values into Eq. (38). Similarly, if Eq. (28) is satisfied with the $P_i^{[k]}$ obtained, the corresponding k will be n and this $P_i^{[k]}$ will be P_i . Otherwise another iteration step needs to be performed until an ideal $P_i^{[k]}$ is found which satisfies Eq. (28). $Y(X)$ is therefore obtained.

Case 3: Hinged at left end $X = 0$ ($x = 0$).

In this case, the deflection and bending moment are zero at the left end $X = 0$, that is $Y(0) = 0, Y^{(2)}(0) = 0$, and $Y^{(1)}(0), Y^{(3)}(0)$ are arbitrary constants that depend on the given boundary conditions at the right end $X = 1$ ($x = l$). Hence Eq. (26) can be simplified as

$$f_{r0}^{[n]}(P)Y(0) + f_{r2}^{[n]}(P)Y^{(2)}(0) = 0, \quad r = 1, 2. \tag{41}$$

For nontrivial solutions, the equation is given as

$$\begin{vmatrix} f_{10}^{[n]}(P) & f_{12}^{[n]}(P) \\ f_{20}^{[n]}(P) & f_{22}^{[n]}(P) \end{vmatrix} = 0. \tag{42}$$

From Eq. (42), the i th estimated eigenvalue $P_i^{[k]}$ corresponding to k is obtained for this case. Similarly, this beam can be considered as connected to a translational spring and a rotational spring at its right end $X = 1$ ($x = l$). Then the boundary conditions at the right end are specified in Eqs. (33) and (34).

Starting from $k = 1$ and still assuming $Y_k(X)$ as $Y(X)$, we can obtain the following equations by substituting $Y_k(1)$ from Eq. (24) into Eqs. (33) and (34):

$$Y_k^{(2)}(1) + \beta_{RR}Y_k^{(1)}(1) = 0 \tag{43}$$

$$Y_k^{(3)}(1) - \beta_{TR}Y_k(1) = 0. \tag{44}$$

By comparing Eqs. (43) and (44) to Eq. (41), the values of $f_{10}, f_{11}, f_{21}, f_{22}$ can be determined. Next, one substitutes these values into Eq. (42) and the i th estimated eigenvalue $P_i^{[k]}$ can be obtained along with its corresponding k . Like in the previous cases, if this $P_i^{[k]}$ satisfies Eq. (28) the current k is considered as n and $P_i^{[k]}$ is considered as P_i . Otherwise the same process needs to be repeated with increasing k until Eq. (28) is satisfied, and then $Y(X)$ can be obtained from Eq. (26).

4. Numerical examples

In this section the feasibility and the efficiency of He's variational iteration method is illustrated with two examples of the above cases. By following the procedures described in case 2 and case 3 the i th estimated eigenvalues $P_i^{[k]}$ for $i = 1, 2, 3, 4, 5, 6$ are calculated. The values obtained here are compared to those yielded from the ADM method [7] to validate He's variational iteration method.

Example 1. A clamped-free beam with elastic spring restraints at $X = 1$ ($x = l$).

Consider a clamped-free beam as discussed in case 2, whose right end $X = 1$ is connected to a translational spring and a rotational spring. It is assumed that $\beta_{RR} = 1, \beta_{TR} = 1$, and the small preset value $\epsilon = 0.0001$; then Eq. (28) becomes

$$|P_i^{[n]} - P_i^{[n-1]}| \leq 0.0001. \tag{45}$$

The algorithm introduced in case 2 is implemented into a Matlab program (Appendix A) to find the eigenvalue P_i . The calculated values of P_i are given in Table 1.

Here the first estimated eigenvalue P_1 is obtained in the fifth iteration and the second eigenvalue P_2 is obtained in the eighth iteration. The remaining eigenvalues are found in the same way, and these are listed in Table 1. These results are exactly the same as the results obtained by using the ADM method (see Table 1 of Ref. [7]). On the basis of the calculated eigenvalues, the natural frequencies and mode shapes for this clamped-free beam can be correctly obtained by using He's variational iteration method, as illustrated in [7].

Example 2. A pinned-free beam with elastic spring restraints at $X = 1$ ($x = l$).

Table 1Results for the i th eigenvalue $P_i^{[k]}$ for k up to 18 approximate terms – Example 1.

k	$P_1^{[k]}$	$P_2^{[k]}$	$P_3^{[k]}$	$P_4^{[k]}$	$P_5^{[k]}$	$P_6^{[k]}$
2	22.3736	231.5189				
3	21.3335	472.0628				
4	21.3305	562.1272				
5	21.3305	565.6123				
6	21.3305	565.6379	4049.8336	9 339.5961		
7	21.3305	565.6380	4030.2659	13 572.9073		
8	21.3305	565.6380	4030.0576	14 998.4559		
9	21.3305	565.6380	4030.0563	15 064.5559		
10	21.3305	565.6380	4030.0563	15 065.4757	40 917.1710	62 059.3503
11	21.3305	565.6380	4030.0563	15 065.4834	40 698.4889	81 698.4012
12	21.3305	565.6380	4030.0563	15 065.4834	40 695.3486	89 737.9000
13	21.3305	565.6380	4030.0563	15 065.4834	40 695.3164	90 259.8484
14	21.3305	565.6380	4030.0563	15 065.4834	40 695.3161	90 269.0215
15	21.3305	565.6380	4030.0563	15 065.4834	40 695.3161	90 269.1302
16	21.3305	565.6380	4030.0563	15 065.4834	40 695.3161	90 269.1311
17	21.3305	565.6380	4030.0563	15 065.4834	40 695.3161	90 269.1311
18	21.3305	565.6380	4030.0563	15 065.4834	40 695.3161	90 269.1311

Table 2Results for the i th eigenvalue $P_i^{[k]}$ for k up to 18 approximate terms – Example 2.

k	$P_1^{[k]}$	$P_2^{[k]}$	$P_3^{[k]}$	$P_4^{[k]}$	$P_5^{[k]}$	$P_6^{[k]}$
2	73.5340	92.2059				
3	47.5095	298.9577				
4	47.4140	353.6648				
5	47.4138	355.1710	3130.0422	3 437.2527		
6	47.4138	355.1791	2605.7908	7 663.6966		
7	47.4138	355.1791	2602.0470	10 423.4768		
8	47.4138	355.1791	2602.0176	10 955.5542		
9	47.4138	355.1791	2602.0174	10 969.4425		
10	47.4138	355.1791	2602.0174	10 969.5922	31 928.3732	55 908.0995
11	47.4138	355.1791	2602.0174	10 969.5931	31 881.6235	70 289.5517
12	47.4138	355.1791	2602.0174	10 969.5931	31 881.0367	73 969.0165
13	47.4138	355.1791	2602.0174	10 969.5931	31 881.0316	74 099.4366
14	47.4138	355.1791	2602.0174	10 969.5931	31 881.0316	74 101.3371
15	47.4138	355.1791	2602.0174	10 969.5931	31 881.0316	74 101.3566
16	47.4138	355.1791	2602.0174	10 969.5931	31 881.0316	74 101.3568
17	47.4138	355.1791	2602.0174	10 969.5931	31 881.0316	74 101.3568
18	47.4138	355.1791	2602.0174	10 969.5931	31 881.0316	74 101.3568

In this example, there is a pinned beam as described in case 3, whose right end $X = 1$ is connected to a translational spring and a rotational spring. Similarly, we assume that $\beta_{RR} = 0$, $\beta_{TR} = 25$, and $\epsilon = 0.0001$. The algorithm presented in case 3 is implemented in a Matlab program ([Appendix B](#)) in order to obtain the eigenvalues P_i , which are displayed in [Table 2](#).

In this problem, the first estimated eigenvalue P_1 is obtained in the sixth iteration and the second one P_2 is obtained in the seventh iteration. Following the same approach, the remaining eigenvalues are obtained, and these are listed in [Table 2](#). These results are the same as those obtained from the ADM method (see [Table 3](#) of Ref. [7]). Hence, the natural frequencies and mode shapes for the pinned–free beam can be obtained on the basis of these eigenvalues. The detailed solution approach is presented in [7].

5. Conclusion

The two numerical examples verify that the vibration problem for the Euler–Bernoulli beam can be solved by using He’s variational iteration method, accurately and efficiently. The eigenvalues of the Euler–Bernoulli beam calculated in this study are in good agreement with the results obtained from the ADM method. Although both He’s variational iteration method and the ADM method are efficient and powerful methods for solving the Euler–Bernoulli beam problem, He’s method leads to fewer calculations compared to the ADM method. As shown in the examples, each iteration step of He’s method gives a direct approximate solution to the problem, whereas each iteration step of the ADM method only gives components of the approximate solution. These components need to be summed together to obtain the approximate solution. Therefore, in using the ADM method, the summation of those components is required after each iteration step in order to attest to the accuracy of that step. Both examples presented in this paper require 17 iterations to get the sixth eigenvalue with a precision of 0.0001 by using either He’s variational iteration method or the ADM method. This means that the ADM method

requires 17 summations additionally for the iterations while He's variational method only needs one more calculation to find the Lagrange multiplier λ (Eq. (21)) to start the iterations. This therefore verifies that He's variational method takes less computational work than the ADM method.

Appendix A. Matlab program for Example 1

```
syms c d a b p x t b1 b2 f f11 f12 f21 f22 pe ps;
y(1)=a+b*x+c/2*x^2+d/6*x^3;
for i=1:18
    ya(i)=subs(y(i),x,t);
    y(i+1)=y(i)+int((t-x)^3/6*(diff(ya(i),t,4)-p*ya(i)),t,0,x);
z0(i)=y(i+1);z1(i)=diff(z0(i));z2(i)=diff(z1(i));z3(i)=diff(z2(i));
i1(i)=simplify(b1*subs(z1(i),{x,a,b},{1,0,0})+subs(z2(i),{x,a,b},{1,0,0}));
i2(i)=simplify(-
b2*subs(z0(i),{x,a,b},{1,0,0})+subs(z3(i),{x,a,b},{1,0,0}));
f=coeffs(i1(i),c);f11(i)=f(2);
f=coeffs(i1(i),d);f12(i)=f(2);
f=coeffs(i2(i),c);f21(i)=f(2);
f=coeffs(i2(i),d);f22(i)=f(2);
pe(i)=vpa(simplify(subs(f11(i)*f22(i)-
f12(i)*f21(i),{b1,b2},{1,1})),10);
end
```

Appendix B. Matlab program for Example 2

```
syms c d a b p x t b1 b2 f f11 f12 f21 f22 pe ps;
y(1)=a+b*x+c/2*x^2+d/6*x^3;
for i=1:18
    ya(i)=subs(y(i),x,t);
    y(i+1)=y(i)+int((t-x)^3/6*(diff(ya(i),t,4)-p*ya(i)),t,0,x);
z0(i)=y(i+1);z1(i)=diff(z0(i));z2(i)=diff(z1(i));z3(i)=diff(z2(i));
i1(i)=simplify(b1*subs(z1(i),{x,a,c},{1,0,0})+subs(z2(i),{x,a,c},{1,0,0}));
i2(i)=simplify(-
b2*subs(z0(i),{x,a,c},{1,0,0})+subs(z3(i),{x,a,c},{1,0,0}));
f=coeffs(i1(i),b);f11(i)=f(2);
```

```

f=coeffs(i1(i),d);f12(i)=f(2);
f=coeffs(i2(i),b);f21(i)=f(2);
f=coeffs(i2(i),d);f22(i)=f(2);
pe(i)=vpa(simplify(subs(f11(i)*f22(i)-
f12(i)*f21(i),{b1,b2},{0,25})),10);
end

```

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