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Fundamental frequency of a circular membrane with a strip of small length

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Abstract. A circular membrane with an arbitrarily placed internal strip of small length is concerned in this article. A two-term asymptotic expansion for the fundamental frequency of the membrane, as the length of the strip approaching to zero, is specified. Comparing it with the one [8] derived for the membrane with an internal circular core, it is found that the position of the internal constraint has more effect than the shape of the internal constraint on the fundamental frequency. The asymptotic approximation is also compared with the numerical data computed by the dual boundary element method [2] for a circular membrane of radius 1 with a radially placed internal strip of length $2c$. These two sets of data are in good agreement. The relative error is less than 3% as c is less than or equal to 0.1, for all positions of the strip. Moreover, the relative error is less than 1% as c is less than or equal to 0.01.

Keywords. Asymptotic approximation, fundamental frequency, strip, circular membrane.

1. Introduction

Vibration of a fixed membrane is important in the theory of sound [5]. The resulting eigenvalue problem of negative Laplace operator with Dirichlet boundary conditions also represents the propagation of TM modes in an electromagnetic waveguide [3].

Related literature on the vibration of a circular membrane with internal line constraints are few. Veselov & Gaydar [6] considered a circular waveguide with a central cross-shaped conductor, Wang [7] investigated on a circular membrane with radial constraints on the boundary, Chen *et al.* [2] studied on a membrane with internal stringers, and Yu & Wang [10] computed the fundamental frequency of a circular membrane with a centered strip. To the author's best knowledge, an asymptotic case for a circular membrane with an arbitrarily placed internal strip of small length has not been dealt explicitly in the literature.

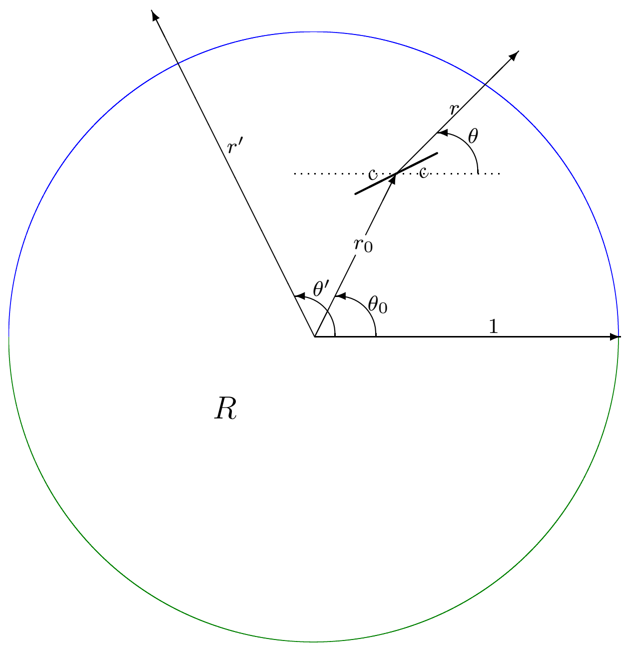

Figure 1. A circular membrane R of radius 1 with an internal strip of small length $2c$ ($c \ll 1$) centered at (r_0, θ_0).

2. Asymptotic formulation

Consider a circular membrane R of radius 1 with an arbitrarily placed internal strip of small length $2c$, $c \ll 1$, as sketched in Figure 1. The governing equation is

$$\triangle U + K^2 U = 0 \ , \tag{1}$$

where all lengths have been normalized by the radius L of the membrane, U is the normalized vertical displacement (mode shape) and K is the normalized vibrational frequency, $K = \textit{principal frequency} \cdot L \cdot \sqrt{\textit{density/tension per length}}$. The boundary conditions are that $U = 0$ on all the boundaries.

For the fundamental frequency (the smallest normalized vibrational frequency) K of the membrane R, an implicit asymptotic formula ($c \to 0$) (equation 3.87) [9] was derived by author,

$$K = K_0 + \left(\frac{\pi B_0^2}{K_0 \displaystyle\int_{R_0} U_0^2 dA} \right) \frac{1}{|\ln c|} + \cdots \ , \tag{2}$$

where R_0 is the circular membrane without the internal strip, K_0 is the fundamental frequency of the membrane R_0, and U_0 is its corresponding mode shape assumed to be

$$U_0(r,\theta) = B_0 J_0(K_0 r) + \sum_{m=1}^{\infty} J_m(K_0 r)\left(A_m \sin(m\theta) + B_m \cos(m\theta)\right) , \tag{3}$$

J_n is the n^{th} order *Bessel* function.

A circular membrane enclosed by $r' = 1$ has [3, 4] the fundamental frequency $K_0 \approx 2.4048$ which is the *first* zero of J_0 and the corresponding mode shape $U_0 = J_0(K_0 r')$. By a translational addition theorem for circular cylindrical wave functions [4],

$$J_0(K_0 r') = \sum_{l=-\infty}^{\infty} J_l(K_0 r_0) J_l(K_0 r) \cos\left(l\theta - l(\theta_0 + \pi)\right) , \tag{4}$$

where (r_0, θ_0) describes the position of the strip as shown in Figure 1, and an identity [1],

$$J_{-m}(z) = (-1)^m J_m(z),\ m = 1, 2, 3, \dots , \tag{5}$$

the mode shape U_0 can be expressed as

$$\begin{aligned} U_0 &= J_0(K_0 r_0) J_0(K_0 r) \\ &+ \sum_{m=1}^{\infty} \left[(-1)^m 2 J_m(K_0 r_0) \sin(m\theta_0)\right] J_m(K_0 r) \sin(m\theta) \\ &+ \sum_{m=1}^{\infty} \left[(-1)^m 2 J_m(K_0 r_0) \cos(m\theta_0)\right] J_m(K_0 r) \cos(m\theta) . \end{aligned} \tag{6}$$

Also, an integral of products of *Bessel* functions [1],

$$\int_0^z t J_0^2(t)\ dt = \frac{z^2}{2}\left[J_0^2(z) + J_1^2(z)\right] , \tag{7}$$

gives the value $\pi J_1^2(K_0)$ to the double integral in (2).

Thus, a two-term asymptotic expansion ($c \to 0$) for the fundamental frequency K of a circular membrane of radius 1 with an arbitrarily placed internal strip of small length $2c$ is determined explicitly as

$$K = K_0 + \left(\frac{J_0^2(K_0 r_0)}{K_0 J_1^2(K_0)}\right) \frac{1}{|\ln c|} + \cdots , \tag{8}$$

where r_0 is the distance between the center of the strip and that of the circular membrane, $K_0 \approx 2.4048$ is the *first* zero of J_0, and J_n is the n^{th} order *Bessel* function, $n = 0, 1$.

3. Results and discussion

A two-term asymptotic expansion for the fundamental frequency K of a circular membrane of radius 1 with an arbitrarily placed internal strip of small length $2c$ is determined explicitly as (8). Comparing it with the one (equation(109)) [8] derived for the membrane with an internal circular core of radius c, it is found that the position of the internal constraint has more effect than the shape of the internal constraint on the fundamental frequency. Moreover, the fundamental frequency is decreasing as the internal constraint is moving from the center of the membrane toward the boundary of the membrane.

Comparing the asymptotic approximation (8) with the numerical result obtained in the author's previous work [10] for a circular membrane with a centered strip, a good agreement is found. It is shown in Table 1. The asymptotic approximations (8) for various c are also compared with the numerical data computed by the dual boundary element method [2] for a circular membrane of radius 1 with a radially placed internal strip of length $2c$ centered at distance r_0 from the center of the circular membrane. The comparisons are shown in Figure 2. These two sets of data are in good agreement, as verified by the relative percentage of error reported in Table 2 and Table 3. The relative error is less than 3% as c is less than or equal to 0.1, for all positions of the strip. Moreover, the relative error is less than 1% as c is less than or equal to 0.01.

c	0.1	0.01	0.001	10^{-4}	10^{-5}	10^{-6}
analytical (8)	3.075	2.740	2.628	2.572	2.539	2.517
Yu & Wang	3.061	2.741	2.629	2.573	2.539	2.517
error(%)	0.46	0.04	0.04	0.04	0	0

Table 1. The fundamental frequency K for a circular membrane of radius 1 with a centered strip of length $2c$.

r_0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
d.b.e.m.	3.063	3.025	2.937	2.835	2.737	2.650	2.575	2.514	2.466	2.431
analytical (8)	3.075	3.056	3.001	2.917	2.814	2.706	2.603	2.516	2.453	2.416
error(%)	0.39	1.02	2.18	2.89	2.81	2.11	1.09	0.08	0.53	0.62

Table 2. The fundamental frequency K for a circular membrane of radius 1 with a radially placed internal strip of length $2c$ centered at distance r_0 from the center of the circular membrane, when $c = 0.1$.

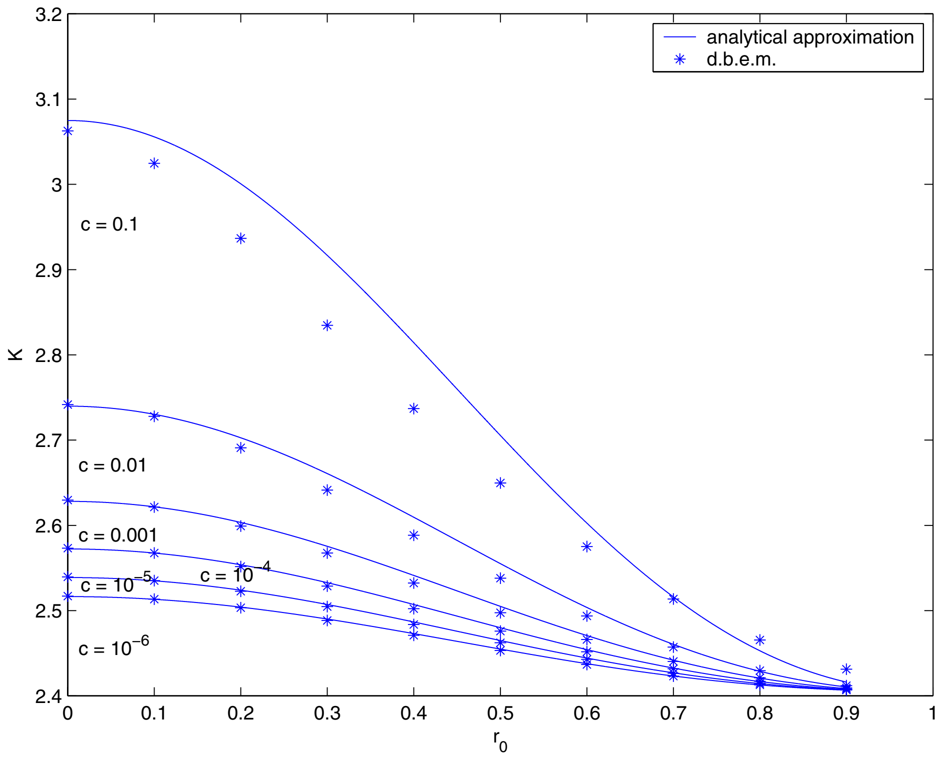

Figure 2. Comparisons between analytical approximation, (8), and numerical data (d.b.e.m.) for various c.

r_0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
d.b.e.m.	2.742	2.728	2.691	2.642	2.589	2.538	2.494	2.458	2.430	2.412
analytical (8)	2.740	2.730	2.703	2.661	2.610	2.555	2.504	2.461	2.429	2.411
error(%)	0.07	0.07	0.45	0.72	0.81	0.67	0.40	0.12	0.04	0.04

Table 3. The fundamental frequency K for a circular membrane of radius 1 with a radially placed internal strip of length $2c$ centered at distance r_0 from the center of the circular membrane, when $c = 0.01$.

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References

[1] M. Abramowitz and I. A. Stegun (eds), *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, Dover, New York 1965.

[2] J. T. Chen and M. T. Liang and et al, Dual boundary element analysis of wave scattering from singularities, *Wave Motion* **30** (1999), 367–381.

[3] J. R. Kuttler and V. G. Sigillito, Eigenvalues of the laplacian in two dimensions, *Siam Review* **26**(2) (1984), 163–193.

[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics 1*, McGraw-Hill, New York 1953.

[5] Lord Rayleigh, *The Theory of Sound 1*, Dover, New York 1945.

[6] G. I. Veselov and V. I. Gaydar, Analysis of a circular waveguide with an internal cross-shaped conductor, *Radio Eng.*, **25**(11) (1970), 147–149.

[7] C. Y. Wang, Fundamental modes of a circular membrane with radial constraints on the boundary, *J. Sound Vib.*, **220**(3) (1999), 559–563.

[8] L. H. Yu, Fundamental frequency of a doubly connected membrane: a modified perturbation method, *IMA J. Appl. Math.* **68**(4) (2003), 329–354.

[9] L. H. Yu, The Lowest Eigenvalue of the Negative Laplacian in Two Dimensions: A Modified Perturbation Method, (Ph.D Dissertation) Michigan State University, U.S.A. 2001.

[10] L. H. Yu and C. Y. Wang, Fundamental frequencies of a circular membrane with a centered strip, *J. Sound Vib.*, **239**(2) (2001), 363–368.

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