Applied Mathematics Letters 92 (2019) 99–107

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Revisit of the degenerate scale for an infinite plane problem containing two circular holes using conformal mapping

Shyh-Rong Kuo [a], Shing-Kai Kao [a], Yi-Ling Huang [a], Jeng-Tzong Chen [a,b,c,d,*]

[a] Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, 20224, Taiwan
[b] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung, 20224, Taiwan
[c] Department of Civil Engineering, National Cheng Kung University, Tainan, 70101, Taiwan
[d] Center of Excellence for Ocean Engineering, National Taiwan Ocean University, Keelung, 20224, Taiwan

A R T I C L E I N F O

Article history:
Received 21 September 2018
Received in revised form 26 November 2018
Accepted 26 November 2018
Available online 7 December 2018

Keywords:
Degenerate scale
Exterior problems
Complex variables
Conformal mapping

A B S T R A C T

It is well known that a degenerate scale results in a non-uniqueness solution in the BEM/BIEM. Study on the degenerate scale mainly focused on interior problems. Exterior problems were rarely discussed. In this paper, we revisited the problem of an infinite plane with two identical circular holes by using the complex variables instead of using the degenerate kernel. The domain was mapped to an annulus and the points at infinity were mapped to a pole through the conformal mapping. A boundary value problem was transformed into a Green's function. Hence, we needed to consider the pole's influence to the field. The complex variables provide us another way to solve these problems and it was easier than the degenerate kernel to understand. The reason why we use the conformal mapping is that the degenerate kernel may not be available and cannot be employed to derive the degenerate scale of general geometric shapes. Finally, we analytically derived the degenerate scale and compare the present result with that of the degenerate kernel. The equivalence is proved.

1. Introduction

The boundary element method (BEM) or the boundary integral equation method (BIEM) is efficient and accurate to solve the two-dimensional Laplace equation. However, it may result in a non-uniqueness solution for the two-dimensional Dirichlet problem at some critical size. This special domain is called the degenerate scale. Fig. 1 shows six kinds of the degenerate scale that we had analytically studied including circle [1], ellipse [2,3], regular N-gons [4] and infinite domain containing two circular cavities [5]. When the size of the boundary is the degenerate scale (solid line), the influence matrix of the BEM/BIEM is rank deficient.

* Corresponding author at: Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, 20224, Taiwan.
E-mail address: jtchen@mail.ntou.edu.tw (J.-T. Chen).

https://doi.org/10.1016/j.aml.2018.11.023
0893-9659/

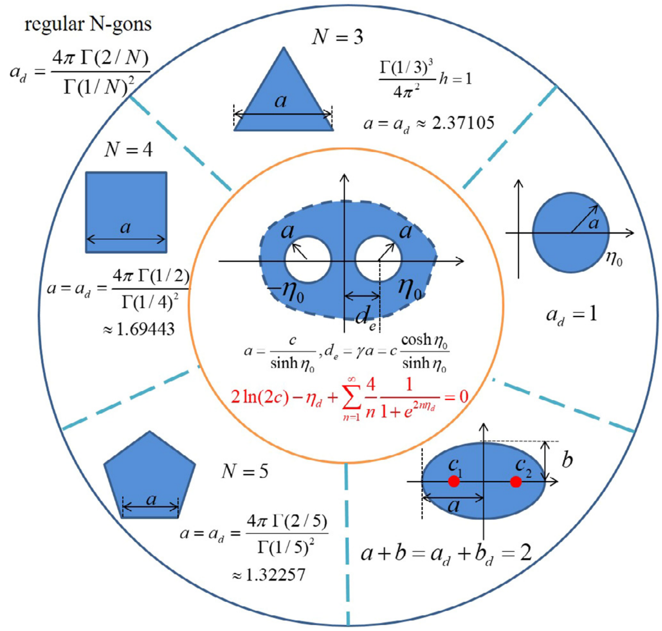


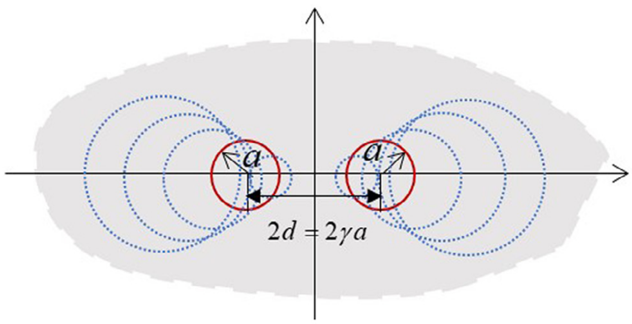
Fig. 1. Six cases of degenerate scale.

But the ordinary scale (dotted line) is full rank. It is shown in Fig. 2 including ordinary and degenerate scales where $\gamma = 2$. From the viewpoint of mathematics, there are two ways to understand the degenerate scale. One is the non-uniqueness solution in the BIEM/BEM. The other is the unit logarithmic capacity corresponding to the conformal radius in the complex analysis [6]. The definition of the logarithmic capacity is given in [7]. Many researchers paid attention to this issue in recent years [8,9]. The linkage between the degenerate scale and the logarithmic capacity was discussed in [10–12].

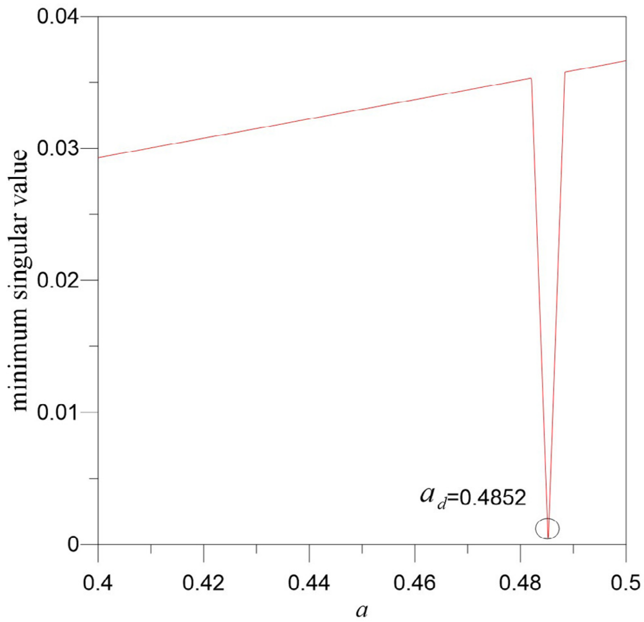
The theory of complex variables is an analytical tool for two-dimensional problems. Chao et al. [13] employed the method of analytical continuation and the conformal mapping to obtain the field solution for the temperature and stresses in the compact complex form. Rumely [14] used the conformal mapping to derive the logarithmic capacity of many shapes, such as the circle, the ellipse, and so on. Kuo et al. [4] used the Riemann conformal mapping (Schwarz–Christoffel transformation) to study the regular N-gon domains. The degenerate scale of the method agrees well with the numerical results. In 2013, Kuo et al. [15] employed the Riemann conformal mapping to connect the unit logarithmic capacity and the degenerate scale.

However, the degenerate scale for the exterior problem is rarely discussed. Rumely used elliptic functions to obtain the unit logarithmic capacity for an infinite plane with two identical circular holes [14]. Corfdir and Bonnet [16] studied the Laplace problem of degenerate scale for a half-plane domain. They claimed that the degenerate scale depends on the type of the boundary condition on the line bounding the half-plane. Later, Chen [17] employed a null-field BIEM to study the same problem. Numerical results [17] also support the finding in [16]. In these two papers [16,17], they both employed the image method to construct the corresponding Green's function. The boundary condition on the line bounding the half-plane can be satisfied in advance by using the Green's function in their BEM formulations. In this way, the degenerate scale is free for the Dirichlet condition on the line bounding the half-plane. Chen et al. [5] found the analytical formula of the degenerate scale of the infinite domain containing two equal circular holes by using the degenerate kernel.

In this paper, we would construct a complex variables system to revisit the problem. We analytically rederive the degenerate scale through the conformal mapping of complex variables instead of using the



(a) Ordinary (dotted line) and degenerate scales (solid line).



(b) Rank deficiency of the influence matrix for a degenerate scale.

Fig. 2. Sketch of the degenerate scales ($\gamma = 2$) (a) Ordinary (dotted line) and degenerate scales (solid line) (b) Rank deficiency for a degenerate scale.

degenerate kernel in the bipolar coordinates. We compare with the present results with that of using degenerate kernel. The equivalence is examined.

2. Analytical derivation of the degenerate scale by using the conformal mapping

In 2013, Kuo et al. [15] employed the Riemann conformal mapping to link the degenerate scale in the BEM/BIEM and the logarithmic capacity in the theory of complex variables. Since the degenerate scale results in the range deficiency by a constant term, the potential is simplified to

$$\begin{aligned} &\Phi(x, y) = C_{\mathrm{T}}, (x, y) \in D_{in}, \\ &\Phi_{\Gamma}(\alpha, \beta) = C_{\mathrm{T}}, (\alpha, \beta) \in \Gamma, \end{aligned} \tag{1}$$

where $\Phi(x, y)$ is the potential and Γ is the boundary to separate the inner and outer domains as shown in Fig. 3. When the size of the domain is a degenerate scale, $C_{\mathrm{T}} = 0$. In this paper, we would use the conformal mapping to revisit the degenerate scale in the exterior problem. The bilinear transformation, w, is given

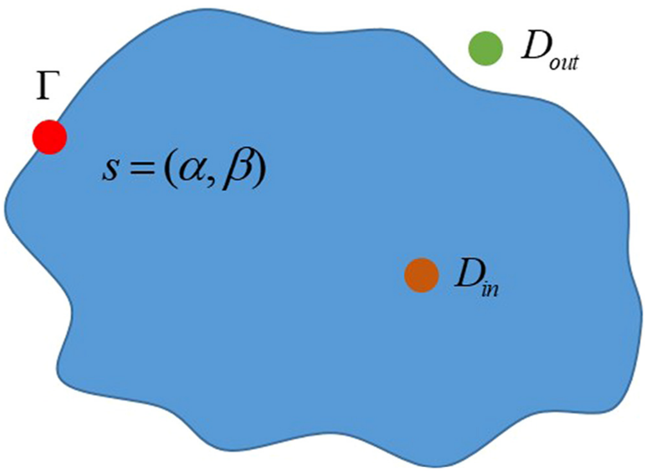


Fig. 3. The figure sketch for the domains D_{in} and D_{out} separated by Γ.

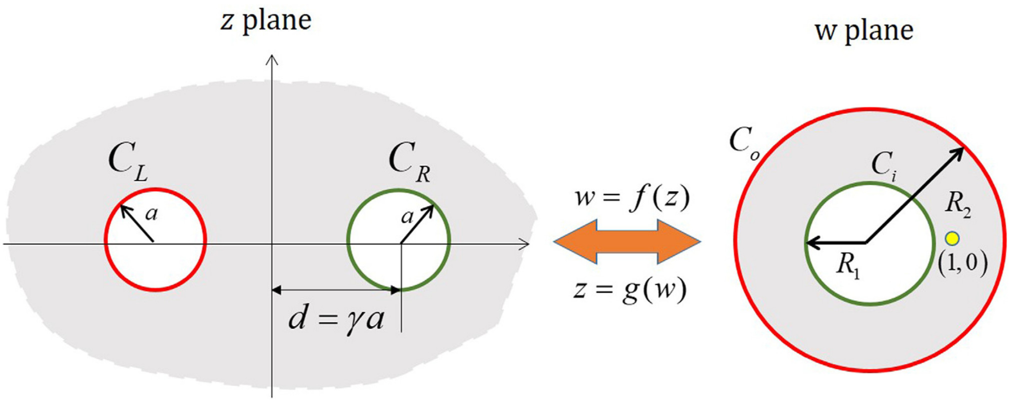


Fig. 4. Mapping of two-circular cavities in infinite domain to annulus.

below

$$w = f(z) = \frac{z - a\sqrt{\gamma^2 - 1}}{z + a\sqrt{\gamma^2 - 1}}, \tag{2}$$

where a is the radius of two circular holes and γ is a geometry parameter as shown in Fig. 4. Eq. (2) is also called a linear fractional transformation or Möbius transformation. The inverse transformation of Eq. (2) is

$$z = g(w) = a\sqrt{\gamma^2 - 1}\left(\frac{1 + w}{1 - w}\right). \tag{3}$$

By using the bilinear transformation, an infinite plane with two identical circular holes is mapped to an annulus. The right circle is defined by

$$z = a\gamma + ae^{i\theta}, z \in C_R. \tag{4}$$

By substituting Eq. (4) into Eq. (2), we have

$$w = \frac{a\gamma - a\sqrt{\gamma^2 - 1} + ae^{i\theta}}{a\gamma + a\sqrt{\gamma^2 - 1} + ae^{i\theta}} = \left(\gamma - \sqrt{\gamma^2 - 1}\right) e^{i\phi} = be^{i\phi}, \tag{5}$$

where b is the radius of the inner circle in the w plane and

$$b = \gamma - \sqrt{\gamma^2 - 1}, \tag{6}$$

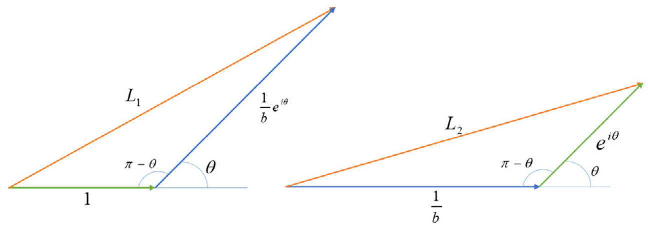


Fig. 5. Sketch of the relation between $|L_1|$ and $|L_2|$.

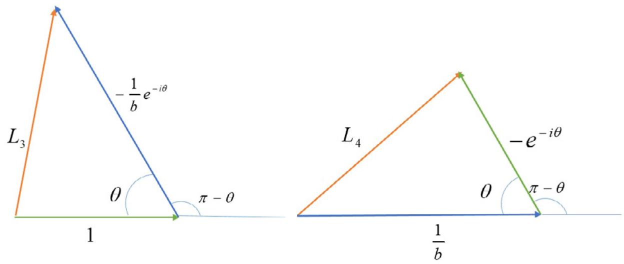


Fig. 6. Sketch of the relation between $|L_3|$ and $|L_4|$.

and

$$e^{i\phi}=\frac{1+\left(\gamma+\sqrt{\gamma^2-1}\right)e^{i\theta}}{\left(\gamma+\sqrt{\gamma^2-1}\right)+e^{i\theta}}=\frac{L_1}{L_2}, \tag{7}$$

in which L_1 and L_2 are shown in Fig. 5. Based on the congruence in the geometry, $|L_1|$ is equal to $|L_2|$. It represents that $\left|e^{i\phi}\right|=1$. The left circle is defined by

$$z=-a\gamma+ae^{i\theta}, z\in C_L \tag{8}$$

By substituting Eq. (8) into Eq. (2), we have

$$w=\frac{-a\gamma-a\sqrt{\gamma^2-1}+ae^{i\theta}}{-a\gamma+a\sqrt{\gamma^2-1}+ae^{i\theta}}=\left(\gamma+\sqrt{\gamma^2-1}\right)e^{i\psi}=\frac{1}{b}e^{i\psi}, \tag{9}$$

where $1/b$ is the radius of the outer circle in the w plane and

$$e^{i\psi}=\frac{1-\left(\gamma+\sqrt{\gamma^2-1}\right)e^{-i\theta}}{\left(\gamma+\sqrt{\gamma^2-1}\right)-e^{-i\theta}}=\frac{L_3}{L_4}, \tag{10}$$

in which L_3 and L_4 are shown in Fig. 6. Based on the congruence in the geometry, $|L_3|$ is equal to $|L_4|$. It represents that $\left|e^{i\psi}\right|=1$. A dictionary [18] of conformal mapping can be consulted. When $|z|\to\infty$, $w=1$, it results in a pole in the annulus domain Ω. We consider the pole in $(1,0)$, then we have

$$\Phi(w)|_{w=(1,0)}\approx\lim_{|z|\to\infty}\ln(z)=\ln\left(g(w)\right)=\ln 2a\left(\sqrt{\gamma^2-1}\right)-\ln\left(1-w\right), |z|\to\infty, w=(1,0), \tag{11}$$

and the Taylor series of $\ln(1-w)$ yields

$$\ln(1-w)=-\sum_{n=1}^{\infty}\frac{w^n}{n}=-\sum_{n=1}^{\infty}\frac{b^n e^{in\phi}}{n}, |w|<1, \tag{12}$$

$$\ln(1-w) = \ln(-w) + \ln(1-\frac{1}{w}) = \pi i + \ln(w) - \sum_{n=1}^{\infty} \frac{w^{-n}}{n} = i\,(\pi+\psi) \\ - \ln(b) - \sum_{n=1}^{\infty} \frac{b^n e^{-in\psi}}{n}, |w| > 1. \tag{13}$$

By using the Taylor series of the fundamental solution, $\ln r$ where r is the distance between source and field points, the solution representations inside and outside the circular boundary are expressed as

$$\Phi(w) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n w^n,\ \ |w| < \frac{1}{b}. \tag{14}$$

and

$$\Phi(w) = \beta_0 \ln w + \sum_{n=1}^{\infty} \beta_n w^{-n},\ \ |w| > b. \tag{15}$$

The series in terms of the bases (w^n and w^{-n}) are convergent in the corresponding region. According to Eqs. (11), (14) and (15), the solution representation of the annulus domain can be expressed as

$$\Phi(w) = \ln 2a\left(\sqrt{\gamma^2-1}\right) - \ln(1-w) + \sum_{n=1}^{\infty}\left(\alpha_n w^n + \beta_n w^{-n}\right) + \alpha_0 + \beta_0 \ln w,\ w \in \Omega. \tag{16}$$

By substituting Eq. (11) into Eq. (16), we have

$$\sum_{n=1}^{\infty}\left(\alpha_n w^n + \beta_n w^{-n}\right) + \alpha_0 = 0, w = (1,0). \tag{17}$$

Therefore, we could easily find that

$$\alpha_0 = -\sum_{n=1}^{\infty}\left(\alpha_n + \beta_n\right). \tag{18}$$

By substituting Eq. (12) into Eq. (16), we have

$$\Phi_i(w) = \ln\left(2a\sqrt{\gamma^2-1}\right) + \sum_{n=1}^{\infty}\left(\alpha_n b^n e^{in\phi} + \beta_n b^{-n} e^{-in\phi} + \frac{b^n e^{in\phi}}{n}\right) + \beta_0 \ln\left(be^{i\phi}\right) + \alpha_0,\ w \in C_i. \tag{19}$$

By substituting Eq. (13) into Eq. (16), we have

$$\Phi_o(w) = \ln\left(2a\sqrt{\gamma^2-1}\right) - i\pi + (1-\beta_0)(\ln b - i\psi) \\ + \sum_{n=1}^{\infty}\left(\alpha_n b^{-n} e^{in\psi} + \beta_n b^n e^{-in\psi} + \frac{1}{n} b^n e^{-in\psi}\right) + \alpha_0,\ w \in C_o. \tag{20}$$

If a is a degenerate scale, we have

$$\mathrm{Re}\ (\Phi_i(w)) = C_{\mathrm{T}} = 0, w \in C_i, \tag{21}$$

$$\mathrm{Re}\ (\Phi_o(w)) = C_{\mathrm{T}} = 0, w \in C_o. \tag{22}$$

Since $\mathrm{Re}\,(\Phi_i(w)) = \mathrm{Re}\,(\Phi_o(w)) = C_{\mathrm{T}}$, we have

$$\ln\left(2a\sqrt{\gamma^2-1}\right) + \beta_0 \ln b + \alpha_0 = \ln\left(2a\sqrt{\gamma^2-1}\right) + (1-\beta_0)\ln b + \alpha_0 \tag{23}$$

and

$$\sum_{n=1}^{\infty}\left(\alpha_n b^n \cos n\phi + \beta_n b^{-n} \cos n\phi + \frac{1}{n} b^n \cos n\phi\right) = 0, \tag{24}$$

$$\sum_{n=1}^{\infty}\left(\alpha_n b^{-n}\cos n\psi+\beta_n b^{n}\cos n\psi+\frac{1}{n}b^{n}\cos n\psi\right)=0. \tag{25}$$

After simplification, Eq. (23) could be reduced to

$$\beta_0=\frac{1}{2}. \tag{26}$$

By solving Eqs. (24) and (25), we have

$$\alpha_n=\beta_n=\left(\frac{-1}{n}\right)\frac{b^{2n}}{1+b^{2n}}=H_n. \tag{27}$$

By substituting Eq. (27) into Eq. (18), we have

$$\alpha_0=-2\sum_{n=1}^{\infty}H_n. \tag{28}$$

According to the above coefficients, α_0, α_n, β_0 and β_n, Eqs. (19) and (20) could be reduced as

$$\operatorname{Re}\left(\Phi_i\left(w\right)\right)=\ln\left(2a\sqrt{\gamma^2-1}\right)-2\sum_{n=1}^{\infty}H_n+\frac{1}{2}\ln\left(b\right),\ w\in C_i, \tag{29}$$

$$\operatorname{Re}\left(\Phi_o\left(w\right)\right)=-2\sum_{n=1}^{\infty}H_n+\frac{1}{2}\ln\left(b\right)+\ln\left(2a\sqrt{\gamma^2-1}\right),\ w\in C_o,\text{respectively.} \tag{30}$$

Therefore, the analytical formula of degenerate scale for the infinite plane with two identical circular boundaries could be shown as

$$\ln\left(2a\sqrt{\gamma^2-1}\right)+\frac{1}{2}\ln\left(b\right)-2\sum_{n=1}^{\infty}H_n=0. \tag{31}$$

Chen et al. [5] expressed the kernel function in terms of the bipolar coordinates (η,ξ) to analytically study the degenerate scale of the BIE. The relationship between the Cartesian coordinates $(x_1,\, x_2)$ and the bipolar coordinates $(\eta_x,\, \xi_x)$ is

$$x_1=c\frac{\sinh\eta_x}{\cosh\eta_x-\cos\xi_x}, \tag{32}$$

$$x_2=c\frac{\sin\xi_x}{\cosh\eta_x-\cos\xi_x}, \tag{33}$$

where c is the half distance between the two foci of the bipolar coordinates. The analytical formula of degenerate scale was expressed as

$$2\ln(2c)-\eta_0+\sum_{n=1}^{\infty}\frac{4}{n}\left(\frac{e^{-2n\eta_0}}{1+e^{-2n\eta_0}}\right)=0. \tag{34}$$

The radius of the circle, a, in the bipolar coordinates is expressed by

$$a=\frac{c}{\sinh\eta_0},\eta_0>0. \tag{35}$$

In the bipolar coordinates, the distance between centers of the two circles is defined by

$$2\gamma a=2c\frac{\cosh\eta_0}{\sinh\eta_0}, \tag{36}$$

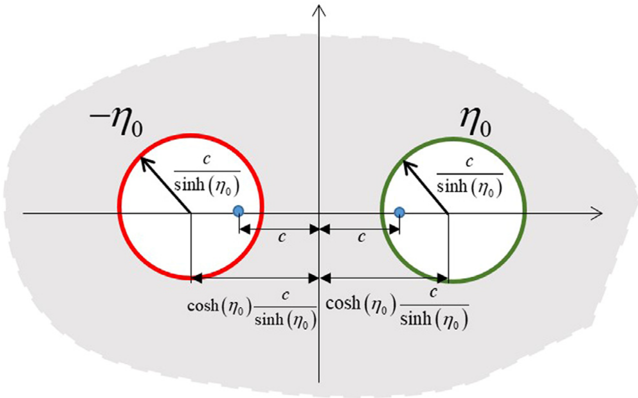


Fig. 7. An infinite plane containing two identical circular holes in the bipolar coordinates.

Table 1
Formula of degenerate scales derived by using different approaches.

Formula	Method			
	Bipolar coordinates degenerate kernel [5] (Analytical formula)	Elliptic functions [14] (Analytical formula)	Image method [16] (Asymptotic formula)	The present method conformal mapping (Analytical formula)
Degenerate scale	$2\ln(2c) - \eta_0 + \sum_{n=1}^{\infty} \frac{4}{n}\frac{e^{-2n\eta_0}}{1+e^{-2n\eta_0}} = 0$ where $a = \frac{c}{\sinh \eta_0}$ $2\gamma a = 2c\frac{\cosh \eta_0}{\sinh \eta_0}$ $\gamma = \cosh \eta_0$	$\frac{KC\sqrt{1-k^2}}{\lambda} = 1$ where $\lambda = \log((a+C)/R)$ $C = \sqrt{a^2 - R^2}$ $q = e^{2\pi i\tau}$ $\tau = \frac{i\pi}{\lambda}$ $k = \left[\sum_{n\in z} q^{\frac{1}{2}(n+\frac{1}{2})^2}\right]^2 / \left[\sum_{n\in z} q^{\frac{1}{2}n^2}\right]^2$ $K = \sum_{n=0}^{\infty}\left[((2n)!)^2/(2^n n!)^4\right]k^{2n}$	$a_d \approx \sqrt{1/2\gamma}$	$\ln\left(2a\sqrt{\gamma^2-1}\right) + \frac{1}{2}\ln(b) - 2\sum_{n=1}^{\infty} H_n = 0$ where $b = \gamma - \sqrt{\gamma^2-1}$ $H_n = -\frac{1}{n}\frac{b^{2n}}{1+b^{2n}}$

as shown in Fig. 7, i.e.

$$\gamma = \cosh \eta_0. \tag{37}$$

According to Eq. (37), we have

$$b = \gamma - \sqrt{\gamma^2 - 1} = \cosh \eta_0 - \sinh \eta_0 = e^{-\eta_0}. \tag{38}$$

Therefore, the coefficient H_n in Eq. (27) could be expressed as

$$H_n = -\frac{1}{n}\frac{b^{2n}}{1+b^{2n}} = -\frac{1}{n}\frac{e^{-2n\eta_0}}{1+e^{-2n\eta_0}}. \tag{39}$$

By substituting Eqs. (35) and (37)–(39) into Eq. (31), the analytical formula of Eq. (31) could be rewritten as

$$\ln(2c) + \frac{1}{2}\ln\left(e^{-\eta_0}\right) + \sum_{n=1}^{\infty}\frac{2}{n}\frac{e^{-2n\eta_0}}{1+e^{-2n\eta_0}} = 0. \tag{40}$$

Therefore, it is proved that Eqs. (31) and (34) are equivalent. The comparison for degenerate scales derived by Rumely [14], Corfdir & Bonnet [16] and the present approach is shown in Table 1.

3. Conclusions

This study focused on the degenerate scale problem of the infinite plane containing two identical circular holes. We used the complex variables to study the phenomenon of non-uniqueness solution. The analytical result matches well with that of using the degenerate kernel. It provides an alternative way by using the conformal mapping to analytically derive the degenerate scale of the problem. Comparatively speaking, the way of using complex variables does not have too much complicated mathematical formula than the way of using degenerate kernel. On the other hand, it is easier to understand than using the degenerate kernel from the beginner's viewpoint. Besides, we can extend to solve problems of more general geometric shapes that the degenerate kernel may not be available.

Acknowledgments

The authors wish to thank the financial support from the Ministry of Science and Technology, Taiwan under Grant No. MOST106-2221-E-019-009-MY3 and the support under Grant No. MOST107-2813-C-019-015-E.

References

[1] J.T. Chen, J.H. Lin, S.R. Kuo, Y.P. Chiu, Analytical study and numerical experiments for degenerate scale problems in boundary element method using degenerate kernels and circulants, Eng. Anal. Bound. Elem. 25 (2001) 819–828.
[2] J.T. Chen, S.R. Lin, K.H. Chen, Degenerate scale problem when solving Laplace's equation by BEM and its treatment, Internat. J. Numer. Methods Engrg. 62 (2005) 233–261.
[3] J.T. Chen, Y.T. Lee, S.R. Kuo, Y.W. Chen, Analytical derivation and numerical experiments of degenerate scale for an ellipse in BEM, Eng. Anal. Bound. Elem. 36 (2012) 1397–1405.
[4] S.R. Kuo, J.T. Chen, J.W. Lee, Y.W. Chen, Analytical derivation and numerical experiments of degenerate scale for regular N-gon domains in BEM, Appl. Math. Comput. 219 (2013) 5668–5683.
[5] J.T. Chen, S.K. Kao, J.W. Lee, Analytical derivation and numerical experiment of degenerate scale by using the degenerate kernel of the bipolar coordinates, Eng. Anal. Bound. Elem. 85 (2017) 70–86.
[6] Y.Z. Chen, X.Y. Lin, Z.X. Wang, Evaluation of the degenerate scale for BIE in plane elasticity and antiplane elasticity by using conformal mapping, Eng. Anal. Bound. Elem. 33 (2009) 147–158.
[7] E. Hille, Analytic Function Theory, vol. 2, Gin and Company, New York, 1962.
[8] R. Vodička, V. Mantič, On solvability of a boundary integral equation of the first kind for Dirichlet boundary value problems in plane elasticity, Comput. Mech. 41 (2008) 817–826.
[9] R. Vodička, M. Petrik, Degenerate scales for boundary value problems in anisotropic elasticity, Int. J. Solids Struct. 52 (2015) 209–219.
[10] I.H. Sloane, A. Spence, The Galerkin method for integral equations of the first kind with logarithmic kernel: Theory, IMA J. Numer. Anal. 8 (1988) 105–122.
[11] C. Constanda, Direct and Indirect Boundary Integral Equation Methods, Chapman & Hall/CRC, 2000.
[12] G. Hsiao, R.C. Maccamy, Solution of boundary value problems by integral equations of the first kind, SIAM Rev. 15 (4) (1973) 687–705.
[13] C.K. Chao, S.C. Tsai, C.H. Wang, Thermal stresses in an infinite body with a partially reinforced elliptic hole — a type of mixed boundary value problem, J. Chin. Inst. Eng. 22 (3) (1999) 285–292.
[14] R.S. Rumely, Capacity theory on algebraic curves, in: Lecture Notes in Mathematics, vol. 1378, Springer-Verlag, Berlin, 1989.
[15] S.R. Kuo, J.T. Chen, S.K. Kao, Linkage between the unit logarithmic capacity in the theory of complex variables and the degenerate scale in the BEM/BIEMs, Appl. Math. Lett. 26 (9) (2013) 929–938.
[16] A. Corfdir, G. Bonnet, Degenerate scale for the Laplace problem in the half plane; approximate logarithmic capacity for two distant boundaries, Eng. Anal. Bound. Elem. 37 (2014) 836–841.
[17] Y.Z. Chen, Numerical solution for the degenerate scale in 2D Laplace equation for notch in half-plane using null field BIE, Eng. Anal. Bound. Elem. 70 (2016) 126–133.
[18] H. Kober, Dictionary of Conformal Representations, Dover Publications, 1957.