ELASTIC WEDGE SUBJECTED TO ANTIPLANE SHEAR TRACTIONS—A PARADOX EXPLAINED

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SUMMARY

The elementary solution for the stress distribution near the apex of an elastic wedge subjected to a uniform antiplane shear traction on one side of the wedge becomes infinite for every point (r, θ) in the wedge when the wedge angle 2α approaches π or 2π. The paradox can be resolved by adding to the elementary solution a homogeneous solution which satisfies the traction-free boundary condition at the sides of the wedge. By letting the coefficient of the homogeneous solution depend on α and approach infinity with opposite sign as 2α approaches π or 2π, one obtains a bounded solution. Similar procedures were used in resolving the paradoxes in other related problems. However, the addition of a homogeneous solution with its coefficient tending to infinity, though mathematically permissible, appears to have an arbitrariness which defies explanation. For the present problem, we are able to obtain an explicit solution for the wedge which is fixed at $r = r_0$ in a form of infinite series. As 2α approaches π or 2π, the first two terms of the solution have opposite signs and tend to infinity. Hence, the homogeneous solution with its coefficient tending to infinity as 2α tends to π or 2π exists. We also show that the infinite series can be reduced to integrals. For 2α equal to π or 2π the integrals yield a closed-form solution. Finally, for the wedge angle 2π subjected to non-symmetric tractions, the stresses have $O(\ln r)$ as well as $O(r^{-\frac{1}{2}})$ singularities.

1. Introduction

IN THE cylindrical coordinate system (r, θ, z), let an elastic wedge of wedge angle 2α occupy the region

$$0 \leqslant r \leqslant r_0, \qquad -\alpha \leqslant \theta \leqslant \alpha, \qquad -\infty < z < \infty, \tag{1}$$

where r_0 is a constant. Let $w(r, \theta)$ be the only non-zero antiplane displacement component in the z-direction (**1**). The non-zero antiplane shear stresses τ_{rz} and $\tau_{\theta z}$ are given by

$$\tau_{rz} = \mu \frac{\partial w}{\partial r}, \qquad \tau_{\theta z} = \mu \frac{\partial w}{r\,\partial \theta}, \tag{2}$$

where μ is the shear modulus. The equation of equilibrium reduces to (**2**)

$$\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0. \tag{3}$$

The fundamental solutions to (2) and (3) can be written as

$$\left.\begin{aligned} \mu w &= r^{1-\lambda}\{A\sin(1-\lambda)\theta + B\cos(1-\lambda)\theta\}/(1-\lambda),\\ \tau_{rz} &= r^{-\lambda}\{A\sin(1-\lambda)\theta + B\cos(1-\lambda)\theta\},\\ \tau_{\theta z} &= r^{-\lambda}\{A\cos(1-\lambda)\theta - B\sin(1-\lambda)\theta\},\end{aligned}\right\} \tag{4}$$

where λ, A and B are arbitrary constants. For the strain energy to be integrable in the region near $r=0$, we must have $\lambda<1$.

Let τ^+ and τ^- be the uniform shear tractions applied at $\theta=\alpha$ and $\theta=-\alpha$ respectively:

$$\tau_{\theta z}(r,\alpha)=\tau^+, \qquad \tau_{\theta z}(r,-\alpha)=-\tau^-. \tag{5}$$

Using $(4)_3$ to determine A and B with $\lambda=0$, we obtain the elementary solution

$$\left.\begin{aligned} \mu w &= r\left\{\tau_a\frac{\sin\theta}{\cos\alpha}-\tau_s\frac{\cos\theta}{\sin\alpha}\right\},\\ \tau_{rz} &= \tau_a\frac{\sin\theta}{\cos\alpha}-\tau_s\frac{\cos\theta}{\sin\alpha},\\ \tau_{\theta z} &= \tau_a\frac{\cos\theta}{\cos\alpha}+\tau_s\frac{\sin\theta}{\sin\alpha},\end{aligned}\right\} \tag{6}$$

where

$$\tau_s=\frac{\tau^+ + \tau^-}{2}, \qquad \tau_a=\frac{\tau^+ - \tau^-}{2}. \tag{7}$$

Thus τ_s and τ_a are, respectively, the symmetric and antisymmetric loadings.

We see that the solution given by (6) becomes infinite when 2α is equal to π or 2π. Similar paradoxes can be found in wedges subjected to in-plane tractions (**3**), wedges subjected to a concentrated couple (**4, 5**), and flow injected into a wedge region (**6**). These paradoxes were investigated in (**6** to **8**) and resolved in (**9** to **11**). We shall resolve the present paradox by a procedure similar to the one employed in (**10, 11**). We shall add to the solution obtained in (6) a homogeneous solution which satisfies zero traction conditions on $\theta=\pm\alpha$. By letting the coefficient of this homogeneous solution depend on α and tend to infinity as 2α approaches π or 2π, we obtain a bounded solution. This is presented in sections 2 and 3. For the problems studied in (**3** to **11**), a solution which also satisfies a specified boundary condition at $r=r_0$ where r_0 is finite was not available. In section 4 we obtain for the present problem the solution which satisfies a zero displacement condition at $r=r_0$. We show that indeed there is a term which represents a homogeneous solution and which tends to infinity as 2α approaches π or 2π. For 2α equal to π or 2π, we obtain a closed-form solution for the wedge which is fixed at $r=r_0$. In section 5, we present the solution for the wedge angle $2\alpha=2\pi$ which is subjected to non-symmetric tractions at the sides of the wedge. We show that the stresses have $O(\ln r)$ as well as $O(r^{-\frac{1}{2}})$ singularities.

2. Antisymmetric deformations

In this section we consider the cases in which the tractions at $\theta=\alpha$ and $\theta=-\alpha$ are antisymmetric, that is,

$$\tau_{\theta z}(r,\alpha)=\tau_{\theta z}(r,-\alpha)=\tau_a. \tag{8}$$

We therefore use the solution associated with τ_a in (6). Before we present a bounded solution for 2α near π, we consider homogeneous solutions which satisfy zero surface tractions at $\theta=\pm\alpha$. We use the terms associated with the constant A in (4) since these terms represent antisymmetric deformations. The conditions that $\tau_{\theta z}=0$ at $\theta=\pm\alpha$ yield

$$(1-\lambda_n)\alpha=(n+\tfrac{1}{2})\pi, \qquad n=0,1,2,\dots. \tag{9}$$

This is shown in Fig. 1. Hence the general solution for antisymmetric deformations satisfying the boundary conditions (8) is

$$\left.\begin{aligned}\mu w&=\tau_a r\frac{\sin\theta}{\cos\alpha}+\sum_{n=0}^{\infty}A_n r^{1-\lambda_n}\frac{\sin(1-\lambda_n)\theta}{1-\lambda_n},\\ \tau_{rz}&=\tau_a\frac{\sin\theta}{\cos\alpha}+\sum_{n=0}^{\infty}A_n r^{-\lambda_n}\sin(1-\lambda_n)\theta,\\ \tau_{\theta z}&=\tau_a\frac{\cos\theta}{\cos\alpha}+\sum_{n=0}^{\infty}A_n r^{-\lambda_n}\cos(1-\lambda_n)\theta,\end{aligned}\right\} \tag{10}$$

where A_n, $n=0,1,2,\dots$, are arbitrary constants.

Notice that $\lambda_0=0$ when $2\alpha=\pi$ and the term associated with A_0 in each equation in (10) is identical to the term associated with τ_a except for the factor $(\cos\alpha)^{-1}$. For the solution near $r=0$, we may ignore the terms associated with A_n for $n>0$. To obtain a bounded solution for 2α near π, we let $A_0=-f(\lambda_0)\tau_a/\cos\alpha$ where f is a continuous and continuously differentiable function of λ_0 with the property that $f(0)=1$. We have

$$\left.\begin{aligned}\mu w&=\tau_a r\{\sin\theta-f(\lambda_0)r^{-\lambda_0}[\sin(1-\lambda_0)\theta]/(1-\lambda_0)\}/\cos\alpha,\\ \tau_{rz}&=\tau_a\{\sin\theta-f(\lambda_0)r^{-\lambda_0}\sin(1-\lambda_0)\theta\}/\cos\alpha,\\ \tau_{\theta z}&=\tau_a\{\cos\theta-f(\lambda_0)r^{-\lambda_0}\cos(1-\lambda_0)\theta\}/\cos\alpha.\end{aligned}\right\} \tag{11}$$

It can be shown that the solution given by (11) remains bounded as 2α approaches π. In the limit as $2\alpha\to\pi$, $\lambda_0\to 0$ by (9), and it is readily shown that, at $2\alpha=\pi$,

$$\left.\begin{aligned}\mu w&=2\tau_a r\{(1-\ln r)\sin\theta-\theta\cos\theta+f'(0)\sin\theta\}/\pi,\\ \tau_{rz}&=2\tau_a\{-(\ln r)\sin\theta-\theta\cos\theta+f'(0)\sin\theta\}/\pi,\\ \tau_{\theta z}&=2\tau_a\{-(\ln r)\cos\theta+\theta\sin\theta+f'(0)\cos\theta\}/\pi,\end{aligned}\right\} \tag{12}$$

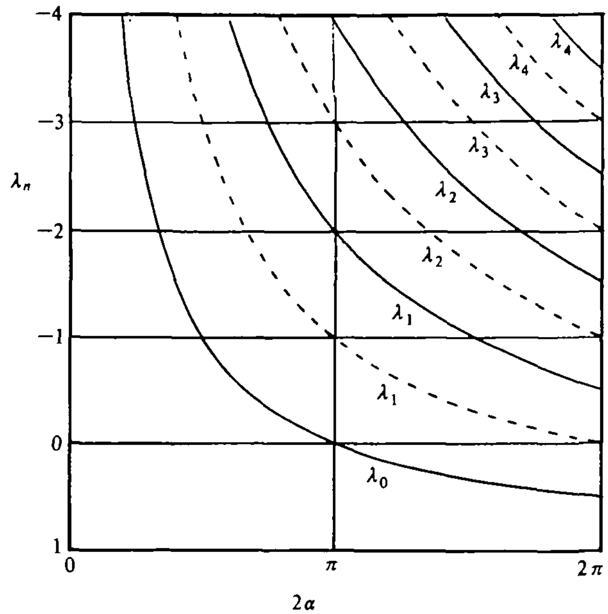

FIG. 1. $(1-\lambda_n)\alpha=(n+\frac{1}{2})\pi$ for antisymmetric deformations (solid lines); $(1-\lambda_n)\alpha=n\pi$ for symmetric deformations (dotted lines)

where $f'=df/d\lambda_0$. The solution given by (12) contains the arbitrary constant $f'(0)$. The non-uniqueness is expected since the boundary condition at $r=r_0$ has yet to be specified. In section 4 we shall present the solution which satisfies the zero-displacement condition at $r=r_0$.

3. Symmetric deformations

For a symmetric deformation, we have

$$\tau_{\theta z}(r,\alpha)=-\tau_{\theta z}(r,-\alpha)=\tau_s. \tag{13}$$

We therefore use the solution associated with τ_s in (6). The homogeneous solutions which represent symmetric deformations satisfying the zero tractions at $\theta=\alpha$ and $\theta=-\alpha$ can be obtained from (4) with $A=0$. The conditions that $\tau_{\theta z}=0$ at $\theta=\pm\alpha$ yield

$$(1-\lambda_n)\alpha=n\pi, \qquad n=1,2,3,\ldots, \tag{14}$$

and the general solution for a symmetric deformation which satisfies (13) can be written as

$$\left.\begin{aligned} \mu(w-w_0) &= -\tau_s r\frac{\cos\theta}{\sin\alpha} + \sum_{n=1}^{\infty} B_n r^{1-\lambda_n}\frac{\cos(1-\lambda_n)\theta}{1-\lambda_n}, \\ \tau_{rz} &= -\tau_s\frac{\cos\theta}{\sin\alpha} + \sum_{n=1}^{\infty} B_n r^{-\lambda_n}\cos(1-\lambda_n)\theta, \\ \tau_{\theta z} &= \tau_s\frac{\sin\theta}{\sin\alpha} - \sum_{n=1}^{\infty} B_n r^{-\lambda_n}\sin(1-\lambda_n)\theta, \end{aligned}\right\} \quad (15)$$

where w_0 is the displacement at $r=0$. We see from Fig. 1 that $\lambda_1=0$ when $2\alpha=2\pi$. The term associated with B_1 in each equation is identical to the term associated with τ_s except for the factor $(\sin\alpha)^{-1}$. For the solution near $r=0$, we may ignore the terms associated with B_n for $n>1$. We let $B_1=g(\lambda_1)\tau_s/\sin\alpha$ where g is a continuous and continuously differentiable function of λ_1 with the property that $g(0)=1$. We have

$$\left.\begin{aligned} \mu(w-w_0) &= -\tau_s r\{\cos\theta - g(\lambda_1)r^{-\lambda_1}[\cos(1-\lambda_1)\theta]/(1-\lambda_1)\}/\sin\alpha, \\ \tau_{rz} &= -\tau_s\{\cos\theta - g(\lambda_1)r^{-\lambda_1}\cos(1-\lambda_1)\theta\}/\sin\alpha, \\ \tau_{\theta z} &= \tau_s\{\sin\theta - g(\lambda_1)r^{-\lambda_1}\sin(1-\lambda_1)\theta\}/\sin\alpha. \end{aligned}\right\} \quad (16)$$

It can be shown that the solution given by (16) is bounded for 2α near 2π. In the limit as $2\alpha\to2\pi$, $\lambda_1\to0$ by (14). It is readily shown that when $2\alpha=2\pi$, we have

$$\left.\begin{aligned} \mu(w-w_0) &= \tau_s r\{(\ln r-1)\cos\theta - \theta\sin\theta - g'(0)\cos\theta\}/\pi, \\ \tau_{rz} &= \tau_s\{(\ln r)\cos\theta - \theta\sin\theta - g'(0)\cos\theta\}/\pi, \\ \tau_{\theta z} &= -\tau_s\{(\ln r)\sin\theta + \theta\cos\theta - g'(0)\sin\theta\}/\pi. \end{aligned}\right\} \quad (17)$$

As before, (17) contains the arbitrary constant $g'(0)$. The non-uniqueness is due to the fact that the boundary condition at $r=r_0$ has yet to be specified.

4. The wedge fixed at $r=r_0$

In this section we let

$$w(r_0,\theta)=0, \qquad -\alpha\leqslant\theta\leqslant\alpha, \quad (18)$$

where r_0 is a constant. As before, we shall consider the symmetric and antisymmetric deformations separately.

(a) *Antisymmetric deformation*

Application of the boundary condition (18) to $(10)_1$ yields

$$0=\tau_a r_0\frac{\sin\theta}{\cos\alpha} + \sum_{n=0}^{\infty} A_n r_0^{1-\lambda_n}\frac{\sin(1-\lambda_n)\theta}{1-\lambda_n}. \quad (19)$$

With λ_n defined in (9), equation (19) is in a form of a Fourier sine series in which the terms $\sin(1-\lambda_n)\theta$ are orthogonal to each other for different n. Hence, multiplying (19) by $\sin(1-\lambda_m)\theta$ and integrating with respect to θ from $-\alpha$ to α, we obtain

$$A_n = \tau_a \frac{2}{\alpha} \frac{(-1)^n(1-\lambda_n)}{\lambda_n(2-\lambda_n)} r_0^{\lambda_n}. \tag{20}$$

Equations (10) can now be written as

$$\left.\begin{aligned}
\mu w &= \tau_a r\left\{\frac{\sin\theta}{\cos\alpha} + \frac{2}{\alpha} R^{-\lambda_0} \frac{\sin(1-\lambda_0)\theta}{\lambda_0(2-\lambda_0)} + \frac{2}{\alpha} I_1(R,\theta)\right\},\\
\tau_{rz} &= \tau_a\left\{\frac{\sin\theta}{\cos\alpha} + \frac{2}{\alpha} R^{-\lambda_0} \frac{(1-\lambda_0)\sin(1-\lambda_0)\theta}{\lambda_0(2-\lambda_0)} + \frac{2}{\alpha} I_2(R,\theta)\right\},\\
\tau_{\theta z} &= \tau_a\left\{\frac{\cos\theta}{\cos\alpha} + \frac{2}{\alpha} R^{-\lambda_0} \frac{(1-\lambda_0)\cos(1-\lambda_0)\theta}{\lambda_0(2-\lambda_0)} + \frac{2}{\alpha} I_3(R,\theta)\right\},
\end{aligned}\right\} \tag{21}$$

where

$$\left.\begin{aligned}
R &= r/r_0,\\
I_1 &= \sum_{n=1}^{\infty} (-1)^n R^{-\lambda_n} \frac{\sin(1-\lambda_n)\theta}{\lambda_n(2-\lambda_n)},\\
I_2 &= \sum_{n=1}^{\infty} (-1)^n R^{-\lambda_n} \frac{(1-\lambda_n)\sin(1-\lambda_n)\theta}{\lambda_n(2-\lambda_n)},\\
I_3 &= \sum_{n=1}^{\infty} (-1)^n R^{-\lambda_n} \frac{(1-\lambda_n)\cos(1-\lambda_n)\theta}{\lambda_n(2-\lambda_n)}.
\end{aligned}\right\} \tag{22}$$

Thus R, for $0 \leq R \leq 1$, is the dimensionless radial distance. Notice that I_1, I_2 and I_3 vanish when $R=0$. Hence I_1, I_2 and I_3 can be ignored for small R and equations (21) are identical to (11) if we let

$$f(\lambda_0) = -\frac{2}{\alpha} r_0^{\lambda_0} \frac{(1-\lambda_0)\cos\alpha}{\lambda_0(2-\lambda_0)}. \tag{23}$$

The second term in each part of (21) represents the homogeneous solution which tends to infinity as $2\alpha \to \pi$, i.e., $\lambda_0 \to 0$. It can be shown that $f(0)=1$ and

$$f'(0) = (\ln r_0) - \tfrac{1}{2}. \tag{24}$$

Introducing the complex variable ζ and the function $U(\zeta)$ by

$$\zeta = R(\cos\theta + i\sin\theta), \tag{25}$$

$$U = \sum_{n=1}^{\infty} (-1)^n \frac{\zeta^{1-\lambda_n}}{\lambda_n(2-\lambda_n)}, \tag{26}$$

(22) can be partly rewritten as

$$\left.\begin{aligned} I_1 &= \frac{1}{R}\,\mathrm{Im}\,(U), \\ I_2 &= \frac{1}{R}\,\mathrm{Im}\left(\zeta\frac{dU}{d\zeta}\right), \\ I_3 &= \frac{1}{R}\,\mathrm{Re}\left(\zeta\frac{dU}{d\zeta}\right). \end{aligned}\right\} \tag{27}$$

We write U as

$$U = -\frac{\zeta}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\zeta^{-\lambda_n}}{-\lambda_n} + \frac{\zeta^{-1}}{2}\sum_{n=1}^{\infty}(-1)^n\frac{\zeta^{2-\lambda_n}}{2-\lambda_n}. \tag{28}$$

Using the definition of λ_n in (9), we have

$$\left.\begin{aligned} \frac{d}{d\zeta}\sum_{n=1}^{\infty}(-1)^n\frac{\zeta^{-\lambda_n}}{-\lambda_n} &= \sum_{n=1}^{\infty}(-1)^n\zeta^{-\lambda_n-1} = \frac{-\zeta^{(3\pi/2\alpha)-2}}{1+\zeta^{\pi/\alpha}}, \\ \frac{d}{d\zeta}\sum_{n=1}^{\infty}(-1)^n\frac{\zeta^{2-\lambda_n}}{2-\lambda_n} &= \sum_{n=1}^{\infty}(-1)^n\zeta^{1-\lambda_n} = \frac{-\zeta^{3\pi/2\alpha}}{1+\zeta^{\pi/\alpha}}. \end{aligned}\right\} \tag{29}$$

Hence,

$$U = \frac{\zeta}{2}\int_0^{\zeta}\frac{x^{(3\pi/2\alpha)-2}}{1+x^{\pi/\alpha}}\,dx - \frac{\zeta^{-1}}{2}\int_0^{\zeta}\frac{x^{3\pi/2\alpha}}{1+x^{\pi/\alpha}}\,dx. \tag{30}$$

The integrals in (30) can be integrated in closed forms for some integer values of π/α. In particular, when $\pi/\alpha = 2$, we have

$$\left.\begin{aligned} U &= \tfrac{1}{4}[(\zeta+\zeta^{-1})\ln(1+\zeta^2)-\zeta], \\ \zeta\frac{dU}{d\zeta} &= \tfrac{1}{4}[(\zeta-\zeta^{-1})\ln(1+\zeta^2)+\zeta], \end{aligned}\right\} \tag{31}$$

and the solution for the wedge of wedge angle $2\alpha = \pi$ which is fixed at $r = r_0$ is

$$\left.\begin{aligned} \mu w &= \tau_a r\{-2(\ln R)\sin\theta - 2\theta\cos\theta + (R^{-2}+1)\psi\cos\theta \\ &\qquad\qquad - (R^{-2}-1)(\ln\rho)\sin\theta\}/\pi, \\ \tau_{rz} &= \tau_a\{-2(\ln R)\sin\theta - 2\theta\cos\theta - (R^{-2}-1)\psi\cos\theta + \\ &\qquad\qquad + (R^{-2}+1)(\ln\rho)\sin\theta\}/\pi, \\ \tau_{\theta z} &= \tau_a\{-2(\ln R)\cos\theta + 2\theta\sin\theta - (R^{-2}+1)\psi\sin\theta \\ &\qquad\qquad - (R^{-2}-1)(\ln\rho)\cos\theta\}/\pi, \end{aligned}\right\} \tag{32$_1$}$$

where

$$\left.\begin{aligned} \rho^2 &= (1-R^2)^2+(2R\cos\theta)^2, \\ \tan\psi &= \frac{R^2\sin 2\theta}{1+R^2\cos 2\theta}. \end{aligned}\right\} \qquad (32)_2$$

Notice that

$$\left.\begin{aligned} &\psi = 0, \qquad \text{at } \theta = \pm\tfrac{1}{2}\pi, \\ &\rho = 2\cos\theta, \quad \psi = \theta, \qquad \text{at } R=1, \\ &R^{-2}\ln\rho = \cos 2\theta, \quad R^{-2}\psi = \sin 2\theta, \qquad \text{at } R=0. \end{aligned}\right\} \qquad (32)_3$$

It can be verified easily that equations $(32)_1$ satisfy the boundary conditions (8) and (18), and that the only singularity for the stresses at $r=0$ is the logarithmic singularity. It can also be verified that the first few terms of the asymptotic expansion for $r=0$ are identical to (12), in which $f'(0)$ is given by (24).

If one uses the stress components τ_{xz} and τ_{yz} instead of τ_{rz} and $\tau_{\theta z}$, one can deduce from $(32)_1$ that the logarithmic singularity appears only for the τ_{yz}-component. The τ_{xz}-component is bounded and assumes the value $-2\tau_a\theta/\pi$ at $R=0$.

(b) *Symmetric deformation*

From $(15)_1$, the zero displacement at $r=r_0$ implies that

$$-\mu w_0 = -\tau_s r_0\frac{\cos\theta}{\sin\alpha}+\sum_{n=1}^{\infty} B_n r_0^{1-\lambda_n}\frac{\cos(1-\lambda_n)\theta}{1-\lambda_n}, \qquad (33)$$

where λ_n is defined by (14). Integrating the equation from $\theta=-\alpha$ to $\theta=\alpha$, we obtain

$$\mu w_0 = \tau_s r_0/\alpha. \qquad (34)$$

If we multiply each term by $\cos(1-\lambda_m)\theta$ and integrate from $\theta=-\alpha$ to $\theta=\alpha$, we have

$$B_n = \tau_s\frac{2}{\alpha}\frac{(-1)^n(1-\lambda_n)}{\lambda_n(2-\lambda_n)}r_0^{\lambda_n}. \qquad (35)$$

Equations (15) can now be written as

$$\left.\begin{aligned} \mu w-\frac{\tau_s r_0}{\alpha} &= -\tau_s r\left\{\frac{\cos\theta}{\sin\alpha}+\frac{2}{\alpha}R^{-\lambda_1}\frac{\cos(1-\lambda_1)\theta}{\lambda_1(2-\lambda_1)}-\frac{2}{\alpha}J_1(R,\theta)\right\}, \\ \tau_{rz} &= -\tau_s\left\{\frac{\cos\theta}{\sin\alpha}+\frac{2}{\alpha}R^{-\lambda_1}\frac{(1-\lambda_1)\cos(1-\lambda_1)\theta}{\lambda_1(2-\lambda_1)}-\frac{2}{\alpha}J_2(R,\theta)\right\}, \\ \tau_{\theta z} &= \tau_s\left\{\frac{\sin\theta}{\sin\alpha}+\frac{2}{\alpha}R^{-\lambda_1}\frac{(1-\lambda_1)\sin(1-\lambda_1)\theta}{\lambda_1(2-\lambda_1)}-\frac{2}{\alpha}J_3(R,\theta)\right\}, \end{aligned}\right\} \qquad (36)$$

where R is defined in (22),

$$\left.\begin{aligned} J_1 &= \frac{1}{R}\,\mathrm{Re}\,(V), \\ J_2 &= \frac{1}{R}\,\mathrm{Re}\left(\zeta\frac{dV}{d\zeta}\right), \\ J_3 &= \frac{1}{R}\,\mathrm{Im}\left(\zeta\frac{dV}{d\zeta}\right), \end{aligned}\right\} \tag{37}$$

$$V = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta^{1-\lambda_n}}{\lambda_n(2-\lambda_n)}, \tag{38}$$

and ζ is defined in (25). We see that for small R, J_1, J_2 and J_3 can be ignored and (36) and (16) are identical if we let

$$g(\lambda_1) = -\frac{2}{\alpha}\, r_0^{\lambda_1} \frac{(1-\lambda_1)\sin\alpha}{\lambda_1(2-\lambda_1)}. \tag{39}$$

The second term in each part of (36) represents the homogeneous solution which tends to infinity as $2\alpha \to 2\pi$, i.e., $\lambda_1 \to 0$. It can be shown that $g(0) = 1$ and that

$$g'(0) = (\ln r_0) - \tfrac{1}{2}, \tag{40}$$

which is identical to $f'(0)$ in (24).

As before, V can be expressed in terms of integrals. Omitting the detailed derivations, we have

$$V = -\frac{\zeta}{2}\int_0^{\zeta} \frac{x^{(2\pi/\alpha)-2}}{1+x^{\pi/\alpha}}\,dx + \frac{\zeta^{-1}}{2}\int_0^{\zeta} \frac{x^{2\pi/\alpha}}{1+x^{\pi/\alpha}}\,dx. \tag{41}$$

In particular, for $\pi/\alpha = 1$, we have

$$\left.\begin{aligned} V &= -\tfrac{1}{2}[(\zeta-\zeta^{-1})\ln(1+\zeta) + 1 - \tfrac{1}{2}\zeta], \\ \zeta\frac{dV}{d\zeta} &= -\tfrac{1}{2}[(\zeta+\zeta^{-1})\ln(1+\zeta) - 1 + \tfrac{1}{2}\zeta], \end{aligned}\right\} \tag{42}$$

and the solution for the wedge with wedge angle $2\alpha = 2\pi$ which is fixed at $r = r_0$ is

$$\left.\begin{aligned} \mu w - (\tau_s r_0/\pi) &= \tau_s r\{(\ln R)\cos\theta - \theta\sin\theta + (R^{-2}+1)\psi\sin\theta + \\ &\qquad + (R^{-2}-1)(\ln\rho)\cos\theta - R^{-1}\}/\pi, \\ \tau_{rz} &= \tau_s\{(\ln R)\cos\theta - \theta\sin\theta - (R^{-2}-1)\psi\sin\theta \\ &\qquad - (R^{-2}+1)(\ln\rho)\cos\theta + R^{-1}\}/\pi, \\ \tau_{\theta z} &= -\tau_s\{(\ln R)\sin\theta + \theta\cos\theta - (R^{-2}+1)\psi\cos\theta + \\ &\qquad + (R^{-2}-1)(\ln\rho)\sin\theta\}/\pi, \end{aligned}\right\} \tag{43$_1$}$$

where

$$\left.\begin{aligned} \rho^2 &= 1+2R\cos\theta+R^2, \\ \tan\psi &= \frac{R\sin\theta}{1+R\cos\theta}. \end{aligned}\right\} \qquad (43)_2$$

It can be shown that

$$\left.\begin{aligned} &\psi=0, \quad \text{at } \theta=\pm\pi, \\ &\rho=2\cos\tfrac{1}{2}\theta, \quad \psi=\tfrac{1}{2}\theta, \quad \text{at } R=1, \\ &R^{-1}\ln\rho=\cos\theta, \quad R^{-1}\psi=\sin\theta, \quad \text{at } R=0. \end{aligned}\right\} \qquad (43)_3$$

Therefore, equations $(43)_1$ satisfy the boundary conditions (13) and (18), and the only singularity at $r=0$ is the logarithmic singularity. Moreover, the first few terms of the asymptotic solution are identical to (17), in which $g'(0)$ is given by (40).

Although the logarithmic singularity appears in both stress components τ_{rz} and $\tau_{\theta z}$, it can be shown that in terms of the τ_{xz}- and τ_{yz}-components the logarithmic singularity appears only for the τ_{xz}-component. The τ_{yz}-component remains bounded and assumes the value $-\tau_s\theta/\pi$ at $R=0$.

5. A crack subjected to non-symmetric tractions

Consider the wedge of wedge angle 2π subjected to the boundary conditions (18) and (5). Using (7), the boundary conditions (5) can be divided into a symmetric loading τ_s and an antisymmetric loading τ_a. The solution for the symmetric loading τ_s is presented in $(43)_1$ where the stresses have the logarithmic singularity. We now present the solution for the antisymmetric loading τ_a for $\alpha=\pi$.

By letting $\alpha=\pi$ in (30), we have

$$U=(\zeta-\zeta^{-1})\tan^{-1}(\sqrt{\zeta})+\zeta^{-\frac{1}{2}}-\tfrac{1}{3}\zeta^{\frac{1}{2}}. \qquad (44)$$

Noticing that (12)

$$\tan^{-1}(\sqrt{\zeta})=\tfrac{1}{2}i\ln\frac{1-i\sqrt{\zeta}}{1+i\sqrt{\zeta}}, \qquad (45)$$

and making use of (9) and (27), equations (21) yield

$$\left.\begin{aligned} \mu w &= \tau_a r\{[(R^{-2}+1)\psi-\pi]\sin\theta-(R^{-2}-1)(\ln\rho)\cos\theta \\ &\qquad -2(R^{-\frac{3}{2}}-R^{-\frac{1}{2}})\sin\tfrac{1}{2}\theta\}/\pi, \\ \tau_{rz} &= \tau_a\{-[(R^{-2}-1)\psi+\pi]\sin\theta+(R^{-2}+1)(\ln\rho)\cos\theta+ \\ &\qquad +2(R^{-\frac{3}{2}}+R^{-\frac{1}{2}})\sin\tfrac{1}{2}\theta\}/\pi, \\ \tau_{\theta z} &= \tau_a\{[(R^{-2}+1)\psi-\pi]\cos\theta+(R^{-2}-1)(\ln\rho)\sin\theta \\ &\qquad -2(R^{-\frac{3}{2}}-R^{-\frac{1}{2}})\cos\tfrac{1}{2}\theta\}/\pi, \end{aligned}\right\} \qquad (46)_1$$

where

$$\left.\begin{aligned}\rho^2 &= (1+2\sqrt{R}\sin\tfrac{1}{2}\theta+R)/(1-2\sqrt{R}\sin\tfrac{1}{2}\theta+R),\\ \tan\psi &= 2\sqrt{R}(\cos\tfrac{1}{2}\theta)/(1-R).\end{aligned}\right\} \qquad (46)_2$$

It is readily shown that

$$\left.\begin{aligned}&\psi = 0, \quad \text{at } \theta = \pm\pi,\\ &\psi = \tfrac{1}{2}\pi, \quad \text{at } R = 1,\\ &R^{-\frac{1}{2}}\psi = 2\cos\tfrac{1}{2}\theta, \quad R^{-\frac{1}{2}}\ln\rho = 2\sin\tfrac{1}{2}\theta, \quad \text{at } R = 0.\end{aligned}\right\} \qquad (46)_3$$

With $(46)_3$, it is not difficult to see that equations $(46)_1$ satisfy the boundary conditions (8) and (18). Moreover, as $R \to 0$, the R^{-2} and $R^{-\frac{3}{2}}$ terms in $(46)_1$ disappear and the only stress singularities at $R = 0$ are the $R^{-\frac{1}{2}}$ terms.

For the general loading given by (5), the solution is the sum of equations $(43)_1$ and $(46)_1$. The stresses therefore have the $(\ln r)$ and $r^{-\frac{1}{2}}$ singularities at $r = 0$.

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