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CRACKS AT THE EDGE OF AN ELLIPTIC HOLE IN OUT OF PLANE SHEAR

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Abstract—In this paper integral transform techniques are used to find the mode III stress intensity factors for cracks at the edge of an elliptic hole in an infinite elastic solid.

1. INTRODUCTION

WE PROPOSE to determine the stress intensity factors for one and two cracks at the edge of an elliptic hole in an infinite elastic solid which is subject to out of plane shear. These problems were first investigated by Yokobori *et al.*[1, 2] who solved them by means of continuous dislocations. The purpose of the present paper is to verify Yokobori's solutions and to show that they can be obtained elegantly via integral transforms.

In Cartesian coordinates (x, y) our ellipse is given by the equation

$$\frac{x^2}{c^2}+\frac{y^2}{h^2}=1. \tag{1.1}$$

In problem 1 there is an edge crack defined by the relations $c \leqslant x < b, y = 0$ while, in problem 2, there are two edge cracks defined by $c \leqslant |x| < b, y = 0$. The cracks and the hole are assumed to be traction free while the solid is subject to a uniform out of plane shear load T as shown in Fig. 1. In the interest of brevity, solution details are provided only for the case in which $h < c$. It should be noted, however, that the case $h \geqslant c$ can be dealt with similarly and yields precisely the same expressions for the stress intensity factors and other field quantities.

To facilitate the use of integral transforms we introduce elliptic coordinates (ξ, η) which are defined by

$$x = R \operatorname{ch} \xi \cos \eta, \quad y = R \operatorname{sh} \xi \sin \eta \tag{1.2}$$

where $\xi \geqslant 0$, $0 \leqslant \eta < 2\pi$ and $R = (c^2 - h^2)^{1/2}$, so that our ellipse becomes the coordinate line $\xi = \gamma = \operatorname{ch}^{-1}(c/R), 0 \leqslant \eta < 2\pi$. Since the presence of the cracks and the hole perturbs the uniform field, our solution must take the form

$$u_\xi = u_\eta = 0, \quad u_z = \frac{TR}{\mu}[\operatorname{sh} \xi \sin \eta + \phi(\xi, \eta)] \tag{1.3}$$

where $\phi(\xi, \eta)$ is a harmonic function. It follows at once that the only non-zero stresses are given by

$$\sigma_{\xi z} = \frac{T}{K}\left[\operatorname{ch} \xi \sin \eta + \frac{\partial \phi}{\partial \xi}\right] \tag{1.4}$$

and

$$\sigma_{\eta z} = \frac{T}{K}\left[\operatorname{sh} \xi \cos \eta + \frac{\partial \phi}{\partial \eta}\right] \tag{1.5}$$

where

$$K = (\operatorname{ch}^2 \xi - \cos^2 \eta)^{1/2} \tag{1.6}$$

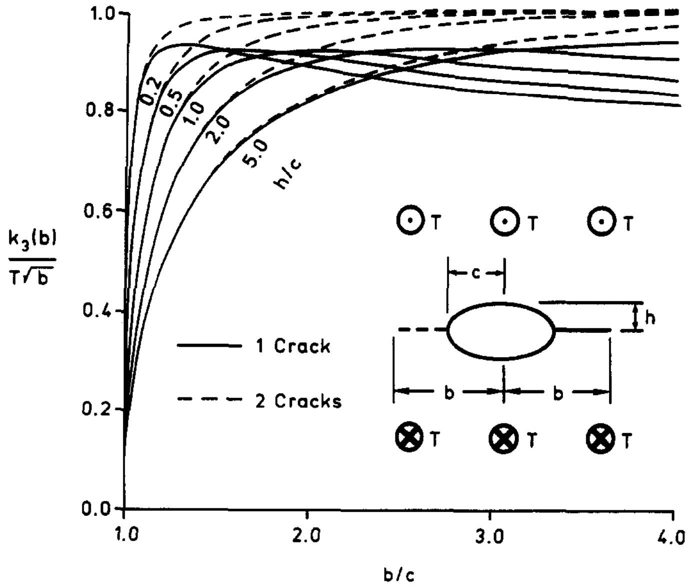

Fig. 1. The variation of $k_3(b)/T\sqrt{b}$ with b/c for various values of h/c.

2. STATEMENT AND SOLUTION OF PROBLEM 1

By symmetry problem 1 reduces to that of finding a harmonic function $\phi(\xi, \eta)$ in the strip $\gamma < \xi < \infty, 0 < \eta < \pi$ subject to the conditions

$$\begin{array}{lll} (1) & \phi(\xi, \eta) \to 0 & \text{as } \xi \to \infty \\ (2) & \phi(\xi, \pi) = 0 & \xi > \gamma \\ (3) & \dfrac{\partial \phi}{\partial \xi}(\gamma, \eta) = -\text{ch}\,\gamma \sin \eta & 0 < \eta < \pi \\ (4) & \lim\limits_{\xi \to \gamma^+} \dfrac{\partial \phi}{\partial \xi}(\xi, 0) = 0 & \\ & \dfrac{\partial \phi}{\partial \eta}(\xi, 0) = -\text{sh}\,\xi & \gamma < \xi < \beta \\ & \phi(\xi, 0) = 0 & \beta < \xi < \infty \end{array}$$

where $\beta = \text{ch}^{-1}(b/R)$

On introducing new variables $X = \xi - \gamma$, $Y = \eta$, $B = \beta - \gamma$ and $\psi(X, Y) = \phi(\xi, \eta)$ we obtain the equivalent problem

$$\begin{array}{llll} \text{P.D.E.} & & \dfrac{\partial^2 \psi}{\partial X^2} + \dfrac{\partial^2 \psi}{\partial Y^2} = 0 & 0 < X < \infty, 0 < Y < \pi \\ \text{B.C.} & (1) & \psi(X, Y) \to 0 & \text{as } X \to \infty \\ & (2) & \psi(X, \pi) = 0 & X > 0 \\ & (3) & \dfrac{\partial \psi}{\partial X}(0, Y) = -\text{ch}\,\gamma \sin Y & \text{as } 0 < Y < \pi \\ & (4) & \lim\limits_{X \to 0^+} \dfrac{\partial \psi}{\partial X}(X, 0) = 0 & \\ & & \dfrac{\partial \psi}{\partial Y}(X, 0) = -\text{sh}(X + \gamma) & 0 < X < B \\ & & \psi(X, 0) = 0 & B < X < \infty \end{array}$$

whose solution is clearly given by

$$\psi(X, Y) = \mathscr{F}_c\left[\rho^{-1}\Omega(\rho)\frac{\operatorname{sh}\rho(\pi - Y)}{\operatorname{sh}\rho\pi}; \rho \to X\right] + \operatorname{ch}\gamma e^{-X}\sin Y, \tag{2.1}$$

where $\mathscr{F}_c$ is the Fourier cosine transform

$$\mathscr{F}_c[f(\rho); \rho \to x] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(\rho)\cos\rho x \, \mathrm{d}\rho,$$

provided $\Omega(\rho)$ satisfies the dual integral equations

$$F(X) = \mathscr{F}_c[\Omega(\rho)\coth\pi\rho; \rho \to X] = e^\gamma \operatorname{ch} X \quad 0 < X < B$$

$$G(X) = \mathscr{F}_c[\rho^{-1}\Omega(\rho); \rho \to X] = 0 \qquad B < X < \infty$$

$$\lim_{X \to 0+}\frac{\mathrm{d}G(X)}{\mathrm{d}X} = 0. \tag{2.2}$$

Assuming a representation of the form

$$\Omega(\rho) = \sqrt{\frac{2}{\pi}}\int_0^B p(t)\sin\rho t \, \mathrm{d}t \tag{2.3}$$

we find that

$$F(X) = \frac{1}{\pi}\int_0^B \frac{\operatorname{sh} t \, p(t)\,\mathrm{d}t}{\operatorname{ch} t - \operatorname{ch} X} \tag{2.4}$$

and

$$G(X) = H(B - X)\int_X^B p(t)\,\mathrm{d}t \tag{2.5}$$

and hence that (2.3) satisfies the dual equations if $p(t)$ is given by the integral equation

$$\frac{1}{\pi}\int_0^B \frac{\operatorname{sh} t \, p(t)\,\mathrm{d}t}{\operatorname{ch}t - \operatorname{ch}X} = e^\gamma \operatorname{ch} X \quad 0 < X < B \tag{2.6}$$

with subsidiary condition

$$p(0) = 0. \tag{2.7}$$

A simple change of variable reduces (2.6) to the finite Hilbert Transform discussed by Tricomi[3], from which we obtain

$$p(t) = \frac{e^\gamma}{2}\left(\frac{\operatorname{ch} t - 1}{\operatorname{ch} B - \operatorname{ch} t}\right)^{1/2}\{2\operatorname{ch} t - \operatorname{ch} B + 1\}. \tag{2.8}$$

The stress intensity factor at the tip $(b, 0)$ is defined by the limit

$$k_3(b) = -\lim_{x \to b^-}\mu[2(b - x)]^{1/2}\frac{\partial u_z}{\partial x}(x, 0). \tag{2.9}$$

Therefore, if we let $k_0 = T\sqrt{b}$, it is readily seen that

$$\frac{k_3(b)}{k_0} = \lim_{\xi \to \beta^-}\left(\frac{2\{\operatorname{ch}\beta - \operatorname{ch}\xi\}}{\operatorname{ch}\beta}\right)^{1/2}\frac{1}{\operatorname{sh}\xi}p(\xi - \gamma). \tag{2.10}$$

After a little algebra we find, in agreement with Yokobori, Ichikawa, Konosu and Takahashi[1], that

$$\frac{k_3(b)}{k_0} = \frac{(c + h)(s + 1)}{2}\left(\frac{s^2 - 1}{bs\{(c + h)s^2 - c + h\}}\right)^{1/2} \tag{2.11}$$

where

$$s = \frac{b + (b^2 - c^2 + h^2)^{1/2}}{c + h}. \tag{2.12}$$

3. STATEMENT AND SOLUTION OF PROBLEM 2

Since the configuration to be investigated in problem 2 is symmetric with respect to both the x and y axes, we require to find a function $\phi(\xi, \eta)$ which is harmonic in the strip $\gamma < \xi < \infty, 0 < \eta < \pi/2$ and satisfies the conditions

$$\begin{array}{lll}
(1) & \phi(\xi,\eta)\to 0 & \text{as } \xi\to\infty \\
(2) & \dfrac{\partial\phi}{\partial\eta}\left(\xi,\dfrac{\pi}{2}\right)=0 & \xi>\gamma \\
(3) & \dfrac{\partial\phi}{\partial\xi}(\gamma,\eta)=-\text{ch}\,\gamma\sin\eta & 0<\eta<\dfrac{\pi}{2} \\
(4) & \lim\limits_{\xi\to\gamma^+}\dfrac{\partial\phi}{\partial\xi}(\xi,0)=0 & \\
 & \dfrac{\partial\phi}{\partial\eta}(\xi,0)=-\text{sh}\,\xi & \gamma<\xi<\beta \\
 & \phi(\xi,0)=0. & \beta<\xi<\infty
\end{array}$$

As in the previous case the change of variables $X = 2(\xi - \gamma)$, $Y = 2\eta$, $B = 2(\beta - \gamma)$ and $\psi(X, Y) = \phi(\xi, \eta)$ produces the equivalent problem

$$\begin{array}{llll}
\text{P.D.E.} & & \dfrac{\partial^2\psi}{\partial X^2}+\dfrac{\partial^2\psi}{\partial Y^2}=0 & 0<X<\infty, 0<Y<\pi \\
\text{B.C.} & (1) & \psi(X,Y)\to 0 & \text{as } X\to\infty \\
 & (2) & \dfrac{\partial\psi}{\partial Y}(X,\pi)=0 & X>0 \\
 & (3) & \dfrac{\partial\psi}{\partial X}(0,Y)=-\dfrac{1}{2}\text{ch}\,\gamma\sin\dfrac{Y}{2} & 0<Y<\pi \\
 & (4) & \lim\limits_{X\to 0+}\dfrac{\partial\psi}{\partial X}(X,0)=0 & \\
 & & \dfrac{\partial\psi}{\partial Y}(X,0)=-\dfrac{1}{2}\text{sh}\left(\dfrac{X}{2}+\gamma\right) & 0<X<B \\
 & & \psi(X,0)=0. & B<X<\infty
\end{array}$$

It is readily seen that this problem has solution

$$\psi(X, Y) = \mathscr{F}_c\left[\rho^{-1}\Omega(\rho)\frac{\text{ch}\,\rho(\pi - Y)}{\text{ch}\,\rho\pi}; \rho\to X\right] + e^{-X/2}\sin\frac{Y}{2}\text{ch}\,\gamma \tag{3.1}$$

provided $\Omega(\rho)$ satisfies the dual equations

$$\begin{array}{ll}
F(X)=\mathscr{F}_c[\Omega(\rho)\text{th}\,\rho\pi;X]=\dfrac{1}{2}e^{\gamma}\,\text{ch}\,\dfrac{X}{2} & 0<X<B \\
G(X)=\mathscr{F}_c[\rho^{-1}\Omega(\rho);X]=0 & B<X<\infty
\end{array}$$

$$\lim_{X\to 0+}\frac{\mathrm{d}G}{\mathrm{d}X}=0. \tag{3.2}$$

Again we assume a representation of the form

$$\Omega(\rho) = \sqrt{\frac{2}{\pi}}\int_0^B p(t)\sin\rho t\,\mathrm{d}t \tag{3.3}$$

which yields

$$F(X)=\frac{1}{\pi}\int_0^B \frac{\operatorname{sh}\frac{t}{2}\operatorname{ch}\frac{X}{2}}{\operatorname{sh}^2\frac{t}{2}-\operatorname{sh}^2\frac{X}{2}}p(t)\,\mathrm{d}t \tag{3.4}$$

and

$$G(X)=H(B-X)\int_X^B p(t)\,\mathrm{d}t \tag{3.5}$$

and hence requires that $p(t)$ satisfy the singular integral equation

$$\frac{1}{\pi}\int_0^B \frac{\operatorname{sh}\frac{t}{2}p(t)\,\mathrm{d}t}{\operatorname{sh}^2\frac{t}{2}-\operatorname{sh}^2\frac{X}{2}}=\frac{1}{2}e^{\gamma},\quad 0<X<B \tag{3.6}$$

with subsidiary condition

$$p(0)=0. \tag{3.7}$$

Once more a simple change of variables facilitates the use of the finite Hilbert Transform and leads to the result

$$p(t)=\frac{\frac{1}{2}e^{\gamma}\operatorname{sh}\frac{t}{2}\operatorname{ch}\frac{t}{2}}{\left(\operatorname{sh}^2\frac{B}{2}-\operatorname{sh}^2\frac{t}{2}\right)^{1/2}}. \tag{3.8}$$

This time we find that

$$\frac{k_3(b)}{k_0}=-\lim_{\xi\to\beta^-}\left(\frac{2(\operatorname{ch}\beta-\operatorname{ch}\xi)}{\operatorname{ch}\beta}\right)^{1/2}\frac{1}{\operatorname{sh}\xi}\frac{\partial\phi}{\partial\xi}(\xi,0) \tag{3.9}$$

and hence, in agreement with Yokobori, Kamei and Konosu[2], that

$$\frac{k_3(b)}{k_0}=(c+h)\left(\frac{(s^4-1)}{2bs\{(c+h)s^2-(c-h)\}}\right)^{1/2} \tag{3.10}$$

where s is given by (2.12).

4. CONCLUSION

The results of Sections 2 and 3 are presented graphically in Fig. 1 which shows the variation of $k_3(b)/T\sqrt{b}$ with b/c for several values of h/c. In both the one and two crack cases we find that if $b-c\ll c$ then $k_3(b)\sim\tau\sqrt{b-c}$ where $\tau=T(1+c/h)=\sigma_{yz}(c,0)$ is the stress at the edge of the elliptic hole in the absence of cracks. If $b\gg c$ we find, in the one crack case that $k_3(b)\sim T\sqrt{(b+c)/2}$ and, in the two crack base that $k_3(b)=T\sqrt{b}$. Lastly if $h=0$ we find, as expected, that $k_3(b)=T\sqrt{(b+c)/2}$ in the one crack case and $k_3(b)=T\sqrt{b}$ in the two crack case.

REFERENCES

[1] T. Yokobori, M. Ichikawa, S. Konosu and R. Takahashi, Report of Res. Inst. for Strength and Fracture of Materials, Tohoku University, Vol. 8, p. 1 (1972).
[2] T. Yokobori, A. Kamei and S. Konosu, Report of Res. Inst. for Strength and Fracture of Materials. Tohoku University, Vol. 7, p. 57 (1971).
[3] F. G. Tricomi, *Q. Jl Math.* **2,** 199 (1951).

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