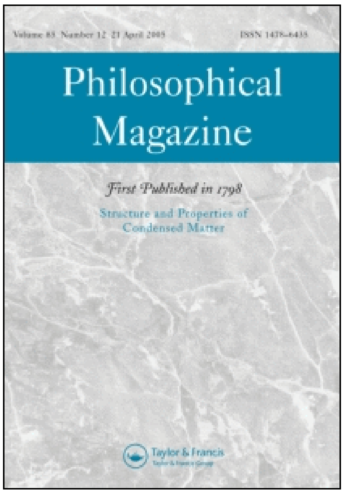

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Numerical solution for curved crack problem in elastic half-plane using hypersingular integral equation

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Numerical solution for curved crack problem in elastic half-plane using hypersingular integral equation

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A hypersingular integral equation for the curved crack problems of an elastic half-plane is introduced. Formulation of the equation is based on the usage of a modified complex potential. The potential is generally expressed in the form of a Cauchy-type integral. The modified complex potential is composed of the principal part and the complementary part. The principal part of the complex potential is actually equivalent to the original complex potential for the curved crack in an infinite plate. The role of the complementary part is to eliminate the boundary traction along the boundary of the half-plane caused by the principal part. From the assumed boundary traction condition, a hypersingular integral equation is obtained for the curved crack problems of an elastic half-plane. The curve length coordinate method is used to obtain a final solution. Several numerical examples are presented that prove the efficiency of the suggested method.

Keywords: hypersingular integral equation; numerical solution; curved crack; elastic half-plane; modified complex potential; crack–boundary interaction

1. Introduction

Previously, a singular integral equation was suggested to solve the curved crack problem in plane elasticity [1]. In the equation, the unknown function is the dislocation distribution along the crack and the right hand term is the traction on the crack face. Problems for slightly curved or kinked cracks were investigated by using this integral equation and the perturbation technique [2,3]. The curve length coordinate method provides an effective way to solve the equation simply because the quadrature rules on the line can be used for integrations along the curve [4].

Subsequently, a weaker singular integral equation with a logarithmic kernel was suggested to solve the curved crack problem [5]. In the equation, the unknown function is the dislocation distribution along the crack, and the right hand term is the resultant force function on the crack face. A particular feature of the equation is that a constant is involved in the right hand term of the equation. A boundary element technique was developed to solve the equation numerically [5].

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Another singular integral equation was suggested to solve the curved crack problem [6]. The particular feature of this type of equation is that the unknown function is the crack opening displacement (COD) distribution along the crack and the right hand term is the resultant force function on the crack face. In addition, there is also a constant involved in the right hand term of the equation. Formulations of the above-mentioned three types of integral equation were summarized in [7].

Comparatively speaking, the hypersingular integral equation for curved cracks is novel, and it was proposed somewhat recently [8–12]. In the equation, the COD is taken as unknown, and the boundary traction is the right hand term. Since the kernel possesses the hypersingularity, the involved hypersingular integral should be understood in the sense of finite part value [13–15]. A numerical technique for solving the equation is suggested by assuming that the CODs function as polynomials [8]. A perturbation method was developed to solve the slightly curved crack [11]. A curve length coordinate method in conjunction with an effective quadrature rule provides an effective way of solving the curved crack using a hypersingular integral equation [9]. Recently, multiple curved crack problems were solved using the hypersingular integral equation [16].

In a short research note [17], a proposal for the formulation of curved crack problem in an elastic half-plane using hypersingular integral equation was suggested, which is based on the usage of modified complex potentials [17]. However, no idea was suggested for a detailed numerical solution for the problem. Therefore, the problem has not been really solved in that note.

Some researchers studied the hypersingular integral equation from the formulation of the dual integral equations [18]. Through three different approaches, the dual integral equations of elasticity problems for domain points were investigated [18]. It was pointed out that the integral equation based on the Somigliana identity is too slim to solve the general elastic crack problems. The authors suggested an additional integral equation, or so-called dual integral equation. On the basis of dual integral equation, a line crack problem was solved using the hypersingular integral equation.

A perspective on the current status of the formulations of dual boundary element methods was presented. The authors emphasized the regularizations of hypersingular integrals and divergent series [19]. For the case of a degenerate boundary, the dual integral representation has been proposed for crack problems in elasticity. The authors paid attention to the second equation of the dual representation. The second equation derived for the secondary field is very popular now and is termed the hypersigular boundary equation [19].

One may formulate the hypersingular integral equation in the curved crack problem using the real variable analysis [11,19]. It was pointed out that, in the formulation, it is a necessary step to let the cavity shrink to a crack [11,19]. However, if one uses the complex variable and formulation based on the dislocation doublet, the formulation of hypersingular integral equation is more compact [7]. This situation can be seen from the relevant hypersingular integral equations solutions, one being Equations (6)–(9) of [11] and the other Equations (62) and (63) of [7]. In the latter case, the kernel is a "two points–two directions" function. Secondly, the hypersigular portion in the kernel is also easy to see.

This paper provides a numerical solution for the hypersingular integral equation for the curved crack problems in an elastic half-plane. In the problem, there are two boundaries along which the boundary conditions should be satisfied. Among them, one boundary is the curved crack face and another is the boundary of the half-plane. The problem is more difficult because of two involved boundary conditions.

In this paper, the hypersingular integral equation for the curved crack problems of an elastic half-plane is introduced. Formulation of the equation is based on the usage of modified complex potential [20]. The potential is generally expressed in a form of the Cauchy-type integral, and the involved density function is the COD function along the crack configuration. The modified complex potential is composed of a principal part and a complementary part. The principal part of the complex potential is actually equivalent to the original complex potential for the curved crack in an infinite medium. The role of the complementary part is to eliminate the boundary traction along the boundary of half-plane caused by the principal part. Therefore, the modified complex potential satisfies the traction-free condition along the boundary of the half-plane automatically. The process for deriving the modified complex potential is similar to the image method in the electrostatics. From the assumed boundary traction condition, a hypersingular integral equation is obtained for the curved crack problems of an elastic half-plane. The curve length coordinate method is used to obtain a final solution. Finally, in order to prove the efficiency of the suggested method, several numerical examples are presented.

2. Theoretical analysis

2.1. *Basic equations in plane elasticity*

The fundamentals of the complex variable function method, which plays an important role in plane elasticity, are briefly introduced in what follows [21]. In the method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of the complex potentials $\phi(z)$ and $\psi(z)$, such that

$$\begin{aligned} \sigma_x + \sigma_y &= 4\,\mathrm{Re}\,\phi'(z), \\ \sigma_y - i\sigma_{xy} &= 2\,\mathrm{Re}\,\phi'(z) + z\overline{\phi''(z)} + \overline{\psi'(z)}, \end{aligned} \tag{1}$$

$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}, \tag{2}$$

$$2G(u + iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \tag{3}$$

where G is the shear modulus of elasticity, $\kappa = (3 - \nu)/(1 + \nu)$ in the plane stress problem, $\kappa = 3 - 4\nu$ in the plane strain problem, ν is the Poisson's ratio, and a bar over a function denotes the conjugated value for the function.

In this paper, a derivative in a specified direction is defined for two analytic functions $f(z)$ and $g(z)$, which is as follows [20]:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{f(z)\overline{g(z)}\right\} = f'(z)\overline{g(z)} + f(z)\overline{g'(z)}\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}. \tag{3a}$$

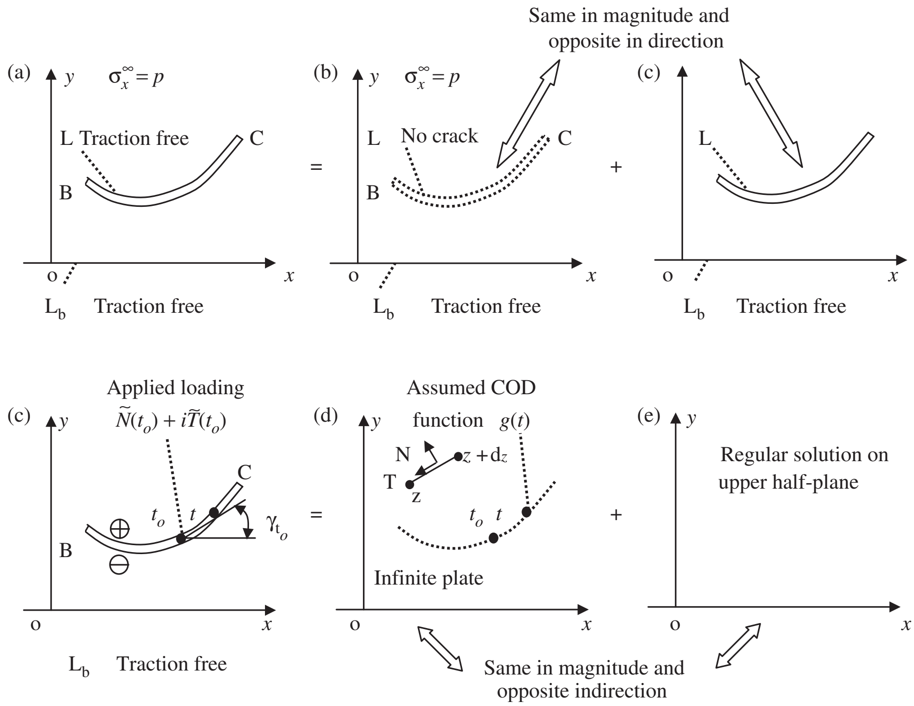

Figure 1. Principle of superposition for the solution of curved crack problem of elastic half-plane with traction-free boundary: (a) the original problem with remote tension $\sigma_x^\infty = p$; (b) an elastic half-plane with remote tension $\sigma_x^\infty = p$; (c) a curved crack problem with loadings on crack face; (d) a curved crack in an infinite plate modeled by the distribution of COD, corresponding to the complex potentials $\phi_p(z)$ and $\psi_p(z)$; (e) a regular solution for the upper half-plane, corresponding to the complex potentials $\phi_c(z)$ and $\psi_c(z)$.

Except for the physical quantities mentioned above, from Equations (2) and (3) two derivatives in a specified direction (DISD) are introduced as follows [20]:

$$J_1(z) = \frac{d}{dz}\{-Y + iX\} = \phi'(z) + \overline{\phi'(z)} + \frac{d\bar{z}}{dz}(z\overline{\phi''(z)} + \overline{\psi'(z)}) = N + iT, \tag{4}$$

$$J_2(z) = 2G\frac{d}{dz}\{u + iv\} = \kappa\phi'(z) - \overline{\phi'(z)} - \frac{d\bar{z}}{dz}(z\overline{\phi''(z)} + \overline{\psi'(z)}) = (\kappa + 1)\phi'(z) - J_1. \tag{5}$$

It is easy to verify that $J_1 = N + iT$ denotes the normal and shear tractions along the segment $\overline{z, z + dz}$ (see Figure 1). Secondly, the J_1 and J_2 values depend not only on the position of a point "z", but also on the direction of the segment or "$d\bar{z}/dz$". The symbol of derivative $d\{\ \}/dz$ is always defined as a DISD [20].

2.2. *Solution of curved crack problem in elastic half-plane using hypersingular integral equation*

The original problem is shown by Figure 1a, where the remote stress is σ_x^∞. In addition, a curved crack is placed in the upper half-plane with the traction-free condition along the boundary of half-plane. The problem shown by Figure 1a can be considered a superposition of two particular problems shown by Figures 1b and 1c.

Further, the problem shown by Figure 1c can be considered a superposition of two particular problems shown by Figures 1d and 1e. In Figure 1c, some traction is applied on the crack face. In Figure 1d, the COD distribution is assumed along the prospective place of crack, which is placed in an infinite plate. The COD distribution is denoted by the function $g(t)$. The relevant complex potentials are denoted by $\phi_p(z)$ and $\psi_p(z)$. A regular (non-singular) stress field is defined for the problem shown by Figure 1e, the relevant complex potentials for the regular field are denoted by $\phi_c(z)$ and $\psi_c(z)$. The regular stress field shown by Figure 1e just compensates the tractions on the boundary of half-plane caused by the stress field in Figure 1d.

Therefore, the complex potentials for the case of Figure 1c can be expressed by

$$\phi(z) = \phi_p(z) + \phi_c(z), \quad \psi(z) = \psi_p(z) + \psi_c(z). \tag{6}$$

In Equation (6), the subscript "p" denotes the principal part of the complex potentials and "c" denotes the complementary part.

The complex potentials for the case of Figure 1d can be expressed through a COD function, $g(t)$, as follows [7]:

$$\phi_p(z) = \frac{1}{2\pi}\int_L \frac{g(t)\mathrm{d}t}{t-z}, \quad \phi_p'(z) = \frac{1}{2\pi}\int_L \frac{g(t)\mathrm{d}t}{(t-z)^2}, \tag{7}$$

$$\psi_p(z) = \frac{1}{2\pi}\int_L \frac{\overline{g(t)}\mathrm{d}t}{t-z} + \frac{1}{2\pi}\int_L g(t)\left(\frac{\mathrm{d}\bar{t}}{t-z} - \frac{\bar{t}\,\mathrm{d}t}{(t-z)^2}\right), \tag{8}$$

where the COD function $g(t)$ is defined as

$$g(t) = \frac{2G}{i(\kappa+1)}\left[(u(t)+iv(t))^{+} - (u(t)+iv(t))^{-}\right], \quad (t \in L). \tag{9}$$

In Equation (9), $(u(t)+iv(t))^{+}$ $[(u(t)+iv(t))^{-}]$ denotes the displacements at a point "t" of the upper (lower) face of crack "L" (Figure 1). From general analysis in the crack problem, the COD function possesses the following properties [7]:

$$\begin{aligned} &g(t) = O[(t-t_B)^{1/2}] \text{ (in the vicinity of the left crack-tip } t_B), \\ &g(t) = O[(t-t_C)^{1/2}] \text{ (in the vicinity of the right crack-tip } t_C). \end{aligned} \tag{10}$$

In this case, two functions $\phi_p(z)$and $\psi_p(z)$ are single-valued, and the single-valuedness condition of the displacement is satisfied automatically.

Substituting Equations (7) and (8) into Equation (4), letting the point "z" approach t_o $(t_o \in L)$ on the crack, changing $\mathrm{d}\bar{z}/\mathrm{d}z$ by $\mathrm{d}\bar{t}_o/\mathrm{d}t_o$, we will find that the $N+iT$ influence at the point t_o from complex potentials $\phi_p(z)$ and $\psi_p(z)$ is as follows [7]:

$$(N(t_o)+iT(t_o))_p = \frac{1}{\pi}\int_L \frac{g(t)\mathrm{d}t}{(t-t_o)^2} + \frac{1}{2\pi}\int_L M_1(t,t_o)\,g(t)\mathrm{d}t + \frac{1}{2\pi}\int_L M_2(t,t_o)\overline{g(t)}\mathrm{d}t, \quad (t_o \in L), \tag{11}$$

where the subscript "p" means that the influence is derived from the principal part of the complex potentials. In Equation (11), the first integral in right hand term represents a hypersingular integral [13–15], and two kernels are defined by

$$M_1(t,t_o) = -\frac{1}{(t-t_o)^2} + \frac{1}{(\bar{t}-\bar{t}_o)^2}\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\frac{\mathrm{d}\bar{t}_o}{\mathrm{d}t_o},$$
$$M_2(t,t_o) = \frac{1}{(\bar{t}-\bar{t}_o)^2}\left(\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}+\frac{\mathrm{d}\bar{t}_o}{\mathrm{d}t_o}\right) - \frac{2(t-t_o)}{(\bar{t}-\bar{t}_o)^3}\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}\frac{\mathrm{d}\bar{t}_o}{\mathrm{d}t_o}. \qquad (12)$$

Note that the kernels $M_1(t,t_o)$ and $M_2(t,t_o)$ depend not only on "t" and "t_o", but also on $\mathrm{d}\bar{t}/\mathrm{d}t$ and $\mathrm{d}\bar{t}_o/\mathrm{d}t_o$. In this sense, the kernels $M_1(t,t_o)$ and $M_2(t,t_o)$ belong to the "two points–two directions" function [7].

From Equation (2), the traction-free condition along the boundary of half-plane (L_b) in Figure 1c will lead to

$$\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = 0 \text{ (for } z \in L_b). \qquad (13)$$

Since along the boundary of half-plane, or the $z \in L_b$, we have $z = \bar{z}$, therefore, the condition (13) can be rewritten as

$$\overline{\phi_p(z)} + \overline{\phi_c(z)} + z(\phi_p'(z) + \phi_c'(z)) + \psi_p(z) + \psi_c(z) = 0 \text{ (for } z \in L_b). \qquad (14)$$

Substituting $\phi_p(z)$ and $\psi_p(z)$ defined by Equations (7) and (8) into Equation (14), after some manipulation, we find [20]

$$\phi_c(z) = -z\bar{\phi}_p'(z) - \bar{\psi}_p(z), \quad \phi_c'(z) = -\bar{\phi}_p'(z) - z\bar{\phi}_p''(z) - \bar{\psi}_p'(z), \qquad (15)$$

$$\psi_c(z) = -\bar{\phi}_p(z) + z\bar{\phi}_p'(z) + z^2\bar{\phi}_p''(z) + z\bar{\psi}_p'(z). \qquad (16)$$

In Equations (15) and (16), for example, $\bar{\phi}_p'(z)$ is an analytic function defined by $\bar{\phi}_p'(z) = \overline{\phi'(\bar{z})}$. From the known complex potentials $\phi_p(z)$ and $\psi_p(z)$ (the principal part), one obtains $\phi_c(z)$ and $\psi_c(z)$ (the complementary part) and then obtains $\phi(z) = \phi_p(z) + \phi_c(z)$, $\psi(z) = \psi_p(z) + \psi_c(z)$. The process was called the modified complex potential method [20].

Similarly, substituting Equations (15) and (16) into Equation (4), letting the point "z" approach $t_o (t_o \in L)$ on the crack, changing $\mathrm{d}\bar{z}/\mathrm{d}z$ by $\mathrm{d}\bar{t}_o/\mathrm{d}t_o$, we will find the $N + iT$ influence at the point t_o from complex potentials $\phi_c(z)$ and $\psi_c(z)$ as follows:

$$(N(t_o) + iT(t_o))_c = \frac{1}{2\pi}\int_L D_1(t,t_o)\,g(t)\mathrm{d}t + \frac{1}{2\pi}\int_L D_2(t,t_o)\,g(t)\mathrm{d}\bar{t}$$
$$+ \frac{1}{2\pi}\int_L D_3(t,t_o)\overline{g(t)}\mathrm{d}t + \frac{1}{2\pi}\int_L D_4(t,t_o)\overline{g(t)}\mathrm{d}\bar{t} \quad (t_o \in L), \qquad (17)$$

where the subscript "c" means that the influence is derived from the complementary part of the complex potentials. In Equation (17), four regular kernels are defined by

$$D_1(t,t_o) = \overline{A_1(t,t_o)} + \frac{\mathrm{d}\bar{t}_o}{\mathrm{d}t_o}B_1(t,t_o),$$
$$D_2(t,t_o) = D_3(t,t_o) = A_2(t,t_o) + \overline{A_2(t,t_o)} + \frac{\mathrm{d}\bar{t}_o}{\mathrm{d}t_o}B_2(t,t_o), \qquad (18)$$
$$D_4(t,t_o) = A_1(t,t_o),$$

$$A_1(t,t_o) = -\frac{1}{(\bar{t}-t_o)^2} - \frac{2(t_o-t)}{(\bar{t}-t_o)^3}, \quad A_2(t,t_o) = -\frac{1}{(\bar{t}-t_o)^2}, \tag{19}$$

$$\begin{aligned} B_1(t,t_o) &= \frac{2(3\bar{t}_o - 2t_o - \bar{t})}{(t-\bar{t}_o)^3} + \frac{6(\bar{t}_o-\bar{t})(\bar{t}_o-t_o)}{(t-\bar{t}_o)^4}, \\ B_2(t,t_o) &= \frac{1}{(t-\bar{t}_o)^2} + \frac{2(\bar{t}_o-t_o)}{(t-\bar{t}_o)^3}. \end{aligned} \tag{20}$$

The boundary condition for the curved crack shown in Figure 1c is as follows:

$$N(t_o) + iT(t_o) = \tilde{N}(t_o) + i\tilde{T}(t_o)\ (t_o \in L), \tag{21}$$

where the right hand term can be evaluated from Figure 1c, which is as follows:

$$\tilde{N}(t_o) + i\tilde{T}(t_o) = -p(\sin^2\gamma_{t_o} + i\sin\gamma_{t_o}\cos\gamma_{t_o})\ (\text{with } \sigma_x^\infty = p), \tag{22}$$

where γ_{t_o} denotes the inclined angle at the observation point t_o.

Since the $N + iT$ influence is composed of those from the principle part and the complementary part of the complex potentials, the boundary condition Equation (21) can be rewritten in the form

$$(N(t_o) + iT(t_o))_p + (N(t_o) + iT(t_o))_c = \tilde{N}(t_o) + i\tilde{T}(t_o). \tag{23}$$

Substituting Equations (11) and (17) into Equation (23) yields the following hypersingular integral equation:

$$\begin{aligned} &\frac{1}{\pi}\int_L \frac{g(t)\mathrm{d}t}{(t-t_o)^2} + \frac{1}{2\pi}\int_L M_1(t,t_o)\,g(t)\mathrm{d}t + \frac{1}{2\pi}\int_L M_2(t,t_o)\overline{g(t)}\mathrm{d}t \\ &\quad + \frac{1}{2\pi}\int_L D_1(t,t_o)\,g(t)\mathrm{d}t + \frac{1}{2\pi}\int_L D_2(t,t_o)\,g(t)\mathrm{d}\bar{t} \\ &\quad + \frac{1}{2\pi}\int_L D_3(t,t_o)\overline{g(t)}\mathrm{d}t + \frac{1}{2\pi}\int_L D_4(t,t_o)\overline{g(t)}\mathrm{d}\bar{t} = \tilde{N}(t_o) + i\tilde{T}(t_o)\ (t_o \in L). \end{aligned} \tag{24}$$

Clearly, the first integral in the left hand term is a hypersingular integral, and remainders are regular integral.

It is assumed that the function $t(s)$ maps the curve L on the interval $(-a, a)$ of the s-axis. In this case, the COD function can be expressed as

$$g(t) = \sqrt{a^2 - s^2}H(s). \tag{25}$$

The curve length coordinate method is used to solve the integral equations numerically. The curved crack is mapped on the real axis, and the detail for this method can be referred to [9]. The available quadrature rules for the hypersingular integral and regular integral can be obtained by referring to the appendix or [15].

After the COD function $g(t)$ is obtained from a solution of Equation (24), the stress intensity factor (SIF) at the left crack-tip "B" can be evaluated by

$$(K_1 - iK_2)_B = \sqrt{2\pi}\lim_{t\to t_B}\sqrt{|t-t_B|}\mathrm{d}g(t)/\mathrm{d}t = (\pi a)^{1/2}H(-a)\exp(-i\gamma_B), \tag{26}$$

where γ_B denotes the tangent angle at the left crack-tip t_B.

Similarly, for the right crack-tip "C", we have

$$(K_1 - iK_2)_C = -\sqrt{2\pi}\,\underset{t\to t_C}{\mathrm{Lim}}\sqrt{|t-t_C|}\,\mathrm{d}g(t)/\mathrm{d}t = (\pi a)^{1/2}H(a)\exp(-i\gamma_C), \tag{27}$$

where γ_C denotes the tangent angle at the right crack-tip t_C (Figure 1).

2.3. *Multiple curved crack problem in an elastic half-plane*

The problem for two curved cracks is introduced below. It is assumed that there are two curved cracks in the upper half-plane with the traction-free boundary of half-plane (Figure 2). The problem can be solved in a similar manner as for the single curved crack problem if only the interaction between two cracks is considered. The COD function for the first curved crack L_1 is denoted by $g_1(t)$, and the COD function for the second curved crack L_2 is denoted by $g_2(t)$. By using the principle of superposition, we will find the hypersingular integral equation for the crack L_1 ($t_{o1} \in L_1$) as follows:

$$\begin{aligned}
&\frac{1}{\pi}\int_{L_1}\frac{g_1(t)\mathrm{d}t}{(t-t_{o1})^2}+\frac{1}{2\pi}\int_{L_1}M_1(t,t_{o1})g_1(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_1}M_2(t,t_{o1})\overline{g_1(t)}\mathrm{d}t\\
&+\frac{1}{2\pi}\int_{L_1}D_1(t,t_{o1})g_1(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_1}D_2(t,t_{o1})g_1(t)\mathrm{d}\bar{t}\\
&+\frac{1}{2\pi}\int_{L_1}D_3(t,t_{o1})\overline{g_1(t)}\mathrm{d}t+\frac{1}{2\pi}\int_{L_1}D_4(t,t_{o1})\overline{g_1(t)}\mathrm{d}\bar{t}\\
&+\frac{1}{\pi}\int_{L_2}\frac{g_2(t)\mathrm{d}t}{(t-t_{o1})^2}+\frac{1}{2\pi}\int_{L_2}M_1(t,t_{o1})g_2(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_2}M_2(t,t_{o1})\overline{g_2(t)}\mathrm{d}t\\
&+\frac{1}{2\pi}\int_{L_2}D_1(t,t_{o1})g_2(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_2}D_2(t,t_{o1})g_2(t)\mathrm{d}\bar{t}\\
&+\frac{1}{2\pi}\int_{L_2}D_3(t,t_{o1})\overline{g_2(t)}\mathrm{d}t+\frac{1}{2\pi}\int_{L_2}D_4(t,t_{o1})\overline{g_2(t)}\mathrm{d}\bar{t}=\tilde{N}(t_{o1})+i\tilde{T}(t_{o1})\,(t_{o1}\in L_1).
\end{aligned} \tag{28}$$

Clearly, the first integral in the left hand term of Equation (28) is a hypersingular integral, and remainders are regular integral.

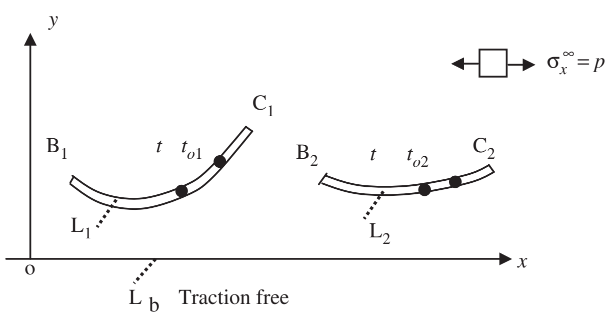

Figure 2. Two curved cracks in an elastic half-plane with traction-free boundary.

Similarly, the hypersingular integral equation for the crack L_2 ($t_{o2} \in L_2$) will be

$$\begin{aligned}
&\frac{1}{\pi}\int_{L_2}\frac{g_2(t)\mathrm{d}t}{(t-t_{o2})^2}+\frac{1}{2\pi}\int_{L_2}M_1(t,t_{o2})g_2(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_2}M_2(t,t_{o2})\overline{g_2(t)}dt\\
&+\frac{1}{2\pi}\int_{L_2}D_1(t,t_{o2})g_2(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_2}D_2(t,t_{o2})g_2(t)\mathrm{d}\bar{t}\\
&+\frac{1}{2\pi}\int_{L_2}D_3(t,t_{o2})\overline{g_2(t)}\mathrm{d}t+\frac{1}{2\pi}\int_{L_2}D_4(t,t_{o2})\overline{g_2(t)}\mathrm{d}\bar{t}\\
&+\frac{1}{\pi}\int_{L_1}\frac{g_1(t)\mathrm{d}t}{(t-t_{o2})^2}+\frac{1}{2\pi}\int_{L_1}M_1(t,t_{o2})g_1(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_1}M_2(t,t_{o2})\overline{g_1(t)}\mathrm{d}t\\
&+\frac{1}{2\pi}\int_{L_1}D_1(t,t_{o2})g_1(t)\mathrm{d}t+\frac{1}{2\pi}\int_{L_1}D_2(t,t_{o2})g_1(t)\mathrm{d}\bar{t}\\
&+\frac{1}{2\pi}\int_{L_1}D_3(t,t_{o2})\overline{g_1(t)}\mathrm{d}t+\frac{1}{2\pi}\int_{L_1}D_4(t,t_{o2})\overline{g_1(t)}\mathrm{d}\bar{t}=\tilde{N}(t_{o2})+i\tilde{T}(t_{o2})\,(t_{o2}\in L_2). \qquad (29)
\end{aligned}$$

It can be seen that the multiple curved crack problem can be solved in a similar manner as in the single curved crack case.

3. Numerical examples

Some numerical examples are given to illustrate the efficiency of the presented method. From the above-mentioned formulation, we see that the conditions of the plane stress or the plane strain do not affect the result for SIFs at the crack-tips.

3.1. *Example 1*

In the first example, a half-circle arc crack with radius R is placed in the upper half-plane. The crack-tip has a distance d from the boundary. The remote tension is $\sigma_x^\infty = p$, and the boundary of the half-plane L_b is traction free (Figure 3a). $M = 155$ is used in the quadrature rule shown in the appendix.

In the condition of $d/R = 0.2, 0.4, \ldots, 2.0$, the calculated results for the SIFs at the crack-tips "B" and "C" are expressed as

$$K_{1B} = K_{1C} = F_{1B}(d/R)\,p\sqrt{\pi R}, \quad K_{2B} = -K_{2C} = -F_{2B}(d/R)\,p\sqrt{\pi R}, \qquad (30)$$

and they are plotted in Figure 4.

The computed results from an integral equation with logarithmic kernel [22] are also plotted in Figure 4. The deviation between the two sources is not too significant.

From the plotted results, it can be seen that in the studied range of d/R, the influence of the ratio d/R is not significant. For example, in the case of an infinite plate with the curved crack (or $d/R = \infty$), we have an exact solution, $F_{1B}(d/R)_{d/R\to\infty} = 0.6482$ and $F_{2B}(d/R)_{d/R\to\infty} = 0.0589$. In addition, in the case of $d/R = 0.2$, we have $F_{1B}(d/R)_{d/R=0.2} = 0.7613$ and $F_{2B}(d/R)_{d/R=0.2} = 0.0056$. In the case of $d/R = 2$, we have $F_{1B}(d/R)_{d/R=2} = 0.6574$ and $F_{2B}(d/R)_{d/R=2} = 0.0621$. Hence, if $d/R > 2$, the influence from the traction-free condition along the boundary can be neglected.

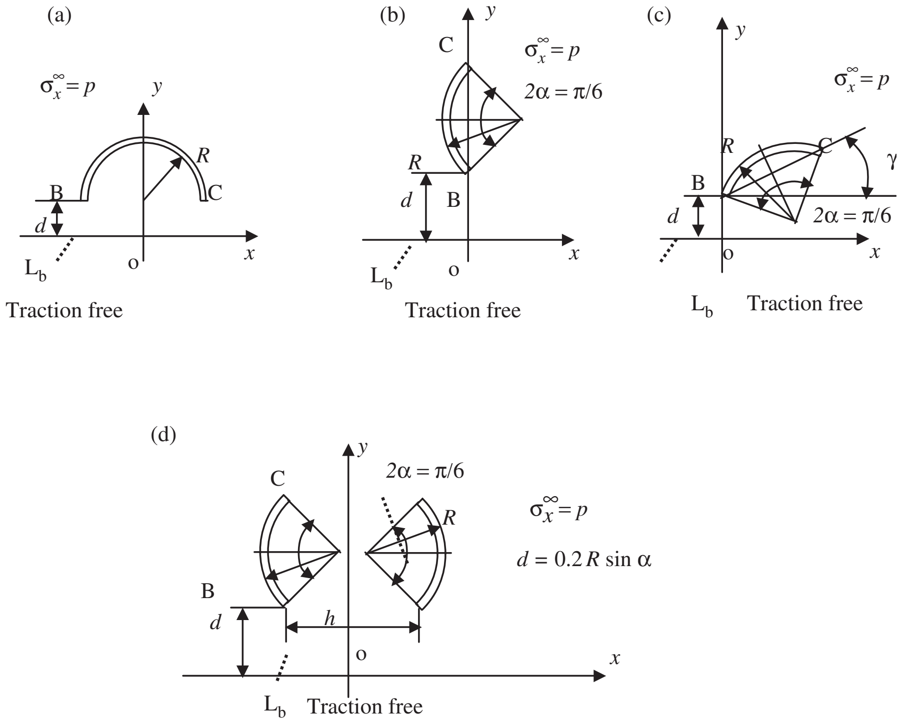

Figure 3. Configurations of the curved cracks in an elastic half-plane with traction-free boundary: (a) a half-circle crack; (b) an arc crack; (c) an arc crack with rotation; (d) two arc cracks.

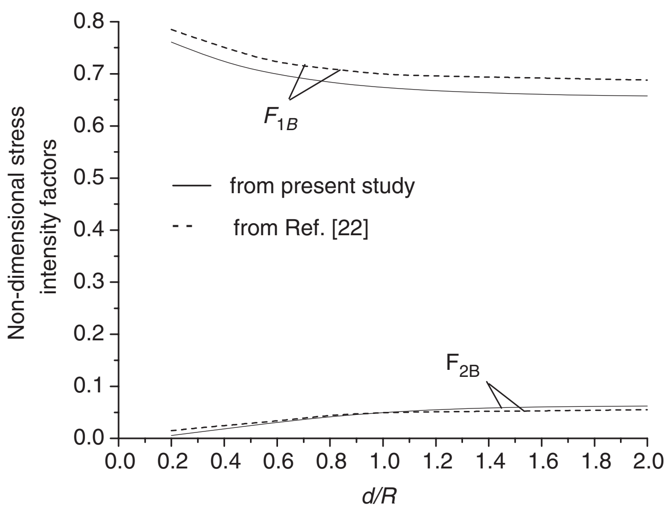

Figure 4. Non-dimensional SIFs $F_{1B}(d/R)$ and $F_{2B}(d/R)$ for a half-circle crack in an elastic half-plane (see Figure 3a and Equation (30)).

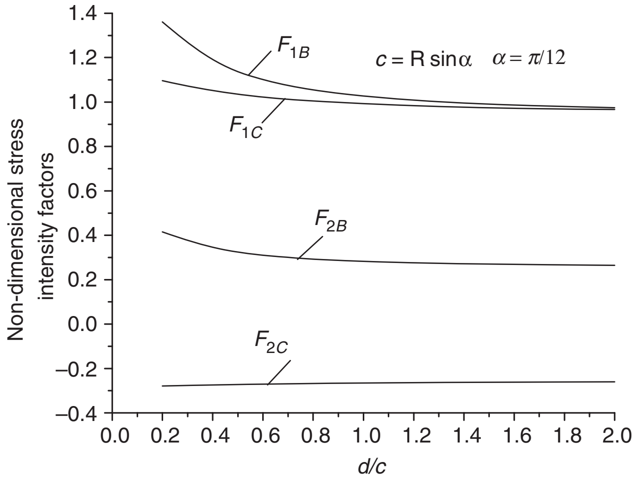

Figure 5. Non-dimensional SIFs $F_{1B}(d/c)$, $F_{2B}(d/c)$, $F_{1C}(d/c)$ and $F_{2C}(d/c)$ ($c = R\sin\alpha$) for an arc crack in an elastic half-plane (see Figure 3b and Equation (31)).

3.2. *Example 2*

In the second example, the half-circle crack with radius R and the spanning angle 2α is placed in the upper half-plane, and $2\alpha = \pi/6$ is taken. The crack-tip "B" has a distance d from the boundary. The remote tension is $\sigma_x^\infty = p$, and the boundary of the half-plane is of traction free (Figure 3b). $M = 155$ is used in the quadrature rule shown in the appendix.

In the condition of $d/c = 0.2, 0.4, \ldots, 2.0$ ($c = R\sin\alpha$), the calculated results for the SIFs at the crack-tip "B" and "C" are expressed as

$$\begin{aligned} K_{1B} &= F_{1B}(d/c)\,p\sqrt{\pi c}, \quad K_{2B} = F_{2B}(d/c)\,p\sqrt{\pi c}, \\ K_{1C} &= F_{1C}(d/c)\,p\sqrt{\pi c}, \quad K_{2C} = F_{2C}(d/c)\,p\sqrt{\pi c}\ (\text{with } c = R\sin\alpha), \end{aligned} \tag{31}$$

and they are plotted in Figure 5.

From the plotted results, it can be seen that in the studied range of d/c, the influence of the ratio d/c is significant. For example, in the case of an infinite plate with the curved crack (or $d/c = \infty$), we have $F_{1B}(d/c)_{d/c\to\infty} = F_{1C}(d/c)_{d/c\to\infty} = 0.9413$, $F_{2B}(d/c)_{d/c\to\infty} = -F_{2C}(d/c)_{d/c\to\infty} = 0.2545$. In addition, in the case of $d/c = 0.2$ in the present example, we have $F_{1B}(d/c)_{d/c=0.2} = 1.3609$, $F_{2B}(d/c)_{d/c=0.2} = 0.4154$, $F_{1C}(d/c)_{d/c=0.2} = 1.0953$ and $F_{2C}(d/c)_{d/c=0} = -0.2792$.

3.3. *Example 3*

In the third example, the arc crack with radius R has spanning angle 2α is placed in the upper half-plane, and $2\alpha = \pi/6$ is taken. The crack-tip "B" has a distance d from the boundary, and $d/(R\sin\alpha) = 0.2$ is assumed. The crack has an inclined angle γ with respect to the horizontal axis. The remote tension is $\sigma_x^\infty = p$ and the boundary

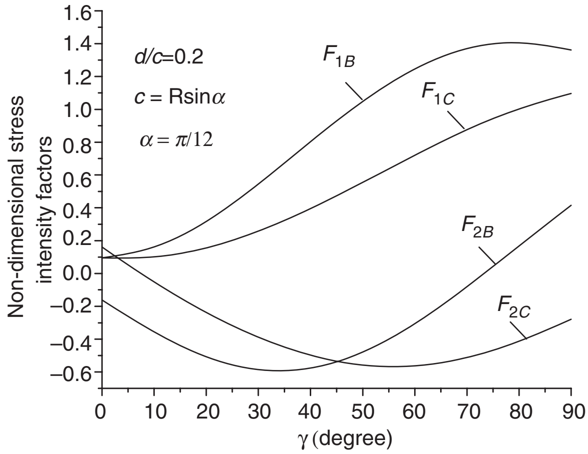

Figure 6. Non-dimensional SIFs $F_{1B}(\gamma)$, $F_{2B}(\gamma)$, $F_{1C}(\gamma)$ and $F_{2C}(\gamma)$ for an arc crack with rotation angle γ in an elastic half-plane (see Figure 3c and Equation (32)).

of the half-plane is traction free (Figure 3c). $M = 155$ is used in the quadrature rule shown in the appendix.

In the condition of $\gamma = 0, 10, 20, \ldots, 90^\circ$, the calculated results for the SIFs at the crack-tip "B" and "C" are expressed as

$$K_{1B} = F_{1B}(\gamma)\, p\sqrt{\pi c}, \quad K_{2B} = F_{2B}(\gamma)\, p\sqrt{\pi c},$$
$$K_{1C} = F_{1C}(\gamma)\, p\sqrt{\pi c}, \quad K_{2C} = F_{2C}(\gamma)\, p\sqrt{\pi c} \text{ (with } c = R\sin\alpha), \tag{32}$$

and they are plotted in Figure 6.

From the plotted results, it can be seen that in the studied range of γ, the influence of the inclined angle γ is significant. For example, in the case of $\gamma = 0$, we have $F_{1B}(\gamma)_{\gamma=0} = 0.0957$, $F_{2B}(\gamma)_{\gamma=0} = -0.1612$. However, in the case of $\gamma = \pi/2$, we have $F_{1B}(\gamma)_{\gamma=\pi/2} = 1.3609$, $F_{2B}(\gamma)_{\gamma=\pi/2} = 0.4154$.

3.4. *Example 4*

The fourth example is devoted to evaluate the interaction of two curved cracks in half-plane. The arc crack with radius R and the spanning angle 2α is placed in the upper half-plane, and $2\alpha = \pi/6$ is taken. The distance between the two cracks is denoted by h. The crack-tip "B" has a distance d from the boundary, and $d/(R\sin\alpha) = 0.2$ is assumed. The remote tension is $\sigma_x^\infty = p$ and the boundary of the half-plane is of traction free (Figure 3d). $M = 75$ is used in the quadrature rule shown in the appendix.

In the condition of $h/R = 0.1, 0.2, \ldots, 1.0$, the calculated results for the SIFs at the crack-tips "B" and "C" are expressed as

$$K_{1B} = F_{1B}(h/R)\, p\sqrt{\pi c}, \quad K_{2B} = F_{2B}(h/R)\, p\sqrt{\pi c},$$
$$K_{1C} = F_{1C}(h/R)\, p\sqrt{\pi c}, \quad K_{2C} = F_{2C}(h/R)\, p\sqrt{\pi c} \text{ (with } c = R\sin\alpha), \tag{33}$$

and they are plotted in Figure 7.

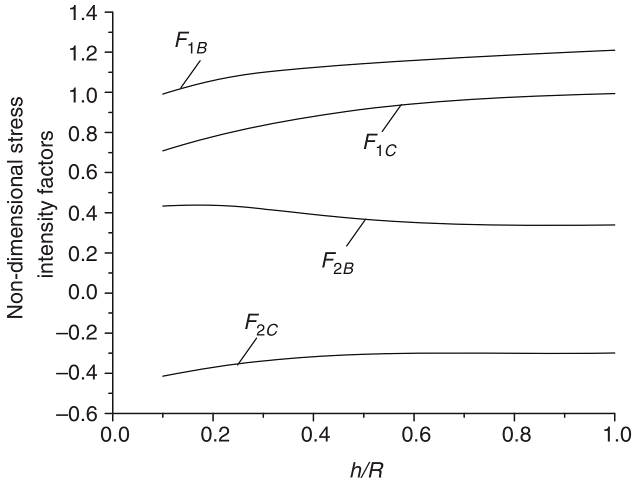

Figure 7. Non-dimensional SIFs $F_{1B}(h/c)$, $F_{2B}(h/c)$, $F_{1C}(h/c)$ and $F_{2C}(h/c)$ ($c = R\sin\alpha$) for two arc cracks in an elastic half-plane (see Figure 3d and Equation (33)).

From the plotted results, it can be seen that in the studied range of h/R, the stacking effect is significant. For example, if $h/R = 1.0$, we have $F_{1B} = 1.2100$ and $F_{1C} = 0.9942$. However, if $h/R = 0.1$, we have $F_{1B} = 0.9919$ and $F_{1C} = 0.7085$. Therefore, if two cracks are in a close position, they are in a safer situation.

4. Conclusions

Particular advantages of the suggested method can be found from the present analysis. Previously, the complex potentials for curved crack were formulated on the distribution of dislocation density along the curved crack [1]. This formulation is invariable to derive a singular integral equation. However, one needs an additional equation to ensure the condition of single-valuedness of displacements.

Alternatively, the curved crack can be modeled by a distribution of the dislocation doublet density, and the relevant complex potentials are obtainable [7,20]. In this case, the condition of single-valuedness of displacements is satisfied automatically. In this case, it is natural to derive a hypersingular integral equation. In the equation, the unknown function is COD and the right hand term is the traction applied on the crack face.

Usually, the image method for 2D Laplace equation is used for the circular region. In the method, the Green's function is composed of two terms. The first term is a singular function and second term is a regular function. The second term can compensate the boundary value from the first term. In this case, the value of the studied function in the domain point can be obtained by an integration of the assumed boundary value to the obtained Green's function. This concept can be used to solve the boundary value problem for an annular region.

The Green's function for the annular Laplace problem is first derived by using the image method [23].

Essentially, the concept of the modified complex potential is similar to the Green's function method for the solution of the Laplace equation. However, when the concept is used for the crack problem, the derivation is lengthier.

Usage of the modified complex potentials is also an important step in the derivation, since the modified complex potentials satisfy the traction free condition along the boundary of half-plane automatically. This can considerably reduce difficulties encountered in derivation and computation. In fact, the complex potential pair ($\phi_p(z)$, $\psi_p(z)$) corresponds to the singular term in the Green's function of 2D Laplace equation, and the complex potential pair ($\phi_c(z)$, $\psi_c(z)$) corresponds to the regular term in the Green's function.

In addition, it is proved that the curve length method is very effective for the problem of curved cracks. In this case, one does not need to construct a boundary element and simply use the available quadrature rule.

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Appendix

It is found that the integration rule suggested previously is very effective to solve the present problem [15]. The integration rule for hypersingular integral with the integrand $\sqrt{a^2 - s^2}\,G(s)$ is as follows:

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2 - s^2}G(s)\mathrm{d}s}{(s - s_o)^2} = \sum_{j=1}^{M+1} W_j(s_o)G(s_j)\ (|\ s_o\ | < a), \tag{A1}$$

where $G(s)$ is a given regular function, M is an assumed integer and

$$s_j = a\cos\left(\frac{j\pi}{M+2}\right), (j = 1, 2, \ldots, M+1), \tag{A2}$$

$$W_j(s_o) = -\frac{2}{M+2}\sum_{n=0}^{M}(n+1)\sin\left(\frac{j\pi}{M+2}\right)\sin\left(\frac{(n+1)j\pi}{M+2}\right)U_n\left(\frac{s_o}{a}\right)(j = 1, 2, \ldots, M+1), \tag{A3}$$

$$U_n(t) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad \text{where } \theta = \arccos(t)\ (|t| \le 1). \tag{A4}$$

In Equation (A4), $U_n(t)$ denotes the Chebyshev polynomials of the second kind.

For the integrand with regular function, the following integration rule is suggested:

$$\frac{1}{\pi}\int_{-a}^{a}\sqrt{a^2 - s^2}G(s)\mathrm{d}s = \frac{1}{M+2}\sum_{j=1}^{M+1}\left(a^2 - s_j^2\right)G(s_j). \tag{A5}$$

One more important equation is introduced below. Suppose $G(s_j)\ [s_j = a\cos(m\pi/(M+2))]$ $(j = 1, 2, \ldots, M+1)$ are known beforehand, the function $G(s)$ will be determined by

$$G(s) = \sum_{n=0}^{M} c_n U_n\left(\frac{s}{a}\right), \quad |\ s\ | \le a, \tag{A6}$$

where

$$c_n = \frac{2}{M+2}\sum_{j=1}^{M+1}\sin\left(\frac{j\pi}{M+2}\right)\sin\left(\frac{(n+1)j\pi}{M+2}\right)G(s_j). \tag{A7}$$