

Elastic waves in nonhomogeneous media under 2D conditions: I. Fundamental solutions

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Received 15 January 1998; received in revised form 15 July 1998; accepted 4 August 1998

Abstract

The purpose of this work is to present three methods of analysis for elastic waves propagating in two dimensional, elastic nonhomogeneous media. The first step, common to all methods, is a transformation of the governing equations of motion so that derivatives with respect to the material parameters no longer appear in the differential operator. This procedure, however, restricts analysis to a very specific class of nonhomogeneous media, namely those for which Poisson's ratio is equal to 0.25 and the elastic parameters are quadratic functions of position. Subsequently, fundamental solutions are evaluated by: (i) conformal mapping in conjunction with wave decomposition, which in principle allows for both vertical and lateral heterogeneities; (ii) wave decomposition into pseudo-dilatational and pseudo-rotational components, which results in an Euler-type equation for the transformed solution if medium heterogeneity is a function of one coordinate only; and (iii) Fourier transformation followed by a first order differential equation system solution, where the final step involving inverse transformation from the wavenumber domain is accomplished numerically. Finally, in the companion paper numerical examples serve to illustrate the above methodologies and to delineate their range of applicability. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: Fundamental solutions; Nonhomogeneous media; Seismic waves; Wave propagation

1. Introduction

The importance of finding fundamental solutions for wave motion problems in nonhomogeneous media has been stressed elsewhere [1,2]. This is especially true for seismic wave propagation in the earth's upper layers (near-field effects), given the high degree of complexity in the geological structure of the ground where heterogeneity is only one of many complications [3,4]. Of major importance in this context is the scalar wave equation with a depth-dependent wavenumber [5], because it corresponds to:

- (i) sound waves where the acoustic medium density variation over the wavelength is important,
- (ii) electromagnetic waves where the electric field is polarised, and
- (iii) horizontally (SH) and, under certain restrictions, vertically (SV) polarised elastic shear waves in a continuously nonhomogeneous medium.

Although many wave propagation phenomena can be explained through recourse to SH wave models, it is still necessary to study the vector wave equation and especially wave motions under two-dimensional conditions as a first step, given the difficulties associated with modeling fully three-dimensional configurations. This is especially true for media where dependence of their material parameters on position is arbitrary, because although the wave field can be represented in terms of a position-dependent amplitude and phase angle, it is not possible to uniquely divide it into the sum of incident plus reflected waves due to continuous scattering of the signal by the inhomogeneities. Thus, many of the solution techniques developed for homogeneous materials [6] (e.g., vector wave decomposition, use of potentials) are no longer valid. The relatively few methods applicable to nonhomogeneous media can be broadly classified as follows [4]: (i) asymptotic methods, which cover a wide range of inhomogeneity and anisotropy but are restricted to high frequencies and to isolated wavefronts, (ii) mode expansions, which are effective when the wavetrain is attributed to a small number of interfering modes, and (iii) generalized ray expansions (e.g., double

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transform methods, Haskell–Thomson matrix formalism, Cagniard–de Hoop inversion) which are effective for wave-trains described in terms of interference patterns generated by large numbers of rays.

The last two groups of methods are restricted to horizontal layering and heterogeneity in the vertical direction.

Of major interest to seismology and earthquake engineering is wave amplification in multi-layered geological media. Solutions for the one-dimensional representation of this problem were provided by Thomson [7] and Haskell [8,9] using transfer matrices which relate forces and displacements between upper and lower interfaces of an individual layer. The complete solution from bedrock to surface is therefore obtained in a step-by-step approach by considering all intermediate layers. In more detail, e.g. Biot [10], the dynamic equilibrium equations of an individual homogeneous layer are solved through decoupling of the wavefield into distortional and dilatational components, followed by an additional decomposition into symmetric and antisymmetric deformation patterns. The final solution is achieved in the form of 2×2 transfer matrices by establishing equilibrium of forces and compatibility of displacements across both upper and lower boundaries of a layer. Furthermore, viscoelastic material behavior can be captured by using modified, frequency dependent material parameters [11]. More recently, Kausel and Roesset [12] introduced a finite element type approach by synthesizing the individual transfer matrices so as to obtain a dynamic stiffness matrix representing the entire layered structure. The practical advantages of this route has provided an impetus for further extension to wave amplification in two-dimensional deposits [13]. Various aspects of wave motion through unidimensional layered media are being continuously explored, e.g., by Chen et al. [14] on regularization of the divergent series which occur in ground motion deconvolution analyses, by Bonnet and Heitz [15] on nonlinear seismic response of soft layers filled with nonlinear materials using the perturbation method, by Pires [16] on the nonlinear stress–strain behavior of layered soil deposits subjected to random seismic loads using equivalent linearization and by Safak [17] on discrete time analysis of seismic site amplification using upgoing and downgoing waves as auxiliary variables and representing each layer by three basic parameters. Finally, a comparison study on the various transfer matrix methods for wave amplification through elastic layers can be found in Urratia et al. [18].

Also of interest to seismology is the generation of synthetic signals in layered media due to various point sources. For instance, a computationally stable solution for determining surface displacements due to buried dislocation sources in a multi-layered elastic medium was developed by Wang and Herrmann [19] based on Haskell's [8,9] work, who employed the Fourier transformation and evaluated signal time histories for the elastic medium by performing contour integration of Bessel functions in the complex wavenumber plane. In general, solution procedures

for wave motions due to buried sources follow along two lines, namely Laplace transform (or Cagniard–de Hoop technique [20]) and the Fourier transform [21]. The former is also known as generalized ray method because the solution is constructed by tracking the individual seismic signal arrivals ray-by-ray from source to receiver. It is valid at high frequencies but not well suited for cases with many layers and large source to receiver distances. In the latter technique, the complete wave solution is expressed in terms of double integral transformations over wavenumber and frequency. The method can handle a large number of plane layers, but requires considerable computational effort at high frequencies. It is also possible to introduce numerical techniques for carrying out the contour integrations [22] or in evaluating the resulting analytical expressions [23].

With the exception of scalar waves [5], relatively little work is available for wave motions in continuously nonhomogeneous media. Of course, approximate solutions can always be generated, e.g., by decomposing the inhomogeneous medium into a stack of vertically varying layers and representing the solution within a layer as a sum of decoupled plane waves [24]. A very general numerical technique for investigating seismic wave propagation in anisotropic and nonhomogeneous materials is presented in Mikhailenko [25,26]. In particular, different families of algorithms are presented based on a combination of finite integral transforms with finite difference techniques for the computation of complete seismograms in complex, three-dimensional subsurface geometries. Of interest here are: (i) the inhomogeneous isotropic 3D medium in which the elastic parameters and the density are functions of depth, [25] and (ii) SH wave propagation in a heterogeneous medium where the wavespeed is a function of two spatial variables [26].

In the former case, the equations of motion are described in cylindrical coordinates and the methodology used is a double Hankel integral transformation with respect to the two spatial variables, followed by a finite difference solution and inverse transformations. The latter case combines a finite Fourier integral transformation in one spatial coordinate with a finite difference scheme in the other coordinate. All spatial derivatives encountered in the finite difference method are approximated by Fourier series [27] for better accuracy.

As far as analytical techniques are concerned, we mention Acharya's [28] method for determining the wavefield due to a point source in an inhomogeneous medium satisfying certain conditions (such as constant density or Poisson's ratio of 0.25 or a linear wavespeed gradient) which allow for independent P and S equations of motion. By assuming cylindrical symmetry and representing the pulse from the point source as a superposition of harmonic waves, the total field is obtained by summing over all plane waves with a given set of direction cosines and then integrating the sum over all values of the direction cosines. Thus, integral expressions are obtained for the compressional and shear

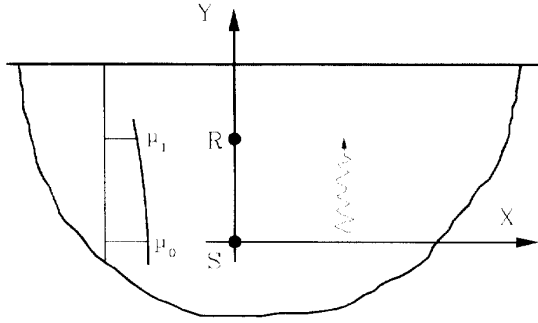


Fig. 1. Non-homogeneous material with depth-dependent material properties.

potentials that are convergent. Other analytical work addressing wave motions in nonhomogeneous media is by Hook [29] on the method of separation of variables in order to recast the vector wave equation with position-dependent material parameters into a system of three linearly independent solutions for the corresponding number of scalar potentials, which in turn satisfy second order wave equations. This can always be achieved for SH waves, while formulations for SV and P wave are possible only for certain functional forms of the material properties, such as power laws for the shear modulus and the density and a fixed value of Poisson's ratio. Furthermore, the wave equations for the latter two potentials are coupled, implying that P and SV waves are no longer purely dilatational and rotational. It therefore becomes necessary to impose further constraints on the material parameters in order to achieve two uncoupled P and SV wave equations. This approach was generalized in a later publication [30] through introduction of a linear transformation for the displacement vector in order to produce a diagonal system matrix for the vector wave equation which was reformulated using matrix notation. The method of separation of variables for nonhomogeneous media was then implemented for the axisymmetric case (with plane strain being a special case) and the resulting constraints which dictate mathematically acceptable material parameter variations with respect to a single spatial coordinate appear in the form of a nonlinear, ordinary differential equation. Other work along the lines of separation of the displacement vector in a nonhomogeneous medium into P and S wave potentials is by Gupta [31], who obtained reflection coefficients for a layer (where the elastic parameters are quadratic functions of the depth coordinate and Poisson's ratio is equal to 0.25) sandwiched between two elastic, homogeneous halfspaces. Furthermore, Payton [32] solved the uni-dimensional wave equation for a pulse travelling in a composite rod exhibiting a constant wavespeed in one part and a quadratically varying one in the other part by using the Laplace transform technique. Finally, a general technique for solving the vector wave equation in an arbitrary nonhomogeneous medium is by Karal and Keller [33], who

introduced an expansion of the solution in terms of asymptotic series. This technique does not require separability in the sense previously discussed, but the calculations required in order to obtain successive terms of the series are very difficult.

In this work, seismic wave propagation in two dimensional, unbounded heterogeneous continuous media is examined, as shown in Fig. 1. More specifically, the paper is structured as follows: First, the governing equations of motion are presented, followed by an algebraic transformation of the displacement vector so that the equilibrium equations attain a form which no longer involves derivatives of the material parameters. This process generates a number of constraints which dictate a Poisson's ratio of 0.25 and a quadratic variation with respect to the depth coordinate of the elastic moduli. Subsequently, three procedures are introduced for generating fundamental solutions for wave motions due to point impulses or initial conditions, namely: (i) conformal mapping followed by decomposition of the displacement vector into dilatational and rotational components, (ii) a pseudo-dilatational and pseudo-rotational decomposition technique in conjunction with an algebraic transformation of the dependent variable, and finally (iii) Fourier transformation followed by a first order differential equation system solution. In the first two cases, referring the transformed solution back to the original displacement vector is rather straightforward, while the last case requires a numerical inverse transformation from the wavenumber to the spatial domain. Furthermore, the first technique cannot be extended to the general three-dimensional situation, whereas the remaining two in principle can. Finally, some numerical examples are presented for specific types of heterogeneities in the companion paper.

2. Governing equations of motion

The dynamic equilibrium equations, the kinematic relations and the constitutive law for a continuous, elastic medium are

$$\left. \begin{aligned} \sigma_{ij,j} + \rho f_i &= \rho \ddot{u}_i \\ \varepsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \sigma_{ij} &= \lambda \theta \delta_{ij} + 2\mu \varepsilon_{ij} \end{aligned} \right\} \quad (1)$$

In the above, u_i , ρf_i , ε_{ij} and σ_{ij} respectively are displacements, body force per unit volume, strain and stress, while λ , μ are the Lamé elastic constants and ρ is the density. Furthermore $\theta = \varepsilon_{kk} = u_{k,k}$ is the dilatation. For 2D conditions, the indices range from 1 to 2, while commas indicate partial differentiation with respect to the spatial coordinates x_i and dots indicate partial derivatives with respect to time t . Finally, the summation convention is implied for repeated indices and δ_{ij} is Kronecker's delta.

If the continuum is heterogeneous, then the material parameters are position dependent (e.g., $\lambda = \lambda(\underline{x})$, $\mu = \mu(\underline{x})$, and $\rho = \rho(\underline{x})$ and the components of Eq. (1) combine to give the governing equations of motion in the form:

$$\{\lambda(\underline{x})u_{k,k}(\underline{x}, t)\}_{,i} + \{\mu(\underline{x})(u_{i,j}(\underline{x}, t) + u_{j,i}(\underline{x}, t))\}_{,j} + \rho(\underline{x})\ddot{f}_i(\underline{x}, t) = \rho(\underline{x})\ddot{u}_i(\underline{x}, t) \quad (2)$$

We note that unless variation of the material parameters over a wavelength is small, there is coupling between pressure and shear waves at every point of the continuum. Thus, vector decomposition methods [6] which form the cornerstone of techniques used for waves in homogeneous media are no longer applicable.

In order to recast for the above dynamic equilibrium equations in a more suitable form which does not involve derivatives of the material parameters, the following transformation [34] is established for the displacement vector $\underline{u} = (u_1, u_2)$:

$$\underline{u}(\underline{x}, t) = T(\underline{x})\underline{U}(\underline{x}, t) \quad (3)$$

where the precise form of T has yet to be determined. Using indicial notation, the various derivatives of u_i with respect to the spatial coordinates are as follows:

$$\left. \begin{aligned} u_{i,j} &= TU_{i,j} + T_{,j}U_i \\ u_{i,jj} &= TU_{i,jj} + 2T_{,j}U_{i,j} + T_{,jj}U_i \\ u_{j,ij} &= TU_{j,ij} + T_{,j}U_{j,i} + T_{,i}U_{j,j} + T_{,ij}U_j \end{aligned} \right\} \quad (4)$$

Substituting Eqs. (3) and (4) in the equations of dynamic equilibrium and collecting terms yields the following equations in terms of the transformed displacement vector U_i :

$$\begin{aligned} &\{T\lambda + T\mu\}U_{i,jj} + \{T\mu\}U_{i,jj} + \{2\mu T_{,j} + \mu_{,j}T\}U_{i,j} \\ &+ \{\lambda T_{,j} + \mu T_{,j} + \mu_{,j}T\}U_{j,i} + \{\lambda T_{,i} + \lambda_{,i}T + \mu T_{,i}\}U_{j,j} \\ &+ \{\mu T_{,jj} + \mu_{,j}T_{,j}\}U_i + \{\lambda T_{,ij} + \lambda_{,i}T_{,j} + \mu T_{,ij} + \mu_{,j}T_{,i}\}U_j \\ &+ \rho f_i = \rho T\ddot{U}_i \end{aligned} \quad (5)$$

If transformation T is chosen such that

$$2\mu T_{,j} + \mu_{,j}T = 0 \quad (6)$$

then

$$T(\underline{x}) = \mu^{-1/2}(\underline{x}) \quad (7)$$

By substituting the above expressions for T and for its spatial derivatives in Eq. (5) and simplifying, we obtain the

following equilibrium equation:

$$\begin{aligned} &\{\lambda + \mu\}U_{i,jj} + \mu U_{i,jj} + \left\{-0.5\frac{\lambda}{\mu}\mu_{,j} + 0.5\mu_{,j}\right\}U_{j,i} \\ &+ \left\{-0.5\frac{\lambda}{\mu}\mu_{,i} - 0.5\mu_{,i} + \lambda_{,i}\right\}U_{j,j} \\ &+ \{0.25\mu^{-1}\mu_{,j}\mu_{,j} - 0.5\mu_{,jj}\}U_i \\ &+ \left\{0.75\left(\frac{\lambda}{\mu^2} + \mu^{-1}\right)\mu_{,i}\mu_{,j} - 0.5\left(\frac{\lambda}{\mu} + 1\right)\mu_{,ij}\right. \\ &\left.- \frac{0.5}{\mu}(\lambda_{,i} + \mu_{,i})\mu_{,j}\right\}U_j + \rho\mu^{1/2}f_i = \rho\ddot{U}_i \end{aligned} \quad (8)$$

In order to remove all terms multiplying the lower order derivatives, constraint equations are identified as follows:

$$\left. \begin{aligned} &\left(-0.5\frac{\lambda}{\mu} + 0.5\right)\mu_{,i} = 0 \\ &\left(-0.5\frac{\lambda}{\mu} - 0.5\right)\mu_{,i} + \lambda_{,i} = 0 \\ &\left(0.25\frac{1}{\mu}\mu_{,j}\mu_{,j} - 0.5\mu_{,jj}\right) = 0 \\ &0.75\left(\frac{\lambda}{\mu^2} + \frac{1}{\mu}\right)\mu_{,i}\mu_{,j} - 0.5\left(\frac{\lambda}{\mu} + 1\right)\mu_{,ij} \\ &- \frac{0.5}{\mu}(\lambda_{,i} + \mu_{,i})\mu_{,j} = 0 \end{aligned} \right\} \quad (9)$$

The first constraint equation requires either a constant μ (i.e., the trivial solution) or $\lambda = \mu$, which corresponds to a Poisson's ratio of 0.25, a rather common value for igneous materials [2]. The second constraint equation is automatically satisfied if $\lambda = \mu$, while the remaining two respectively are:

$$\left. \begin{aligned} &0.25\mu^{-1}\mu_{,j}\mu_{,j} - 0.5\mu_{,jj} = 0 \\ &0.25\mu^{-1}\mu_{,i}\mu_{,j} - 0.5\mu_{,ij} = 0 \end{aligned} \right\} \quad (10)$$

If material parameters λ and μ (and consequently ρ) are assumed to be functions of only one spatial coordinate (depth $y = x_2$ for convenience), both of equations are equivalent to:

$$(\partial\mu(y)/\partial y)^2 - 2\mu(y)\partial^2\mu(y)/\partial y^2 = 0 \quad (11a)$$

whose solution is:

$$\mu(y) = (c_0y + c_1)^2 \quad (11b)$$

where c_0, c_1 are constants, i.e., we obtain a quadratic profile of the shear modulus with respect to the depth coordinate. By taking all the above constraints into account, the final form of the dynamic equilibrium equations is therefore

$$U_{i,jj} + 2U_{j,ij} + \alpha_s^2(y)F_i = \alpha_s^2(y)\ddot{U}_i \quad (12a)$$

where:

$$\underline{F}(x, t) = \mu^{1/2}(y) \underline{f}(x, t) \quad (12b)$$

and

$$\alpha_s(y) = \sqrt{\rho(y)/\mu(y)} \quad (12c)$$

is the shear wave slowness, i.e., the inverse of the shear wavespeed $c_s(y)$. We note that for the particular type of nonhomogeneous material examined herein, the pressure wave slowness (inverse of pressure wavespeed $c_p(y)$) is $\alpha_p(y) = \alpha_s(y)/\sqrt{3}$. At this stage, the usual Helmholtz decomposition of the displacement vector into dilatational and rotational components will still not work due to the presence of the non-constant slowness $\alpha_s(y)$.

Finally, we assume time harmonic conditions for both transformed displacement and forcing function vectors in the form:

$$\left. \begin{aligned} U_i(x, t) &= U_i(x) \exp(i\omega t) \\ F_i(x, t) &= F_i(x) \exp(i\omega t) \end{aligned} \right\} \quad (13)$$

where ω is the circular frequency of vibration. Therefore, the steady-state form of the governing equations of motion is:

$$U_{i,jj} + 2U_{j,ij} + \omega^2 \alpha_s^2(y) U_i = -\alpha_s^2(y) F_i \quad (14a)$$

or, introducing vector notation,

$$\nabla^2 \underline{U} + 2\nabla \nabla \cdot \underline{U} + \omega^2 \alpha_s^2(y) \underline{U} = -\alpha_s^2(y) \underline{F} \quad (14b)$$

In the above, $\nabla = (\partial/\partial x)_i + (\partial/\partial x)_j$ is the gradient vector in two dimensions and $\{\nabla\}^2 = \{\nabla\} \cdot \{\nabla\}$ is the Laplacian. The transient response can then be recovered through Fourier synthesis.

3. Conformal mapping technique

A conformal mapping technique for solving the scalar wave equation in a material with variable elastic parameters and density has been presented by the authors elsewhere [35]. In this section, the technique is generalized for two dimensional elastic waves by introducing an intermediate step involving decomposition of the transformed displacement vector $\underline{U}(x)$ into dilatational (superscript d) and rotational (superscript r) components. More specifically, we define

$$\left. \begin{aligned} \underline{U} &= \underline{U}^d + \underline{U}^r \\ \text{where} \\ \nabla \times \underline{U}^d &= \underline{0}, \nabla \cdot \underline{U}^r = 0 \end{aligned} \right\} \quad (15)$$

Upon substituting the above decomposition into governing equation Eq. (14b) and using vector identity $\nabla^2 \underline{U} = \nabla(\nabla \cdot \underline{U}) - \nabla \times (\nabla \times \underline{U})$, we obtain:

$$3\nabla^2 \underline{U}^d + \nabla^2 \underline{U}^r + \omega^2 \alpha_s^2(y) \{\underline{U}^d + \underline{U}^r\} = -\alpha_s^2(y) \{\underline{F}^d + \underline{F}^r\} \quad (16)$$

where forcing function \underline{F} has also been decomposed in the same manner as \underline{U} .

The next step is to introduce a transformation between two orthogonal coordinate systems (x, y) and (X, Y) based on conformal mapping concepts [36] so that the Laplacian operator transforms as follows:

$$\nabla_{xy}^2 \underline{U}(x, y) = J(x, y) \nabla_{XY}^2 \underline{U}(X, Y) \quad (17)$$

In the above, Jacobian J (the determinant of Jacobi's matrix transforming derivatives between the two coordinate systems) is given by [35]:

$$J(x, y) = \alpha_s^2(x, y) / \alpha_{s0}^2 \quad (18)$$

with $\alpha_{s0} = \alpha_s(x = y = 0)$ being a reference value for the shear wave slowness. The possibility for defining material parameters which are functions of two spatial variables should be noticed, despite the fact that the algebraic transformation procedure used in the previous section constrains variability in terms of a single coordinate, namely depth y . More details on conformal mapping are given later on. Thus, by re-defining Eq. (16) in the (X, Y) coordinate system we have that

$$\begin{aligned} \nabla_{XY}^2 (3\underline{U}^d + \underline{U}^r) &= -\{\omega^2 \alpha_s^2(y)(\underline{U}^d + \underline{U}^r) \\ &\quad + \alpha_s^2(y)(\underline{F}^d + \underline{F}^r)\} / J(x, y) \\ &= -\omega^2 \alpha_{s0}^2 (\underline{U}^d + \underline{U}^r) - \alpha_{s0}^2 \{\underline{F}^d + \underline{F}^r\} \end{aligned} \quad (19)$$

Since we now have a vector wave equation with constant coefficients, it is possible to define a new decomposition involving $\underline{U}^d = \underline{U}^d(X, Y)$ and $\underline{U}^r = \underline{U}^r(X, Y)$ as:

$$E(X, Y) = \nabla_{XY} \cdot \underline{U}^d \text{ and } \underline{\Omega}(X, Y) = \nabla_{XY} \times \underline{U}^r \quad (20a)$$

subject to constraints:

$$\nabla_{XY} \cdot \underline{U}^r = 0 \text{ and } \nabla_{XY} \times \underline{U}^d = 0 \quad (20b)$$

where E and $\underline{\Omega}$ are the dilatation and rotation vectors, respectively. Therefore, by first taking the divergence of Eq. (19), we obtain the governing equation for the dilatation in the (defaultX, defaultY) domain as:

$$\nabla_{XY}^2 E + \omega^2 \alpha_{p0}^2 E = -\alpha_{p0}^2 F_E \quad (21)$$

where body force $F_E = \nabla_{XY} \cdot \underline{F}^d$ and $\alpha_{p0} = \alpha_p(x = y = 0) = \alpha_{s0}/\sqrt{3}$ is the reference pressure wave slowness. Similarly, the curl of Eq. (19) gives the governing equation for the rotation vector as:

$$\nabla_{XY}^2 \underline{\Omega} + \omega^2 \alpha_{s0}^2 \underline{\Omega} = -\alpha_{s0}^2 \underline{F}_\Omega \quad (22)$$

where body force $\underline{F}_\Omega = \nabla_{XY} \times \underline{F}^r$. We observe that E and $\underline{\Omega}$ are associated with pressure and shear waves, respectively, and that both obey the standard, second order time harmonic wave equation. The solutions for freely propagating waves (zero body forces) including both outgoing and incoming

waves are well known [6], i.e.,

$$E(X, Y) = C_1 J_0(k_{p0}R) + C_2 Y_0(k_{p0}R) \quad (23a)$$

and

$$\Omega_z(X, Y) = C_3 J_0(k_{s0}R) + C_4 Y_0(k_{s0}R) \quad (23b)$$

where $k_{p0} = \omega \alpha_{p0}$ and $k_{s0} = \omega \alpha_{s0}$ are the reference pressure and shear wavenumbers, respectively, and $R = \sqrt{X^2 + Y^2}$ is a radial distance. Furthermore, J_0 and Y_0 are Bessel functions of zero order, first and second kind, respectively while constants C_1 through C_4 are determined through recourse to the appropriate boundary conditions. In their most general form, wavenumbers k_{p0} and k_{s0} are complex quantities whose imaginary part is a manifestation of viscoelastic material behavior. Finally, only component Ω_z of rotation vector $\underline{\Omega}$ is relevant in the context of Eq. (20a).

The solutions given by Eqs. (23a) and (23b) however, are not satisfactory. What we seek are fundamental solutions that fulfil the following basic criteria [37]: (i) they are singular at the source, which is taken as the origin ($X = 0$, $Y = 0$) of coordinates, and (ii) at large distances from the source where $R \rightarrow \infty$, the radiation condition is reproduced for outgoing waves only. As things stand, J_0 and Y_0 have to be rejected on account of the first and second criteria, respectively. The Hankel functions $H_0^{(1)} = J_0 + iY_0$ and $H_0^{(2)} = J_0 - iY_0$ are more suitable candidates since they are both singular as $R \rightarrow 0$ and their principal asymptotic forms for large arguments (with the term $\exp(i\omega t)$ included) are [38]: $H_0^{(1)}(kR) \exp(i\omega t) = \sqrt{2/\pi kR} \exp(ik(r + ct - \pi/4k))$ and $H_0^{(2)}(kR) \exp(i\omega t) = \sqrt{2/\pi kR} \exp(-ik(r - ct - \pi/4k))$. Of the two Hankel functions, the latter one corresponds to outgoing waves and is retained so that finally:

$$E(X, Y) = E_0 H_0^{(2)}(k_{p0}R) \quad (24a)$$

and

$$\Omega_z(X, Y) = \Omega_0 H_0^{(2)}(k_{s0}R) \quad (24b)$$

where constants E_0 and Ω_0 are the amount of initial strain prescribed at the source.

The final task is to invert the above solutions to the original (x, y) domain, a process which involves: (i) computing components $\underline{U}^d(R)$ and $\underline{U}^r(R)$ through use of Eqs. (20a) and (20b), (ii) changing back to the original coordinate system (x, y) from (X, Y) so as to synthesize $\underline{U}(r) = \underline{U}^d(r) + \underline{U}^r(r)$, where $r = \sqrt{x^2 + y^2}$, and (iii) invert the algebraic transformation of Eq. (3) so as to recover $\underline{u}(r)$ from $\underline{U}(r)$. It should be noted at this point that the strain fundamental solutions given by Eqs. (24a) and (24b) correspond to a Dirac delta forcing function at the source. The procedure outlined above implies that the final displacement fundamental solution will correspond to a Heaviside forcing function. In principle, it is possible to start with a Dirac delta for the forcing function at the level of the vector wave equation, namely Eq. (14a). It becomes difficult, however, to manipulate this

forcing function into a suitable form so that the final scalar wave equations given by Eqs. (21) and (22) are solvable. It can be done, of course, for a homogeneous medium by combining Helmholtz's decomposition of the body force with a volume integral representation [38], but the presence of a position-dependent wave slowness in Eqs. (14a) and (14b) renders this technique inapplicable. In sum, the solutions derived herein are all classified as Green's function since they are composed of a free space solution which contains the basic singularity and a regular part which accounts for any boundary conditions. Before we proceed with \underline{u} , however, it is necessary to elaborate on the second step involving conformal mapping between two coordinate systems, which also fixes the particular type of heterogeneity manifested by the variable wave slowness α_s .

We define a conformal mapping of the type $f(z)$ which maps complex number $z = x + iy$ into $Z = X + iY$, $i = \sqrt{-1}$. If $f(z)$ is an analytic function of z , then $X(x, y)$ and $Y(x, y)$ are related through the Cauchy–Riemann condition [36], i.e.:

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} \quad \text{and} \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x} \quad (25a)$$

Since this conformal mapping can also be viewed as a transformation between the two orthogonal coordinate systems (x, y) and (X, Y) , then the Jacobian $\partial(X, Y)/\partial(x, y)$ is given as:

$$J(x, y) = (\partial X/\partial x)^2 + (\partial X/\partial y)^2 = (\partial Y/\partial x)^2 + (\partial Y/\partial y)^2 \quad (25b)$$

which is non-dimensional if both coordinate systems have the same units. The Laplacian operator defined for two different coordinate systems is related through the above Jacobian and was given in Eq. (17), while Eq. (18) is the result of specifying a shear wave slowness $\alpha_s(x, y)$ proportional to $J(x, y)$ with the factor of proportionality being a constant, reference-type value α_{s0} . Thus, the choice of mapping $f(z)$ directly determines the particular form of α_s .

A rather general choice for the mapping combines exponential with polynomial terms in z , i.e.

$$f(z) = Z = c_0 z + c_1 e^{-bz} - c_2 e^{-2bz} \quad (26)$$

where $c_0 = 1$, and $c_1 = -c_2 = c$ so as to assure a common origin for both coordinate systems, while b and c are constants still to be specified. Reverting to the transformation, we have that

$$\left. \begin{aligned} X &= x + c \{ e^{-bx} \cos by - e^{-2bx} \cos 2by \} \\ Y &= y - c \{ e^{-bx} \sin by - e^{-2bx} \sin 2by \} \end{aligned} \right\} \quad (27)$$

We observe that both coordinate systems share $(0, 0)$ as

their origin. Radial distance R is defined as

$$\begin{aligned} R^2 = X^2 + Y^2 = r^2 + c^2 \{ e^{-2bx} + e^{-4bx} \} \\ + 2ce^{-bx} \{ x \cos by - y \sin by \} \\ - 2ce^{-2bx} \{ x \cos 2by - y \sin 2by \} \\ - 2c^2 e^{-3bx} \{ \cos by \cos 2by + \sin by \sin 2by \} \end{aligned} \quad (28)$$

where $r^2 = x^2 + y^2$. Finally, the Jacobian is (see Eq. (25b)):

$$\begin{aligned} J(x, y) = 1 + c^2 b^2 e^{-2bx} + 4c^2 b^2 e^{-4bx} \\ - (4c^2 b^2 e^{-3bx} + 2cbe^{-bx}) \cos by \\ + 4cbe^{-2bx} \cos 2by \end{aligned} \quad (29a)$$

The above expressions are general enough to include both horizontal (x) and vertical (y) variations for the wave slowness $\alpha_s^2 = \alpha_{s0}^2 J(x, y)$. Since we are constrained by the algebraic transformation given earlier to vertical heterogeneity only, it follows that:

$$\begin{aligned} \alpha_s^2(y) = \alpha_{s0}^2 J(x=0, y) = \alpha_{s0}^2 (1 + 5c^2 b^2 \\ - 2cb(2cb + 1) \cos by + 4cb \cos 2by) \end{aligned} \quad (29b)$$

Similarly, the radial distance R appearing in the arguments of the fundamental solutions given by Eqs. (23a) and (23b) is also a function of y only, i.e.:

$$R^2(y) = y^2 + 2c^2(1 - \cos by) - 2cy(\sin by - \sin 2by) \quad (29c)$$

Returning to the procedure for recovering \underline{u} , we first focus on Eqs. (20a), (20b), (24a), (24b). It is necessary to adopt polar coordinates (R, Θ) in the transformed coordinate system (X, Y) so that the new definitions $E(R) = \nabla_{R\Theta} \cdot \underline{U}^d(R)$ and $\Omega_z(R) = \nabla_{R\Theta} \times \underline{U}^r(R)$ respectively yield:

$$\left. \begin{aligned} U_R^d(R) &= \int_0^R E(R) dR \\ \text{and} \\ U_\Theta^r(R) &= \int_0^R \Omega_z(R) dR \end{aligned} \right\} \quad (30a)$$

Furthermore, components $U_\Theta^d(R) = U_R^r(R) = 0$ by imposing the constraints given in Eq. (20b) in polar coordinates. Performing the integrations gives: [39]

$$\begin{aligned} U_R^d(R) &= RH_0^{(2)}(k_{p0}R) + \frac{\pi R}{2} \{ \mathcal{H}_0(k_{p0}R) H_1^{(2)}(k_{p0}R) \\ &\quad - \mathcal{H}_1(k_{p0}R) H_0^{(2)}(k_{p0}R) \} \end{aligned}$$

and

$$\begin{aligned} U_\Theta^r(R) &= RH_0^{(2)}(k_{s0}R) + \frac{\pi R}{2} \{ \mathcal{H}_0(k_{s0}R) H_1^{(2)}(k_{s0}R) \\ &\quad - \mathcal{H}_1(k_{s0}R) H_0^{(2)}(k_{s0}R) \} \end{aligned} \quad (30b)$$

where \mathcal{H}_0 and \mathcal{H}_1 are Struve functions, while $H_1^{(2)}$ is the Hankel function of first order and second kind.

Using the coordinate transformation summarized in Eq. (29c), components $U_R^d(R)$ and $U_\Theta^r(R)$ are respectively identified with the pressure wave dilatation component $U_y^d(|y|)$ and the shear wave rotation component $U_x^r(|y|)$. The final step is recovery of \underline{u} from $\underline{U} = \underline{U}^d + \underline{U}^r$, i.e.:

$$\underline{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \mu(y)^{-1/2} \begin{Bmatrix} U_x^r \\ U_y^d \end{Bmatrix} \quad (31)$$

This approach will be further illustrated through the use of a numerical example in the companion paper.

4. Vector decomposition technique

In the vector decomposition technique, we define a dilatation $e(x, y)$ and a rotation vector $\underline{\omega}(x, y)$ (not to be confused with frequency ω) as:

$$\left. \begin{aligned} e &= \nabla \cdot \underline{U} \\ \underline{\omega} &= \nabla \times \underline{U} \end{aligned} \right\} \quad (32)$$

Upon substitution in the equation of motion Eq. (14b) and neglecting the body force for simplicity, we obtain the following two coupled wave equations after applying the divergence and curl operators, respectively:

$$\left. \begin{aligned} \nabla^2 e + \omega^2 \alpha_p^2(y) e + \omega^2 \nabla \alpha_p^2(y) \cdot \underline{U} &= 0 \\ \nabla^2 \underline{\omega} + \omega^2 \alpha_s^2(y) \underline{\omega} + \omega^2 \nabla \alpha_s^2(y) \times \underline{U} &= 0 \end{aligned} \right\} \quad (33)$$

In order to uncouple the above equations, we introduce a transformation of the dependent variable in the form

$$\underline{U}(x, y) = A(\alpha_s(y)) \underline{V}(x, y) \quad (34)$$

along with a modified dilatation $E = \nabla \cdot \underline{V}$ and a modified rotation vector $\underline{\Omega} = \nabla \times \underline{V}$ with $\nabla = (\partial/\partial x)\underline{i} + (\partial/\partial y)\underline{j}$ being the gradient operator in two dimensions. This type of transformation is very versatile in conjunction with the scalar wave equation and for heterogeneity along one spatial coordinate [40], but has not been explored within the context of elastic waves. The specific form of transformation $A(y)$ will be determined later, as part of the solution of a number of constraint-type equations.

Based on this redefinition of displacement vector \underline{U} , the original dilatation and rotation are expressed in terms of the transformed quantities as follows:

$$\left. \begin{aligned} e &= AE + \nabla A \cdot \underline{V} \\ \underline{\omega} &= A \underline{\Omega} + \nabla A \times \underline{V} \end{aligned} \right\} \quad (35)$$

Substituting the above in the wave equation for the dilatation, the first component of Eq. (33), gives:

$$\begin{aligned} \nabla^2 [AE + \nabla A \cdot \underline{V}] + \omega^2 \alpha_p^2(y) [AE + \nabla A \cdot \underline{V}] \\ + \omega^2 \nabla \alpha_p^2(y) \cdot \{A \underline{V}\} = 0 \end{aligned} \quad (36)$$

If we take into account the fact that the pressure wave slowness $\alpha_p(y)$ is a function of a single coordinate only, then the various terms appearing in the above equation are as follows:

$$\begin{aligned}\nabla^2 \alpha_p \{A \underline{V}\} &= \frac{\partial \alpha_p^2}{\partial y} A V_y \underline{A} E + \nabla A \cdot \underline{V} = A E + \frac{\partial A}{\partial y} V_y \\ \nabla^2 [A E + \nabla A \cdot \underline{V}] &= A \nabla^2 E + \frac{\partial A}{\partial y} (\nabla^2 V_y) + 2 \frac{\partial A}{\partial y} \frac{\partial E}{\partial y} \\ &\quad + \frac{\partial^2 A}{\partial y^2} \left(E + 2 \frac{\partial V_y}{\partial y} \right) + \frac{\partial^3 A}{\partial y^3} V_y\end{aligned}\quad (37)$$

Substituting the above in Eq. (36) and re-arranging terms results in:

$$\begin{aligned}\nabla^2 E + \frac{2}{A} \frac{\partial A}{\partial y} \frac{\partial E}{\partial y} + \left(\frac{1}{A} \frac{\partial^2 A}{\partial y^2} + \omega^2 \alpha_p^2 \right) E \\ + \left\{ \frac{1}{A} \frac{\partial^3 A}{\partial y^3} + \omega^2 \alpha_p^2 \left(\frac{1}{A} \frac{\partial A}{\partial y} + \frac{1}{\alpha_p^2} \frac{\partial \alpha_p^2}{\partial y} \right) \right\} V_y \\ + \frac{2}{A} \frac{\partial^2 A}{\partial y^2} \frac{\partial V_y}{\partial y} + \frac{1}{A} \frac{\partial A}{\partial y} \nabla^2 V_y = 0\end{aligned}\quad (38)$$

In order to achieve uncoupling of the above equation so that it pertains to the transformed dilatation E only, we impose the following four constraints:

$$\left. \begin{aligned}\frac{\nabla A}{A} + \frac{\nabla \alpha_p^2}{\alpha_p^2} &= 0 \\ \frac{\nabla^2 (\nabla A)}{A} &= 0 \\ \frac{\nabla^2 A}{A} &= 0 \\ \nabla^2 V_y &= 0\end{aligned}\right\}\quad (39)$$

where $\alpha_p = \alpha_p(y)$ and $A = A(y)$. We observe that the equation governing dilatation $E = E(x, y)$ is no longer a wave equation, but a partial differential equation of the second order with non-constant coefficients, i.e.:

$$\nabla^2 E + \frac{2}{A} \frac{\partial A}{\partial y} \frac{\partial E}{\partial y} + \left(\frac{1}{A} \frac{\partial^2 A}{\partial y^2} + \omega^2 \alpha_p^2(y) \right) E = 0\quad (40)$$

A similar path is followed for the equation governing the rotation vector, namely the second component of Eq. (33). Upon substituting the expression for $\underline{\omega}$ given by the second component of Eq. (35), the aforementioned equation becomes

$$\begin{aligned}\nabla^2 [A \underline{\Omega} + \nabla A \times \underline{V}] + \omega^2 \alpha_s^2(y) [A \underline{\Omega} + \nabla A \times \underline{V}] \\ + \omega^2 \nabla \alpha_s^2(y) \times \{A \underline{V}\} = 0\end{aligned}\quad (41)$$

Expanding the cross products gives:

$$\left. \begin{aligned}\nabla \alpha_s^2 \times \{A \underline{V}\} &= -A \frac{\partial \alpha_s^2}{\partial y} V_y \underline{k} \\ A \underline{\Omega} + \nabla A \times \underline{V} &= \left(A \Omega_z + \frac{\partial A}{\partial y} V_y \right) \underline{k} \\ \nabla^2 [A \underline{\Omega} + \nabla A \times \underline{V}] &= \left(A \nabla^2 \Omega_z - \frac{\partial A}{\partial y} \nabla^2 V_y \right. \\ &\quad \left. + 2 \frac{\partial A}{\partial y} \frac{\partial \Omega_z}{\partial y} + \frac{\partial^2 A}{\partial y^2} \Omega_z - 2 \frac{\partial^2 A}{\partial y^2} \frac{\partial V_y}{\partial y} - \frac{\partial^3 A}{\partial y^3} V_y \right) \underline{k}\end{aligned}\right\}\quad (42)$$

where Ω_z and \underline{k} respectively are the component of $\underline{\Omega}$ and the unit vector along the default z -direction. Substituting the above expressions in Eq. (41) and imposing the same constraints as with the dilatation default E (the only difference being that the first component of Eq. (39) holds true for default α_p replacing default α_s and that the fourth component of Eq. (39) is replaced by $\nabla^2 \text{default } V_x = 0$ so that $\nabla^2 \underline{V} = 0$) is the general constraint on \underline{V} , we recover the following equation governing the only non-zero component of the rotation vector $\underline{\Omega}$, i.e.:

$$\nabla^2 \Omega_z + \frac{2}{A} \frac{\partial A}{\partial y} \frac{\partial \Omega_z}{\partial y} + \left(\frac{1}{A} \frac{\partial^2 A}{\partial y^2} + \omega^2 \alpha_s^2 \right) \Omega_z = 0\quad (43)$$

We observe that both E and Ω_z are governed by the same type of differential equation with $k_p = \omega \alpha_p$ and $k_s = \omega \alpha_s$ as their corresponding wavenumbers, respectively.

The next step is to examine constraint Eq. (39) so as to establish the precise form of wave slowness α_p and α_s for which the uncoupled equations governing dilatation E and rotation Ω_z hold true. The first of the components of Eq. (39) is identically satisfied if:

$$A(y) = \frac{c}{\alpha^2(y)}\quad (44)$$

where $\alpha^2(y)$ is either $\alpha_p^2(y)$ or $\alpha_s^2(y)$ and c is a constant which can be taken as equal to one. The second component of Eq. (39) implies that $\alpha^{-2}(y)$ is a polynomial up to the second degree in y , i.e.:

$$\alpha^2(y) = (c_0 + c_1 y + c_2 y^2)^{-1}\quad (45)$$

where c_0, c_1, c_2 , are constants to be determined. The solution for the third component of Eq. (39) restricts the above polynomial to the first degree in y , while the fourth component of Eq. (39) simply states that the transformed vector \underline{V} must be harmonic. In sum, our constraints yield:

$$A(y) = \alpha^{-2}(y) = c_0 + c_1 y\quad (46)$$

where $\alpha(y)$ is either $\alpha_p(y)$ or $\alpha_s(y)$, with the former wave slowness pertaining to E and the latter to Ω_z .

As previously mentioned, the equations governing dilatation and rotation are essentially equivalent. We will focus on the first one for E , since the same solution is applicable to

Ω_z as well. By substituting Eq. (46) into Eq. (40) and abandoning vector notation, we have that:

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{2c_1}{c_0 + c_1 y} \frac{\partial E}{\partial y} + \frac{\omega^2}{c_0 + c_1 y} E = 0 \quad (47a)$$

Unfortunately, a classical separation of variables of the form $E(x, y) = E_1(x)E_2(y)$ will not work in conjunction with the above equation because of the presence of non-constant coefficients. If however we restrict the solution to the case where elastic waves are propagating along the y direction only ($E = E(x = 0, y)$) we have, upon normalization of Eq. (47a) by introducing $\tilde{y} = c_1 y / c_0$, that

$$(1 + \tilde{y}) \frac{d^2 E}{d\tilde{y}^2} + 2 \frac{dE}{d\tilde{y}} + \frac{\omega^2 c_0}{c_1^2} E = 0 \quad (47b)$$

The final step is to reset the singularity of the above second order, ordinary differential equation at zero by defining $t = 1 + \tilde{y}$. Thus, $E = E(t)$ is the solution of:

$$\frac{d^2 E}{dt^2} + \frac{2}{t} \frac{dE}{dt} + \frac{b}{t} E = 0 \quad (47c)$$

where $b = \omega^2 c_0 / c_1^2$. The above equation has $t = 0$ as a regular singular point and assumes a power series solution [41]. More specifically, the associated indicial equation $r(r - 1) + 2r = 0$ has two roots, $r_1 = 0$ and $r_1 = -1$ (which differ by an integer). The solution which fulfils the criteria set forth in the previous section regarding singular behavior at the origin and a decaying outgoing wave at large distances is [39]

$$E(t) = E_0 H_1^{(2)}(2\sqrt{bt})/\sqrt{t} \quad (48a)$$

In the above, E_0 is the constant initial strain at the source, while $H_1^{(2)}$ is the Hankel function of first order and second kind. The above solution can be verified through backsubstitution in Eq. (47c). Rotation $\Omega_x(t)$ is also governed by the same sequence of equations Eqs. (47a–c), except for the fact that constants c_0, c_1 will now have different values. Thus:

$$\Omega_x(t) = \Omega_0 H_1^{(2)}(\sqrt{b't})/\sqrt{t} \quad (48b)$$

where Ω_0 is an initial strain and $b' = \omega c_0' / (c_1')^2$.

In order to return to the original displacement vector $u(x)$, the following steps are followed: (i) redefine E and $\underline{\Omega}$ in terms of the original spatial variable y ; (ii) integrate relations $E = \nabla \cdot \underline{V}$ and $\underline{\Omega} = \nabla \times \underline{V}$ so as to obtain $\underline{V} = (V_x, V_y)$ in terms of the aforementioned modified dilatation and rotation; (iii) evaluate \underline{U} directly from Eq. (34) since the specific form of $A(y)$ used in the transformation between \underline{U} and \underline{V} is known from Eq. (44); and (iv) use the algebraic scaling given by Eq. (3) to express the original displacement vector \underline{u} in terms of \underline{U} . We note again that the fundamental solutions derived herein for the strains and their corresponding displacements are due to a strain impulse at the source. Finally, the constraint that $\underline{V}(y)$ must be harmonic can be imposed by adjusting the integration constants that appear in the expression for \underline{V} .

The only step that needs elaboration is the second one involving integration of the modified dilatation and rotation. In particular, we have that

$$\left. \begin{aligned} V_y(y) &= \int_0^y E(y) dy \\ \text{and} \\ V_x(y) &= - \int_0^y \Omega_z(y) dy \end{aligned} \right\} \quad (49a)$$

After some algebra and taking into account the recursion formulas for the derivatives of Bessel functions, [39] the final results are given in terms of the physical coordinate y as:

$$V_y(y) = -E_0 \frac{\sqrt{c_0}}{\omega} \left\{ H_0^{(2)} \left(\frac{2\omega}{c_1} \sqrt{c_0 + c_1 y} \right) - H_0^{(2)} \left(\frac{2\omega}{c_1} \sqrt{c_0} \right) \right\}$$

and

$$V_x(y) = -\Omega_0 \frac{\sqrt{c_0'}}{\omega} \left\{ H_0^{(2)} \left(\frac{2\omega}{c_1'} \sqrt{c_0' + c_1' y} \right) - H_0^{(2)} \left(\frac{2\omega}{c_1'} \sqrt{c_0'} \right) \right\} \quad (49b)$$

Recapitulating, we note that the V_y component corresponds to a vertically propagating pressure wave travelling at a wavespeed $c_p^2(y) = c_0 + c_1 y$, while V_x is a horizontally polarized shear wave with $c_s^2(y) = c_0 + c_1 y$ as its wavespeed. The final result is that:

$$\underline{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \mu(y)^{-1/2} A(y) \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} \quad (49c)$$

This procedure will be illustrated in the section on numerical examples in Part II where the particular type of heterogeneity for which the above methodology is valid will also be discussed.

5. First order system solution with Fourier transformation

The last method involves a Fourier transformation with respect to the horizontal spatial coordinate x of the dynamic equilibrium equation Eq. (14a) in order to recover a form which contains derivatives with respect to the depth coordinate only and is therefore amenable to solution via a first order differential equation system formulation. The particular exponential Fourier transformation employed here is defined as

$$\left. \begin{aligned} F(h) &= \bar{h}(k) = \int_{-\infty}^{\infty} h(x) \exp(ikx) dx \\ F^{-1}(\bar{h}) &= h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{h}(k) \exp(-ikx) dk \end{aligned} \right\} \quad (50)$$

where F and F^{-1} respectively are direct and inverse transformations and k is the wavenumber. We mention the operational property for the n -th derivative of function $h(t)$ as:

$$F(h^{(n)}) = (-ik)^n \bar{h}(k) \quad (51)$$

and the transform of the Dirac delta function $\delta(x)$ is $F(\delta) = 1$.

Application of the aforementioned double Fourier transform to Eq. (14a) written in expanded form yields the following system of equations:

$$\begin{aligned} \frac{d^2}{dy^2} \bar{U}_x - 2ik \frac{d}{dy} \bar{U}_y + (-3k^2 + \omega^2 \alpha_s^2(y)) \bar{U}_x &= -\alpha_s^2(y) \bar{F}_x \\ 3 \frac{d^2}{dy^2} \bar{U}_y - 2ik \frac{d}{dy} \bar{U}_x + (-k^2 + \omega^2 \alpha_s^2(y)) \bar{U}_y &= -\alpha_s^2(y) \bar{F}_y \end{aligned} \quad (52)$$

where $\bar{U} = (\bar{U}_x, \bar{U}_y)$. In order to reformulate Eq. (52) as a system consisting of two first order, ordinary differential Eqs. [41], we introduce the following notation:

$$\bar{W}_x = \frac{d}{dy} \bar{U}_x \text{ and } \bar{W}_y = \frac{d}{dy} \bar{U}_y \quad (53)$$

The next step is to combine Eqs. (52), (53) and introduce matrix notation. Thus, we obtain the following 4×4 first order differential equation system

$$\begin{aligned} \frac{d}{dy} \begin{Bmatrix} \bar{U}_x \\ \bar{U}_y \\ \bar{W}_x \\ \bar{W}_y \end{Bmatrix} &= \begin{bmatrix} [0] & [I] \\ -q_1^2(y) & 0 & 0 & 2ik \\ 0 & -q_2^2(y)/3 & \frac{2}{3}ik & 0 \end{bmatrix} \\ &\times \begin{Bmatrix} \bar{U}_x \\ \bar{U}_y \\ \bar{W}_x \\ \bar{W}_y \end{Bmatrix} - \alpha_s^2(y) \begin{Bmatrix} 0 \\ 0 \\ \bar{F}_x \\ \bar{F}_y/3 \end{Bmatrix} \end{aligned} \quad (54)$$

where $[0]$ and $[I]$ are the null and unit submatrices, respectively, and $q_1^2(y) = -3k^2 + \omega^2 \alpha_s^2(y)$, $q_2^2(y) = -k^2 + \omega^2 \alpha_s^2(y)$. Using symbolic notation, Eq. (54) can be written as:

$$\frac{d}{dy} \{\bar{V}\} = [A(y)]\{\bar{V}\} + \{\bar{B}\} \quad (55)$$

in the wavenumber k domain. Since system matrix $[A]$ is non-constant, the usual solution methodology for first order differential equation systems involving the eigensolution of $[A]$ is not applicable and special techniques (series expansions, Picard iterations) must be sought [41,42].

The presence of the load vector $\{\bar{B}\}$ complicates the solution procedure so it becomes necessary to convert to equivalent boundary conditions defined at $y = 0$. In particular, the

original load vector was:

$$\begin{Bmatrix} F_x \\ F_y \end{Bmatrix} = F_0 \delta(x) \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = F_0 \delta(x) \delta(y) \underline{e} \quad (56a)$$

so that the Fourier transform with respect to the x -coordinate yields:

$$\begin{Bmatrix} \bar{F}_x \\ \bar{F}_y \end{Bmatrix} = F_0 \delta(y) \underline{e} \quad (56b)$$

From the theory of generalized functions [41], solution $E^*(y)$ of the n -th order ordinary differential equation $D^n\{E^*(y)\} = \delta(y)$ subject to zero initial conditions can be written as $E^*(y) = H(y)E(y)$, where $H(y)$ is the Heaviside function and $E(y)$ is the solution of the equivalent homogeneous equation $D^n\{E(y)\} = 0$ subject to initial conditions $E^{(n-1)}(0) = 1$, $E^{(n-2)}(0) = \dots = E^{(1)}(0) = E(0) = 0$. Thus Eq. (55) can be re-cast as:

$$\frac{d}{dy} \{\bar{V}\} = [A(y)]\{\bar{V}\}, \quad \{\bar{V}(y=0)\} = \{\hat{V}\} \quad (57a)$$

where the equivalent boundary condition vector is defined as:

$$\{\hat{V}\}^T = -F_0 \alpha_{s0}^2 [0, 0, 1, 1/3] \quad (57b)$$

with reference wave slowness $\alpha_{s0} = \alpha_s(y=0)$.

The solution procedure for the above equation is based on a series expansion of both system matrix and response, i.e.,

$$\begin{aligned} \{\bar{V}(y)\} &= \{\bar{V}\}_0 + \{\bar{V}\}_1 y + \{\bar{V}\}_2 y^2 + \dots \\ &\text{and} \\ [A(y)] &= [A]_0 + [A]_1 y + [A]_2 y^2 + \dots \end{aligned} \quad (58)$$

where subscripts denote the expansion order. By substituting the above in Eq. (57a), matching powers of y and identifying the zeroth order solution with boundary conditions $\{\hat{V}\}$ the first and higher order solutions are obtained as

$$\begin{aligned} \{\bar{V}\}_1 &= [A]_0 \{\hat{V}\} \\ \{\bar{V}\}_2 &= \frac{1}{2} ([A]_1 + [A]_0^2) \{\hat{V}\} = [B]_0 \{\hat{V}\} \\ \{\bar{V}\}_3 &= \frac{1}{3} \left([A]_2 + [A]_1 [A]_0 + \frac{1}{2} [A]_0 [A]_1 + \frac{1}{2} [A]_0^3 \right) \{\hat{V}\} \\ &= [C]_0 \{\hat{V}\} \end{aligned} \quad (59)$$

The final response can then be reconstituted through recourse to Eq. (58) as:

$$\{\bar{V}\} = ([I] + [A]_0 y + [B]_0 y^2 + [C]_0 y^3 + \dots) \{\hat{V}\} \quad (60)$$

This type of approach favors a polynomial structure of the wavespeed slowness $\alpha_s(y)$. Thus, we specify the general form:

$$\alpha_s(y) = \alpha_{s0} (1 + ay)^{-n/2} \quad (61)$$

where $n = 0, 1, 2$ respectively correspond to the quadratic, linear and constant density profiles, while constant $a = (\sqrt{\mu_1}/\sqrt{\mu_0} - 1)/L$ (which is the slope of α_s) results from a quadratic variation of the shear modulus across layer thickness L . For the more general case of $n = 2$, the expansion terms are:

$$[A]_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3k^2 - \omega^2 \alpha_0^2 & 0 & 0 & 2ik \\ 0 & (k^2 - \omega^2 \alpha_0^2)/3 & 2ik/3 & 0 \end{bmatrix} \quad (62a)$$

$$[A]_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\omega^2 \alpha_{s0}^2 a & 0 & 0 & 0 \\ 0 & \frac{2}{3} \omega^2 \alpha_{s0}^2 a & 0 & 0 \end{bmatrix} \quad (62b)$$

and

$$[A]_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3(\omega \alpha_{s0} a)^2 & 0 & 0 & 0 \\ 0 & -(\omega \alpha_{s0} a)^2 & 0 & 0 \end{bmatrix} \quad (62c)$$

With the expansion terms of $[A(y)]$ now available, Eq. (59) can be used to synthesize matrices $[B]_0$ and $[C]_0$, while Eq. (60) gives the solution for $\{\bar{V}\}$ in terms of boundary conditions $\{V^0\}$. The final step in this solution procedure is a numerical inverse transformation of solution vector $\{\bar{V}\}$ back to the spatial domain so as to recover displacement components U_x and U_y corresponding to a forcing function which is a point impulse in space and time harmonic. The Fourier transformation software package given in Ref. [43] can be employed for the inversion part. Finally, one last scaling is required, as dictated by the algebraic transformation procedure and given by Eq. (3). Thus, we obtain a solution vector $u(x, \omega) = [u_x, u_y]^T$ for each of the two point forces, which corresponds to a Green's function $g_{ij}(x, \omega)$, ($i, j = 1, 2$) for the time harmonic vector wave equation defined in a nonhomogeneous medium with Poisson's ratio of 0.25, a quadratic shear modulus profile in depth and constant density. In contrast with the two solutions previously obtained, this is a fundamental solution which corresponds to an initial displacement impulse at the source.

6. Conclusions

In this work, three different methodologies, namely conformal mapping, vector decomposition into pseudo-dilatational and pseudo-rotational components and an integral transformation in conjunction with first order differential equation system solution are developed for constructing

fundamental solutions for wave propagation in two dimensional nonhomogeneous media where the material parameters are functions of the depth coordinate. In particular, constraints dictated by an algebraic transformation applied to the governing equations of motion, which is a common step for all three methods, produce a quadratic depth profile for both elastic parameters plus a Poisson's ratio of 0.25. The density profile remains arbitrary and its particular form depends on the method used. Finally, numerical examples which serve to illustrate the above methodologies are presented in the companion paper. Some improvements, however, need to be introduced to the present methodologies, namely use of the method of images so as to reproduce a traction-free horizontal surface and superposition of a number of point sources (impulses) so as to reproduce the effect of a finite-size source. These issues can be also approached through use of discrete modelling techniques, i.e., integral equation formulations employing the fundamental solutions derived herein as kernel functions.

References

- [1] Kennet BLN. Seismic wave propagation in stratified media. Cambridge: Cambridge University Press, 1983.
- [2] Ewing WM, Jardetzky WS, Press F. Elastic waves in layered media. New York: McGraw-Hill, 1957.
- [3] Ben Menachem A, Singh SJ. Seismic waves and sources. New York: Springer-Verlag, 1981.
- [4] Hanyga A, editor. Seismic wave propagation in the earth. Amsterdam: Elsevier, 1985.
- [5] Brekhovskikh LM, Beyer R. Waves in layered media, 2nd edn. New York: Academic Press, 1980.
- [6] Graff KF. Wave motion in elastic solids. Columbus, OH: Ohio University Press, 1975.
- [7] Thomson W. Transmission of elastic waves through a stratified solid medium. Journal of Applied Physics 1950;21:89–93.
- [8] Haskell NA. The dispersion of surface waves in multilayered media. Bulletin Seismological Society of America 1953;43:17–34.
- [9] Haskell NA. Radiation pattern of surface waves from point sources in multi-layered medium. Bulletin Seismological Society of America 1964;54:337–393.
- [10] Biot MA. Continuum dynamics of elastic plates and multilayered solids under initial stress. Journal of Mathematics and Mechanics 1963;12:793–810.
- [11] Shaw RP, Bugl P. Transmission of plane waves through layered linear viscoelastic media. Journal Acoustical Society of America 1969;46(3):649–654.
- [12] Kausel E, Roesset JM. Stiffness matrices for layered soil. Bulletin Seismological Society of America 1981;71:1743–1761.
- [13] Bravo MA, Sanchez-Sesma F, Chavez-Garcia F. Ground motion on stratified alluvial deposits for incident SH waves. Bulletin Seismological Society of America 1988;78:436–450.
- [14] Chen JT, Chen LY, Hong HK. Regularization method for deconvolution problems in soil dynamics. In: Brebbia CA, Kim S, Osswald TA, Power H, editors. Boundary elements XVII. Southampton: Computational Mechanics Publications, 1995:351–358.
- [15] Bonnet G, Heitz JF. Nonlinear seismic response of a soft layer. In: Duma G, editor. Proceedings 10th European Conference on Earthquake Engineering. Rotterdam: Balkema, 1995:361–364.
- [16] Pires JA. Stochastic seismic response analysis of soft soil sites. In: Kussmaul K, editor. Structural mechanics in reactor technology—12. London: Elsevier Science Publishers, 1993;M/K:219–224.

- [17] Safak E. Discrete time analysis of seismic site amplification. *Journal of Engineering Mechanics ASCE* 1995;121:801–809.
- [18] Urratia-Galicia JL, Ruiz SE, Diederich R. Comparison of matrix methods in one-dimensional amplification problems. In: Cakmak AS, Herrera IS, editors. *Engineering seismology and site response*. Southampton: Computational Mechanics Publications, 1989:89–99.
- [19] Wang CY, Herrmann RB. A numerical study of P-, SV-, and SH-wave generation in a plane layered medium. *Bulletin Seismological Society of America* 1980;70:1015–1036.
- [20] Cagniard L. *Reflection and refraction of progressive seismic waves*. English Translation. New York: McGraw-Hill, 1962.
- [21] Hudson JA. A quantitative evaluation of seismic signals at teleseismic distances—II. Body waves and surface waves from an extended source. *Geophysics Journal* 1969;18:353–370.
- [22] Umek A, Strukelj A. Green's function for layered halfspace. In: Cakmak AS, editor. *Ground motion and engineering seismology*. Amsterdam: Elsevier, 1987:339–345.
- [23] Apsel RJ, Luco JE. On the Green's function for the layered halfspace. Part II. *Bulletin Seismological Society of America* 1983;73:931–951.
- [24] Pai DM. Wave propagation in inhomogeneous media: a planewave layer interaction method. *Wave Motion* 1991;13:205–209.
- [25] Mikhailenko BG. Numerical experiments in seismic investigations. *Journal of Geophysics* 1985;58:101–124.
- [26] Mikhailenko BG. Synthetic seismograms for complex three dimensional geometries using an analytical-numerical technique. *Geophysics Journal Royal Astronomical Society* 1984;79:963–986.
- [27] Gazdag J. Numerical convective schemes based on accurate computation of space derivatives. *Journal Computational Physics* 1971;13:100–113.
- [28] Acharya HK. Field due to a point source in an inhomogeneous elastic medium. *Journal Acoustical Society of America* 1971;50:172–175.
- [29] Hook JF. Separation of the vector wave equation of elasticity for inhomogeneous materials. *Journal Acoustical Society of America* 1961;33:302–313.
- [30] Alverson RC, Gair FC, Hook JF. Uncoupled equations of motion in nonhomogeneous elastic media. *Bulletin Seismological Society of America* 1963;53:1023–1030.
- [31] Gupta RN. Reflection of elastic waves from a linear transition layer. *Bulletin Seismological Society of America* 1966;56:511–526.
- [32] Payton RG. Elastic wave propagation in a nonhomogeneous rod. *Quarterly Journal of Mechanics and Applied Mathematics* 1966;19:83–91.
- [33] Karal FC, Keller JB. Elastic wave propagation in homogeneous and inhomogeneous media. *Journal Acoustical Society of America* 1959;31:694–705.
- [34] Manolis GD, Shaw RP. Green's function for the vector wave equation in a mildly heterogeneous continuum. *Wave Motion* 1996;24:59–83.
- [35] Shaw RP, Manolis GD. Two dimensional acoustic waves in a heterogeneous medium using conformal mapping and dependent variable transformations. In: *Proceedings of Third International Conference on Computational Acoustics*, Newark, July 14–17, 1997 (to appear).
- [36] Churchill RV, Brown JW, Verhey RH. *Complex variables and applications*. New York: McGraw-Hill, 1974.
- [37] Greenberg MD. *Application of Green's functions in science and engineering*. Englewood Cliffs: Prentice-Hall, 1970.
- [38] Sternberg E, Eubanks RA. On the concept of concentrated loads and an extension of the uniqueness theorem in the linear theory of elasticity. *Journal of Rational Mechanics and Analysis* 1955;6:34–50.
- [39] Gradshteyn IS, Ryzhik IM. *Tables of integrals, series and products*. New York: Academic Press, 1980.
- [40] Shaw RP, Manolis GD. Elastic waves in one-dimensionally layered heterogeneous soil media. In: Kausel E, Manolis GD, editors. *Wave motion problems in earthquake engineering*. Southampton: Computational Mechanics Publications, Southampton, 1998 (to appear).
- [41] Braun M. *Differential equations and their applications*. Berlin: Springer, 1975.
- [42] Manolis GD, Shaw RP. Fundamental solutions to Helmholtz's equation for inhomogeneous media by a first-order differential equation system. *Soil Dynamics Earthquake Engineering* 1997;16:81–94.
- [43] Press WH, Flannery BP, Teukolsky SA, Vetterling WT. *Numerical recipes (Fortran Version)*. Cambridge: Cambridge University Press, 1989.