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Boundary integral method applied to the propagation of non-linear gravity waves generated by a moving bottom

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Abstract

A numerical procedure is applied for the solution of the non-linear problem of propagation of waves generated in a homogeneous fluid, occupying an infinite channel, by the bounded motion of the bottom. For the sake of comparison, the analytical solution of the corresponding linearized problem is also given. The obtained results show that for sufficiently small amplitude of the bottom's motion, the predictions of the linear theory are in good agreement with those of the non-linear theory only in some starting time interval, this interval being longer for smaller amplitudes. In the course of time, a growing oscillatory divergence is found to exist between the two theories. This divergence increases significantly with the increase of the amplitude of the bottom's motion. Numerical results are presented and discussed. Unlike results of other publications, the numerical scheme given here proves numerical stability for the considered cases.

Keywords: Ideal fluids; Moving bottom; Boundary integral method; Numerical solution; Linear and nonlinear considerations

1. Introduction

The main difficulties, facing the mathematical treatment of the problem of gravity wave propagation, arise from the non-linearity of the governing system of equations, and also from the fact that the free surface is a priori unknown. Another difficulty may arise from the geometrical complexity of the domain occupied by the fluid. Different theoretical approaches were proposed to overcome these difficulties such as: the linear theory of motion and the theories of higher orders [1], the theory of long waves [2] and the various versions of the shallow water theory [3,4] and the method of fundamental solutions ([5] and references included therein).

The linear theory of motion and the theories of higher orders are inadequate for the description of the propagation of waves with big relative wave lengths.

The theory of long waves replaces the system of equations and conditions of the problem by another approximating system [2]. This theory does not propose a suitable scheme to be followed for the eventual

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improvement of the obtained approximate solutions. This is a capital inconvenience of the theory of long waves.

The different versions of the shallow-water theory are adequate only for the slow time variation phenomena and assume certain physical and geometrical constraints on the theoretical models, such as the smallness of the ratio of the vertical to the horizontal extents of the motion. Further, the waves are assumed to propagate in one and the same direction, otherwise major mathematical difficulties would arise [6,7].

The method of fundamental solutions (MFS) is a technique for the numerical solution of the problem. This method expresses the solution for the velocity potential as a linear combination of fundamental solutions of Laplace's equation with singularities lying outside the domain occupied by the fluid. The corresponding coefficients and the locations of the singularities are then chosen so as to fit the boundary conditions in the sense of least squares approximation. The main disadvantage of the method is that the arising problems may sometimes be unstable and ill-conditioned [8,9]. However, these disadvantages were partially remedied in recent publications due to Chen et al. [10,11] where the authors could locate the sources on the real boundary.

Abou-Dina and Helal [12,13] proposed a boundary integral method for the numerical solution of problems of non-linear wave propagation in fluids with free boundaries. An earlier version of the method designed for linear problems with fixed boundaries has proved to be efficient in treating two-dimensional problems of the theory of electromagnetic current sheets [14] and of the theories of elasticity, thermo-elasticity and thermo-magnetoelasticity [15–17]. The basic idea underlying this technique relies on the transformation of the basic field equations and boundary and initial conditions of the problem into a set of boundary integro-differential equations satisfied by the velocity potential and the free surface elevation. The solution of the latter provides the free surface elevation at any time moment and is also used for the determination of the solution in the bulk. This procedure has the advantage of using the actual boundary (although partially unknown) of the medium and is time saving during the computations.

In the present work, we investigate the linear and the non-linear waves generated in a homogeneous fluid layer, occupying an infinite channel with constant depth, by the bounded motion of the bottom. More precisely, we aim to uncover the non-linear features of the resulting flow and to determine the domain where the linear theory is inadequate for its description and hence the necessity for the non-linear theory. For simplicity, we limit our study to two-dimensional irrotational motions of ideal fluids with constant density. Dimensionless quantities are chosen and the system of equations and conditions is reformulated to fit this choice.

The theoretical model under consideration, which is frequently encountered in oceanography, simulates the propagation of waves generated in oceans following an under-water earthquake [18]. Several types of finite difference schemes, within the frame of the shallow water approximations, have been developed for such initial boundary value problems [19,20].

For the corresponding linear problem, an analytical solution is obtained for a particular choice of the bottom motion and an expression for the resulting free surface elevation, at any instant of time, is determined.

For the study of the non-linear problem with the same particular choice of the bottom motion as that for the linear case, the procedure given by Abou-Dina and Helal [12] seems to be more sound and more practical than other available techniques, since it takes into account the full non-linearity of the system of boundary conditions, while other approaches deal with approximate solutions to approximate systems of equations. As mentioned above, the non-linear system of equations and conditions of the problem is transformed into an equivalent system of integro-differential equations satisfied along the boundary of the fluid domain by the velocity potential and the free surface elevation. Hence, instead of dealing with the field equations in the bulk, one has to solve only some boundary relations. An analytical solution for the resulting system of equations seems improbable and therefore a numerical solution is sought for.

A suitable discretization process applied to the obtained system of boundary equations and conditions. This yields a non-linear system of algebraic equations in the boundary values of the velocity potential and the free surface elevation.

Abou-Dina and Helal [12] used a modification of Gauss-Seidel and Newton methods due to Brown [21,22] for the solution of their non-linear system of algebraic equations. The numerical scheme obtained therein showed a certain instability which did not permit the increase of either the time nor the amplitude of the external pressure causing the motion of the fluid in their model. Here, to overcome this inconvenience, we use a more recent numerical method due to Snyman [23] for the solution of our non-linear system of algebraic equations. The essence of this method is to replace the original problem by another equivalent unconstrained optimization one and to solve this optimization problem.

The free surface elevations obtained using the linear and the non-linear theories are calculated, plotted and compared, for different values of the parameters of the problem, at different instants of time. For sufficiently small amplitude of the bottom's motion, the corresponding results for both theories are shown to be in good agreement in a narrow interval of time following the start of the motion, while a significant oscillating divergence between the two theories appears and grows in the course of time. The noticed divergence between the two theories increases considerably with the increase of the amplitude of the bottom's motion. This clearly indicates the limitations of applicability of the linear theory for the description of the present phenomenon and that the non-linear treatment is necessary.

In addition to the oceanographical importance of the application worked here, it can be used in testing experimental data and in evaluating the validity of the different analytical procedures proposed for the study of the problem.

2. Nomenclature

The following notation is used throughout this paper:

c_0 the critical velocity of waves
g the acceleration of gravity
h the water depth
$\overline{P}(\bar{x},\bar{y},\bar{t})$ the pressure applied to the fluid particle occupying the position $(\bar{x},\bar{y})$ at the instant of time $\bar{t}$
$\bar{s}$ the arc length used as a parameter in the representation of a contour
$\bar{t}$ the time
$(\bar{x},\bar{y})$ Cartesian coordinates of a point
$\bar{x}=\pm\bar{a}$ the equations of the radiational downstream and upstream boundaries, respectively
$\bar{y}=\bar{\eta}(\bar{x},\bar{t})$ the equation of the free surface at the instant of time $\bar{t}$
$V_y=\varepsilon\overline{F}(\bar{x},\bar{t})$ the vertical velocity of the bottom
$\vec{V}$ the velocity vector
$\vec{\nabla}$ the gradient operator
$\overline{\Phi}(\bar{x},\bar{y},\bar{t})$ the velocity potential function
ρ the constant density of the fluid layer
ε a real parameter characterizing the relative amplitude of the bottom's vertical velocity

Superscripts f and b mean restriction of superscripted function on the free surface and on the bottom, respectively.

For convenience, we shall denote by $P,F,\ a,s,x,y,t,\eta,\eta_0$ and Φ the dimensionless quantities $\overline{P}/(\rho gh)$, $\overline{F}/\sqrt{gh}$, $\bar{a}/h$, $\bar{s}/h$, $\bar{x}/h$,$\bar{y}/h$, $\bar{t}\sqrt{g/h}$, $\bar{\eta}/h$, $\bar{\eta}_0/h$ and $\bar{\Phi}/(h\sqrt{gh})$.

3. Problem description

An infinite channel of constant depth is occupied by an incompressible and inviscid homogeneous fluid layer. The layer is bounded from below by the impermeable horizontal bottom of the channel, and from over

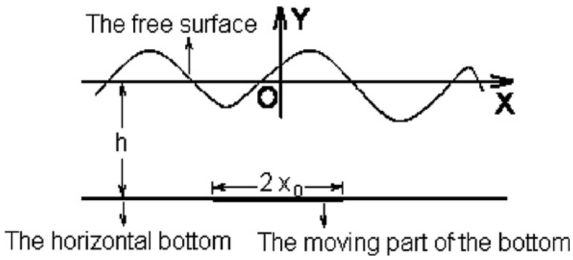

Fig. 1. Problem and frame of reference.

by the free surface which is also impermeable. A portion of the bottom with a finite horizontal extent is set in motion generating progressive gravity waves in the fluid mass. The required is to determine these waves in the course of time.

For simplicity of the formulation and mathematical treatment, the problem is assumed two dimensional and the motion is considered irrotational. Also, the deformation of the horizontal bottom during the motion is neglected and only its vertical velocity is considered.

The two dimensional orthogonal Cartesian system of coordinates shown in Fig. 1 is used.

4. Equations and conditions

The unknowns of the problem are the velocity potential function $\bar{\Phi}(\bar{x}, \bar{y}, \bar{t})$ (with $\vec{V} = \vec{\nabla}\bar{\Phi}$), and the free surface elevation $\bar{\eta}(\bar{x}, \bar{t})$. The physical conditions governing the problem cause the dimensionless functions Φ, η and P to satisfy the following system of equations expressed in terms of the dimensionless variables x, y and t [24]:

(i) In the fluid mass with constant density:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0. \tag{1}$$

The pressure applied to the fluid particle which occupies the position (x, y) at the instant of time t is expressed in terms of the velocity potential as

$$P(x, y, t) = -\rho \left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] + y \right\}. \tag{2}$$

(ii) On the free surface $(y = \eta(x, t))$ the isobarity implies:

$$\eta + \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] = 0, \tag{3}$$

and the impermeability of the free surface leads at $y = \eta(x, t)$ to:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \eta}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial \eta}{\partial t}. \tag{4}$$

(iii) On the horizontal bottom $(y = -1)$:

$$\frac{\partial \Phi}{\partial y} = \varepsilon F(x, t) \quad \text{for } y = -1, \tag{5}$$

where the function $F(x, t)$ (with $F(x, 0) = 0$), characterizing the motion of the bottom, is of finite support (i.e. $F(x, t) = 0$ for $|x| \geqslant x_0$ for a certain prescribed finite value of x_0).

(iv) At infinity $(|x| \to \infty)$:
Since the fluid is initially at rest, and the moving part of the bottom is limited to a finite horizontal extent, the radiation condition implies that at finite instants of time $t > 0$ the fluid at the upstream and the downstream extremities is at rest with horizontal free surface. That is:

$$\lim_{|x| \to \infty} \Phi(x, y, t) = 0, \tag{6a}$$

$$\lim_{|x| \to \infty} \vec{\nabla}\Phi(x, y, t) = 0, \tag{6b}$$

$$\lim_{|x| \to \infty} \eta(x, t) = 0, \tag{6c}$$

$$\lim_{|x| \to \infty} \frac{\partial \eta}{\partial x} = 0. \tag{6d}$$

(v) The initial conditions (at $t = 0$):
The conditions expressing that the fluid layer is initially at rest with a horizontal free surface are:

$$\Phi(x, y, t) = 0 \quad \text{at} \quad = 0, \tag{7a}$$

$$\vec{\nabla}\Phi(x, y, t) = 0 \quad \text{at} \quad t = 0, \tag{7b}$$

$$\frac{\partial \Phi(x, 0, t)}{\partial t} = 0 \quad \text{at} \quad t = 0, \tag{7c}$$

$$\eta(x, t) = 0 \quad \text{at} \quad t = 0, \tag{7d}$$

$$\frac{\partial \eta(x, t)}{\partial x} = 0 \quad \text{at} \quad t = 0, \tag{7e}$$

$$\frac{\partial \eta(x, t)}{\partial t} = 0 \quad \text{at} \quad t = 0. \tag{7f}$$

The main difficulties facing the mathematical study of the system of equations number (1) to (7) arise from the non-linearity of the conditions (3) and (4) and that these conditions are given along the free surface which is itself one of the unknowns of the problem.

5. Linear theory of motion

5.1. The linearization process

Within the frame of the linear theory of motion it is assumed that the fluid motion is slow and the free surface remains always near its position at rest ($y = 0$).

The parameter ε of condition (5), characterizing the amplitude of bottom's motion, is a small parameter and $F(x, t)$ is a prescribed function of the variables x and t. The essence of the linearization process is to put the unknown functions $\Phi(x, y, t)$ and $\eta(x, t)$ as

$$\Phi(x, y, t) = \varepsilon\Phi_1(x, y, t), \tag{8}$$

$$\eta(x, t) = \varepsilon\eta_1(x, t) \tag{9}$$

and to substitute for these functions in the system of Eqs. (1) to (7) keeping only the terms linear in the parameter ε. We develop the conditions on the free surface in terms of powers of the small parameter ε, using Taylor series expansion, in the neighbourhood of $y = 0$ and we keep only the terms linear in ε of this expansion.

The potential function $\Phi_1(x, y, t)$ is found to satisfy the following system of equations:

(i) In the fluid mass:

$$\frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2} = 0 \quad \text{at} \quad -\infty < x < \infty, -1 \leqslant y \leqslant 0. \tag{10}$$

(ii) On the free surface, the condition is replaced by:

$$\frac{\partial \Phi_1}{\partial y} + \frac{\partial^2 \Phi_1}{\partial t^2} = 0 \quad \text{at} \quad y = 0. \tag{11}$$

(iii) On the bottom ($y = -1$):

$$\frac{\partial \Phi_1}{\partial y} = F(x, t) \quad \text{at} \quad y = -1. \tag{12}$$

(iv) At infinity ($|x| \to \infty$):

$$\lim_{|x|\to\infty} \Phi_1(x, y, t) = 0, \tag{13a}$$

$$\lim_{|x|\to\infty} \frac{\partial \Phi_1(x, y, t)}{\partial x} = 0. \tag{13b}$$

(v) The initial conditions (at $t = 0$):

$$\Phi_1(x,y,t) = 0 \quad \text{at } t = 0, \tag{14a}$$

$$\frac{\partial \Phi_1(x,y,t)}{\partial t} = 0 \quad \text{at} \quad t = 0, y = 0. \tag{14b}$$

According to this theory the free surface elevation is given in terms of the velocity potential function as

$$\eta_1(x,t) = -\left.\frac{\partial \Phi_1(x,y,t)}{\partial t}\right|_{y=0}. \tag{15}$$

5.2. Analytical solution

To solve the system of Eqs. (10)–(12), (13a), (13b), (14a), (14b), (15) the functions Φ_1, η_1 and F are assumed to possess complex Fourier transforms with respect to the variable x in the generalized sense of Lighthill [25]. We denote by $\widetilde{\Phi}_1(\xi,y,t)$, $\widetilde{F}(\xi,t)$ the Fourier transforms of the functions $\Phi_1(x,y,t)$, $F(x,t)$ defined by Tranter [26] as

$$\widetilde{\Phi}_1(\xi,y,t) = \int_{-\infty}^{\infty} e^{i\xi x} \Phi_1(x,y,t) dx, \tag{16a}$$

$$\widetilde{F}(\xi,t) = \int_{-\infty}^{\infty} e^{i\xi x} F(x,t) dx. \tag{16b}$$

It can be easily verified that, the solution of the transformed version of Eqs. (10) and (12) takes the form:

$$\widetilde{\Phi}_1(\xi,y,t) = A(\xi,t)\cosh \xi(y+1) + \frac{\widetilde{F}(\xi,t)}{\xi}\sinh \xi(y+1), \tag{17}$$

which is then substituted into the transformed version of Eq. (11) giving:

$$\frac{\partial^2}{\partial t^2}A(\xi,t) + \xi\tanh(\xi)A(\xi,t) + \frac{\tanh(\xi)}{\xi}\frac{\partial^2}{\partial t^2}\widetilde{F}(\xi,t) + \widetilde{F}(\xi,t) = 0, \tag{18}$$

with the initial conditions:

$$A(\xi,t) = 0 \quad \text{at} \quad t = 0, \tag{19a}$$

$$\frac{\partial}{\partial t}A(\xi,t) + \frac{\tanh\xi}{\xi}\frac{\partial}{\partial t}\widetilde{F}(\xi,t) = 0 \quad \text{at} \quad t = 0. \tag{19b}$$

The solution of (18) and (19) is:

$$A(\xi,t) = -\left[\frac{\tanh\xi}{\xi}\widetilde{F}(\xi,t) + \frac{1}{\omega(\cosh\xi)^2}\int_0^t \widetilde{F}(\xi,\tau)\sin\omega(t-\tau)d\tau\right] \tag{20}$$

with

$$\omega^2 = \xi\tanh(\xi). \tag{21}$$

The velocity potential $\Phi_1(x,y,t)$ is obtained, by substitution of (20) in (17) for the function $A(\xi,t)$ and application of the Fourier inversion formula corresponding to (16a), in the form:

$$\Phi_1(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\left[\left\{\frac{\sinh(\xi y)}{\xi\cosh\xi}\widetilde{F}(\xi,t) - \frac{\cosh\xi(y+1)}{\omega(\cosh\xi)^2}\int_0^t \widetilde{F}(\xi,\tau)\sin\omega(t-\tau)d\tau\right\}\right]d\xi. \tag{22}$$

This expression for the velocity potential consists of two parts, the first has no contribution to the free surface elevation and the second has nothing to do with the bottom's vertical velocity.

The free surface elevation is given, up to the first-order, using (9), (15) and (22) as

$$\eta(x,t)=\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left[\frac{e^{-i\xi x}}{\cosh\xi}\int_0^t\widetilde{F}(\xi,\tau)\cos\omega(t-\tau)d\tau\right]d\xi. \tag{23}$$

6. Boundary integro-partial differential equations

The domain of definition of the system of Eqs. (1) to (7) cannot be represented by a separable coordinate system. Therefore, an analytical solution for this non-linear system seems improbable. It is worthy to represent these equations in terms of some boundary integral relations along the boundary of the domain. Accordingly, the domain of definition of the problem is replaced by the region D sketched in Fig. 2, bounded from below by the bottom and from over by the free surface. The radiational boundaries on the right and on the left of the domain D are introduced far enough from the area where the bottom is in motion. We denote by C the boundary of the domain D described in a counter-clockwise direction. The parametric representation of this boundary is written $x = x(s)$, $y = y(s)$ where s is the dimensionless arc length used as a parameter. The parts of the boundary C lying on the free surface, the upstream radiational boundary, the bottom and the downstream radiational boundary are denoted by C_f, C_u, C_b and C_d, respectively.

The function $\Phi(x,y,t)$, which is harmonic inside the region D and differentiable on the boundary C, can be expressed as [27]:

$$\Phi(x,y,t)=\frac{1}{2\pi}\oint_C[\Phi(s',t)\frac{\partial}{\partial n'}\log R-\frac{\partial\Phi}{\partial n'}\log R]\,ds', \tag{24}$$

where $\Phi(s',t)$ is written for $\Phi(x(s',t),y(s',t),t)$, and R represents the dimensionless distance from the position (x,y) inside the domain D to the boundary point with parameter s' and $\frac{\partial}{\partial n'}$ [1] denotes differentiation at this point along the normal to the boundary C.

When the field point (x,y) approaches a boundary point with parameter s relation (24) reduces to:

$$\Phi(s,t)=\frac{1}{\pi}\oint_C[\Phi(s',t)\frac{\partial}{\partial n'}\log R-\frac{\partial\Phi(s',t)}{\partial n'}\log R]\,ds'. \tag{25}$$

It is more convenient, for our purpose, to express the boundary and the initial conditions numbers (3) to (7) in the form:

– On the free surface (C_f with $y=\eta(x,t)$):

$$\left[\frac{\partial\Phi}{\partial y}\right]^f=\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\left[\frac{\partial\Phi}{\partial x}\right]^f, \tag{26a}$$

$$\eta+\left[\frac{\partial\Phi}{\partial t}\right]^f+\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right]^f=0. \tag{26b}$$

The impermeability condition (26a) leads on the free surface to

$$\left[\frac{\partial\Phi}{\partial n}\right]^f=\frac{\frac{\partial\eta}{\partial t}}{\sqrt{1+\left(\frac{\partial\eta}{\partial x}\right)^2}}. \tag{26c}$$

– On the horizontal bottom (C_b with $y=-1$):
Neglecting the deformation of this boundary during the motion, the normal direction associated with the positive sense of describing C_b is directed along the negative y-axis. Hence the boundary condition on C_b is given by (5) as

[1] This derivative is defined in correspondance to the partial derivative $\frac{\partial}{\partial s'}$ with s' as a dimensionless arc length.

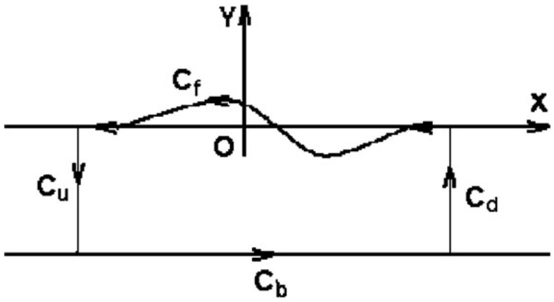

Fig. 2. The domain D occupied by the fluid, bounded by the contour $C = C_f \cup C_u \cup C_b \cup C_d$.

$$\frac{\partial \Phi}{\partial n} = -\varepsilon F(s,t). \tag{27}$$

– On the radiational boundaries (C_u or C_d):

$$\Phi(s,t) = 0, \tag{28a}$$
$$\frac{\partial \Phi(s,t)}{\partial n} = 0. \tag{28b}$$

– At the initial instant ($t = 0$):

$$\Phi(s,0) = 0, \tag{29a}$$
$$\eta(s,0) = 0. \tag{29b}$$

Using the boundary conditions (27) and (28), relation (25) can be put in the form:

$$\Phi(s,t) = \frac{1}{\pi}\int_{C_f}[\Phi^f(s',t)\frac{\partial}{\partial n'}\log R - \left[\frac{\partial \Phi(s',t)}{\partial n'}\right]^f \log R]\,ds' + \frac{1}{\pi}\int_{C_b}\left[\Phi^b(s',t)\frac{\partial \log R}{\partial n'} + \varepsilon F(s',t)\log R\right]ds'. \tag{30}$$

Using the dimensionless abscissa x as a variable of integration and allowing the radiational boundaries to be very far from the origin, relation (30) is written with the help of condition (26a) in the form:

$$\Phi(s,t) = \frac{1}{\pi}\int_{-a}^{a}\left[\Phi^f(x',t)\frac{1}{R_f^2}\left\{\frac{\partial \eta(x',t)}{\partial x'}(x(s)-x') - (y(s)-\eta(x',t))\right\} - \frac{\partial \eta(x',t)}{\partial t}\log R_f\right]dx' + \frac{1}{\pi}\int_{-a}^{a}\left[\Phi^b(x',t)\frac{1}{R_b^2}(y(s)+1) + \varepsilon F(s',t)\log R_b\right]dx', \tag{31a}$$

where Φ^f and Φ^b denote the restrictions of the velocity potential to the free surface and the bottom respectively, $(x(s), y(s))$ are the coordinates of the boundary point with parameter s and

$$R_f = \sqrt{[x(s)-x']^2 + [y(s)-\eta(x',t)]^2}, \tag{31b}$$

$$R_b = \sqrt{[x(s)-x']^2 + [y(s)+1]^2}. \tag{31c}$$

Relations (31) give, at the instant t, on the free surface (i.e. for $x(s) = x, y(s) = \eta(x,t)$):

$$\Phi^{f}(x,t) = \frac{1}{\pi}\int_{-a}^{a}\left[\Phi^{f}(x',t)\frac{\frac{\partial\eta(x',t)}{\partial x'}(x-x') - (\eta(x,t)-\eta(x',t))}{(x-x')^2 + (\eta(x,t)-\eta(x',t))^2}\right.$$
$$-\frac{\partial\eta(x',t)}{\partial t}\log\sqrt{\{(x-x')^2+(\eta(x,t)-\eta(x',t))^2\}} + \Phi^{b}(x',t)\left(\frac{\eta(x,t)+1}{(x-x')^2+(\eta(x,t)+1)^2}\right)$$
$$\left. + \varepsilon F(x',t)\log\sqrt{\{(x-x')^2+(\eta(x,t)+1)^2\}}\right]\mathrm{d}x', \tag{32a}$$

and at the bottom (i.e. for $x(s) = x, y(s) = -1$), relations (31) yield:

$$\Phi^{b}(x,t) = \frac{1}{\pi}\int_{-a}^{a}\left[\Phi^{f}(x',t)\left(\frac{\frac{\partial\eta(x',t)}{\partial x'}(x-x') + (1+\eta(x',t))}{(x-x')^2+(1+\eta(x',t))^2}\right)\right.$$
$$\left. -\frac{\partial\eta(x',t)}{\partial t}\log\sqrt{\{(x-x')^2+(1+\eta(x',t))^2\}} + \varepsilon F(x',t)\log|(x-x')|\right]\mathrm{d}x'. \tag{32b}$$

The free surface conditions number (26) give at the same instant:

$$\eta(x,t) + \left[\frac{\partial\Phi(x,t)}{\partial t}\right]^{f} + \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]^{f} = 0, \tag{32c}$$

where,

$$\left[\frac{\partial\Phi(x,t)}{\partial t}\right]^{f} = \frac{\partial\Phi^{f}(x,t)}{\partial t} - \frac{\left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)\left(\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\Phi^{f}}{\partial x}\right)}{\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2\right]}, \tag{32d}$$

$$\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]^{f} = \frac{\left[\left(\frac{\partial\Phi^{f}}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2\right]}{\left[1+\left(\frac{\partial\eta}{\partial x}\right)^2\right]}. \tag{32e}$$

At the initial instant ($t = 0$), the initial conditions (29) give:

$$\Phi^{f}(x,0) = 0, \tag{33a}$$
$$\Phi^{b}(x,0) = 0, \tag{33b}$$
$$\eta(x,0) = 0. \tag{33c}$$

The system of Eq. (32) reduces the problem under consideration into a non-linear system of integro-partial differential equations subject to the initial conditions (33) in the unknown functions $\Phi^{f}(x,t), \Phi^{b}(x,t)$ and $\eta(x,t)$. Exact analytical solution of this system does not seem possible. Hence, to reveal the non-linear features of the resulting flow, a numerical procedure for the solution is necessary.

7. Algorithm for the numerical solution of the non-linear system of equations

The algorithm, proposed hereafter depends, in essence, on replacing the above non-linear system of integro-partial differential equations and the initial conditions by a suitable system of non-linear algebraic equations whose solution gives appropriate numerical estimates of the unknowns of the original problem.

The first step of this algorithm is to choose the locations of the far boundaries $x = \pm a$ on which the radiation conditions are to be set. If $\bar{t}$ is the time moment at which the solution is required and taking in consideration that the motion starts from rest, then it is sufficient to locate the radiational boundaries C_u and C_d at distances from the nearest part of the moving portion of the bottom slightly greater than $c_0\bar{t}$, since $c_0 = \sqrt{gh}$ is the critical velocity of wave propagation as predicted by the linear theory. This guarantees the complete rest of the fluid at these boundaries, and hence the validity of the imposed boundary conditions up to the considered instant of time. Further increase of the distances at which these radiational boundaries are taken does not affect the actual solution in any way.

The system of integro-differential equations and conditions (32), (33) is then replaced by a suitably chosen approximating system of non-linear algebraic equations according to the following procedure.

(1) divide the interval $(-a < x < a)$ into a finite number N of subintervals with lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_N$,
(2) the time interval from the initial instant $t = 0$ up to the present instant t is divided into a finite number M of subintervals with lengths: $\Delta t_1, \Delta t_2, \ldots, \Delta t_M$,
(3) use any standard technique for numerical integration Isaacson and Keller [28] to replace the integrals in relations (32) by suitable finite sums, and replace the spacial and temporal derivatives of the unknown functions by suitable finite differences,
(4) find the solution of the resulting non-linear system of algebraic equations subject to the initial conditions (33).

8. Numerical scheme for the solution

Following the above algorithm for the numerical solution of the non-linear system of equations, we denote for $i = 1, 2, \ldots, N$ and $k = 0, 1, 2, \ldots, M$:

$$x_i = -a + \Delta x_1 + \Delta x_2 + \ldots + \Delta x_i, \tag{34a}$$

$$t_k = \Delta t_1 + \Delta t_2 + \ldots + \Delta t_k, \quad \text{with} \quad t_0 = 0, \tag{34b}$$

$$\Phi^{\mathrm{f}}_{i,k} = \Phi^{\mathrm{f}}(x_i, t_k), \tag{34c}$$

$$\Phi^{\mathrm{b}}_{i,k} = \Phi^{\mathrm{b}}(x_i, t_k), \tag{34d}$$

$$\eta_{i,k} = \eta(x_i, t_k), \tag{34e}$$

$$F_{i,k} = F(x_i, t_k). \tag{34f}$$

The integral relations (32a) and (32b) and the condition (32c) are approximated, respectively, by the following algebraic equations for $i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, M$:

$$\Phi^{\mathrm{f}}_{i,k} - \frac{1}{\pi} \sum_{j=1}^{N} \left[C_{i,j,k} - D_{i,j,k} + E_{i,j,k} + P_{i,j,k} \right] = 0, \tag{35a}$$

$$\Phi^{\mathrm{b}}_{i,k} - \frac{1}{\pi} \sum_{j=1}^{N} \left[G_{i,j,k} - H_{i,j,k} + Q_{i,j,k} \right] = 0, \tag{35b}$$

$$\eta_{i,k} + A_{i,k} + B_{i,k} = 0, \tag{35c}$$

where

$$A_{i,k} = \frac{\left(\Phi^{\mathrm{f}}_{i,k} - \Phi^{\mathrm{f}}_{i,k-1}\right)}{\Delta t_k} - \frac{\left[\left\{ \frac{(\eta_{i,k} - \eta_{i,k-1})}{\Delta t_k} \right\}^2 + \left\{ \frac{(\eta_{i,k} - \eta_{i,k-1})}{\Delta t_k} \right\} \left\{ \frac{(\eta_{i,k} - \eta_{i-1,k})}{\Delta x_i} \right\} \left\{ \frac{(\Phi^{\mathrm{f}}_{i,k} - \Phi^{\mathrm{f}}_{i-1,k})}{\Delta x_i} \right\} \right]}{\left[1 + \left\{ (\eta_{i,k} - \eta_{i-1,k})/\Delta x_i \right\}^2 \right]}, \tag{36a}$$

$$B_{i,k} = \frac{1}{2} \left[\frac{\left\{ \left(\Phi^{\mathrm{f}}_{i,k} - \Phi^{\mathrm{f}}_{i-1,k}\right)/\Delta x_i \right\}^2 + \left\{ (\eta_{i,k} - \eta_{i,k-1})/\Delta t_k \right\}^2}{\left[1 + \left\{ (\eta_{i,k} - \eta_{i-1,k})/\Delta x_i \right\}^2 \right]} \right], \tag{36b}$$

$$C_{i,j,k} = \Phi^{\mathrm{f}}_{j,k} \left[\frac{(x_i - x_j) \frac{(\eta_{j,k} - \eta_{j-1,k})}{\Delta x_j} - (\eta_{i,k} - \eta_{j,k})}{(x_i - x_j)^2 + (\eta_{i,k} - \eta_{j,k})^2} \right] \Delta x_j, \quad i \neq j, \tag{37a}$$

$$C_{i,i,k} = -\frac{1}{2} \Phi^{\mathrm{f}}_{i,k} \left[\frac{(\eta_{i+1,k} - 2\eta_{i,k} + \eta_{i-1,k})/(\Delta x_i)^2}{\left[1 + \left\{ (\eta_{i,k} - \eta_{i-1,k})/\Delta x_i \right\}^2 \right]} \right] \Delta x_i, \tag{37b}$$

$$D_{i,j,k} = \left(\frac{\eta_{j,k} - \eta_{j,k-1}}{\Delta t_k}\right) \log\left\{\sqrt{(x_i - x_j)^2 + (\eta_{i,k} - \eta_{j,k})^2}\right\}\Delta x_j, \quad i \neq j, \tag{37c}$$

$$D_{i,i,k} = \left(\frac{\eta_{i,k} - \eta_{i,k-1}}{\Delta t_k}\right)\left[\log\left\{0.5\Delta x_i\sqrt{1 + \left(\frac{\eta_{i,k} - \eta_{i-1,k}}{\Delta x_i}\right)^2}\right\} - 1\right]\Delta x_i, \tag{37d}$$

$$E_{i,j,k} = \Phi^{\mathrm{b}}_{j,k}\left[\frac{\eta_{i,k} + 1}{(x_i - x_j)^2 + (\eta_{i,k} + 1)^2}\right]\Delta x_j, \tag{37e}$$

$$P_{i,j,k} = \varepsilon F_{j,k} \log\left\{\sqrt{(x_i - x_j)^2 + (\eta_{i,k} + 1)^2}\right\}\Delta x_j, \tag{37f}$$

$$G_{i,j,k} = \Phi^{\mathrm{f}}_{j,k}\left[\frac{(x_i - x_j)\frac{(\eta_{j,k} - \eta_{j-1,k})}{\Delta x_j} + (\eta_{j,k} + 1)}{(x_i - x_j)^2 + (\eta_{j,k} + 1)^2}\right]\Delta x_j, \tag{38a}$$

$$H_{i,j,k} = \left(\frac{\eta_{j,k} - \eta_{j,k-1}}{\Delta t_k}\right) \log\left\{\sqrt{(x_i - x_j)^2 + (\eta_{j,k} + 1)^2}\right\}\Delta x_j, \tag{38b}$$

$$Q_{i,j,k} = \varepsilon F_{j,k} \log\{|(x_i - x_j)|\}\Delta x_j, \quad i \neq j, \tag{38c}$$

and

$$Q_{i,i,k} = \varepsilon F_{i,k}[\log\{0.5(\Delta x_i)\} - 1]\Delta x_i. \tag{38d}$$

At a fixed time instant t_k $(k = 1, 2, \ldots, M)$, Eqs. (35) and (38) represent a system of $3N$ non-linear algebraic equations in the unknowns $\Phi^{\mathrm{f}}_{i,k}$, $\Phi^{\mathrm{b}}_{i,k}$ and $\eta_{i,k}$ for $i = 1, 2, \ldots, N$. this system is subject to the initial conditions, derived from relations (33) in the form

$$\Phi^{\mathrm{f}}_{i,0} = 0, \tag{39a}$$

$$\Phi^{\mathrm{b}}_{i,0} = 0, \tag{39b}$$

$$\eta_{i,0} = 0. \tag{39c}$$

The following numerical boundary values, approximating the conditions at infinity number (6), are to be taken into account:

$$\Phi^{\mathrm{f}}_{0,k} = \Phi^{\mathrm{f}}_{N+1,k} = 0, \tag{40a}$$

$$\Phi^{\mathrm{b}}_{0,k} = Phi^{\mathrm{b}}_{N+1,k} = 0, \tag{40b}$$

$$\eta_{0,k} = \eta_{N+1,k} = 0. \tag{40c}$$

This scheme will be worked out in the following section for a numerical illustration.

9. Numerical results and discussion

We shall consider the particular case where the moving part of the bottom, initialy at rest:, consists of a sinusoidal hump making a number n of oscillations during a time interval T_0 before coming back again to the state of rest. We therefore may take:

$$V_y(x,t) = \varepsilon F(x,t) = \varepsilon X(x)T(t) \tag{41a}$$

with,

$$X(x) = \begin{cases} \cos\left(\frac{\pi x}{2x_0}\right), & |x| \leqslant x_0 \\ 0, & |x| > x_0 \end{cases} \tag{41b}$$

$$T(t) = \begin{cases} \sin(n\ \frac{\pi t}{T_0}), & t \leqslant T_0 \\ 0, & t > T_0 \end{cases} \tag{41c}$$

for some particular values of the real constants ε and x_0. The function $F(x,t)$ is written, using (34a), (34b), (34c), in the more compact form:

$$F(x,t) = (-1)^n \cos\left(\pi \frac{(|x| + x_0 - ||x| - x_0|)}{4x_0}\right) \sin\left(n\,\pi \frac{(t - T_0) - |t - T_0|}{2T_0}\right). \tag{42}$$

For such a particular choice, the free surface elevation, as calculated using the linear theory of motion, is given by Eqs. (23) and (42) in the following form:

$$\eta(x,t) = \frac{n\pi\varepsilon}{x_0 T_0} \int_0^\infty \frac{\cos(\xi x)\cos(\xi x_0)\left[\cos(\omega t) - \cos\left(\frac{n\pi}{T_0}\tau_0\right)\cos\left(\omega(t-\tau_0)\right)\right]}{\cosh\xi\left[\xi^2 - \left(\frac{\pi}{2x_0}\right)^2\right]\left[\omega^2 - \left(\frac{n\pi}{T_0}\right)^2\right]}\,\mathrm{d}\xi, \tag{43a}$$

where

$$\tau_0 = \frac{(T_0 + t) - |T_0 - t|}{2}. \tag{43b}$$

The elevations of the free surface are calculated within both frames of the linearized theory of motion (using expressions (43)) and the non-linear theory of motion using the algorithm and the numerical scheme given above. For the solution of the resulting non-linear system of algebraic equations, unlike Abou-Dina and Helal [12], the problem is transformed into an equivalent unconstrained optimization one which we solve using the scheme given by Snyman [23]. This procedure proves to be more stable than that followed by Abou-Dina and Helal [12].

During the actual calculations it is found that the variations in the values of the parameters x_0, T_0 and n have no significant effect on the comparison between the linear and non-linear theories. We prescribe, for convenience, the values $x_0 = 0.5$, $T_0 = 2$ and $n = 8$ to these parameters.

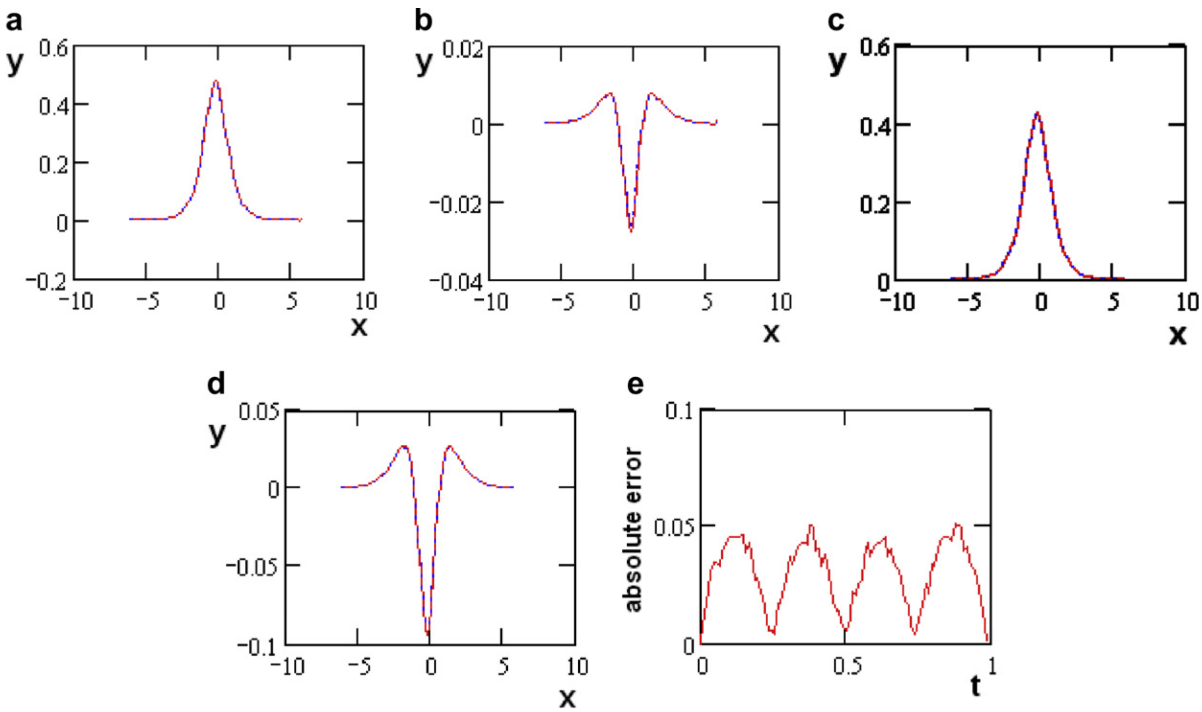

Fig. 3. The variation with x of the free surface elevations (magnified 1000 times) calculated from the non-linear theory (solid curve) and the linear theory (broken line) corresponding to the particular value $\varepsilon = 0.01$ at the instants of time $t = 0.25\sqrt{\frac{h}{g}}$ (case **A**), $0.5\sqrt{\frac{h}{g}}$ (case **B**), $0.75\sqrt{\frac{h}{g}}$ (case **C**), $\sqrt{\frac{h}{g}}$ (case **D**). (e) Gives the variation with time t of the absolute error (the absolute difference between the free surface elevations calculated by the two theories magnified 1000 times) at $x = 0.0$.

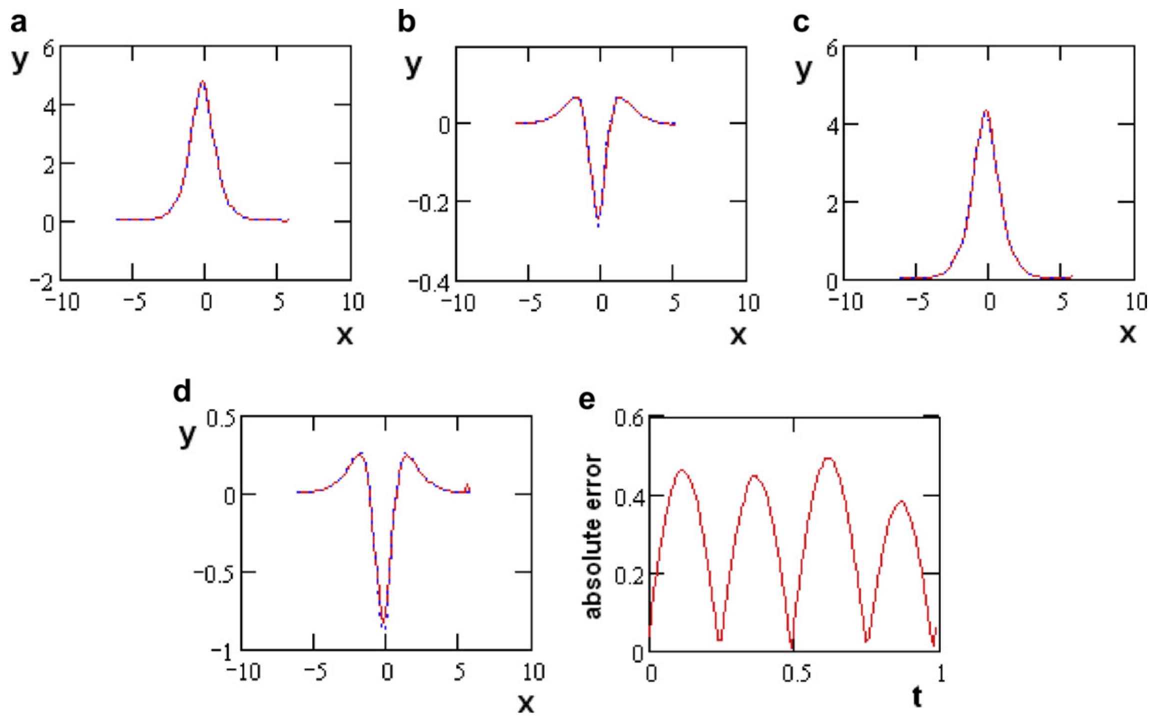

Fig. 4. The free surface elevations, magnified 1000 times, corresponding to the particular value $\varepsilon = 0.1$. The particular values of the remaining parameters are the same as those for Fig. 3.

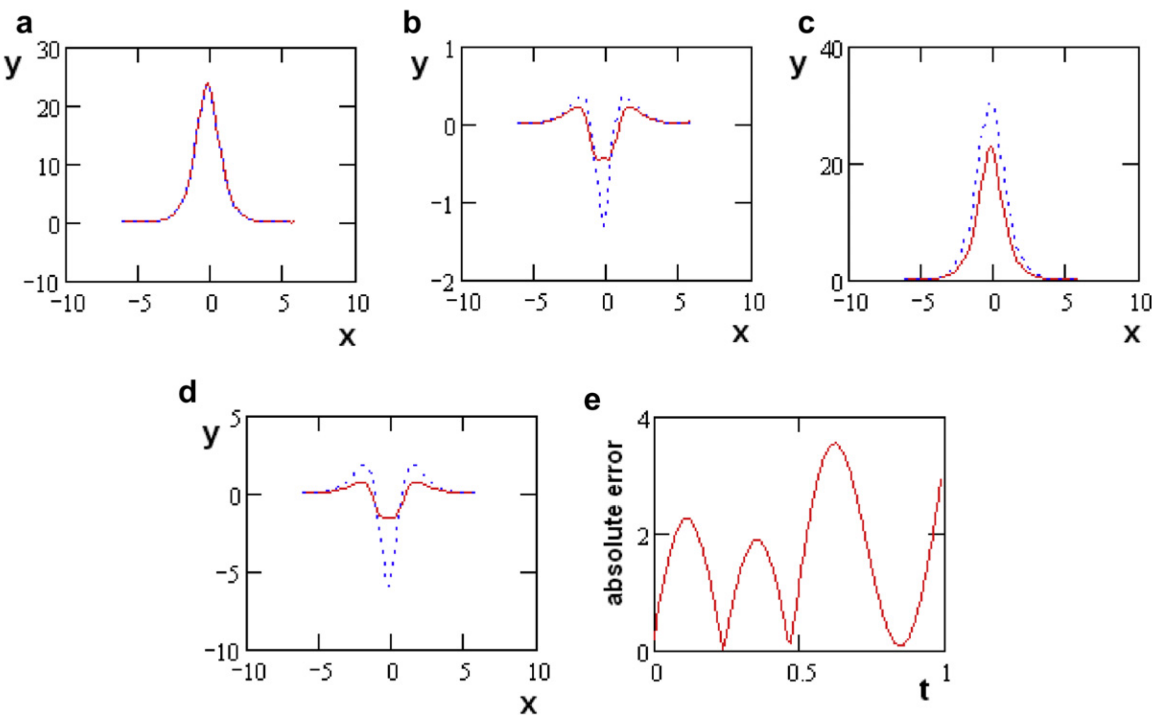

Fig. 5. The free surface elevations, magnified 1000 times, corresponding to the particular value $\varepsilon = 0.5$. The particular values of the remaining parameters are the same as those for Fig. 3.

Figs. 3–7 exhibit the variation with x of the free surface elevation $\eta(x,t)$ at the instants of time $\bar{t} = 0.25\sqrt{\frac{h}{g}}$, $0.5\sqrt{\frac{h}{g}}$, $0.75\sqrt{\frac{h}{g}}$ and $\sqrt{\frac{h}{g}}$, respectively, for the values of $\varepsilon = 0.01$ (Fig. 3), 0.1 (Fig. 4), 0.5 (Fig. 5), 1.0 (Fig. 6) and 2.0 (Fig. 7) of the relative amplitude of the bottom's motion. The solid curves, in each figure, represent the free surface elevation calculated by the non-linear theory, and the broken ones correspond to the linear theory. Figure (e) in each case exhibits the variation with time of the absolute difference between the free surface elevations calculated by the two theories at the position $x = 0$. Therefore, these curves represent the error committed at this position by the linearization of the problem.

Following the above scheme, the radiational boundaries are taken at a distance $a = 6$ on the two sides of the origin and the interval $-a \leqslant x \leqslant a$ is divided using 81 equidistant points (hence the increments Δx_i are taken equal and equal to 0.15) and the increments Δt_i are similarly taken equal and equal to 0.01.

The results indicate the following:

- The linear theory of motion is sufficient to predict the free surface elevation for considerably small values of the amplitude ε of the bottom's motion (Fig. 3) and during a narrow time interval following the start of the motion as this amplitude is slightly increased (Fig. 4).
- The linear theory gives erroneous results and, at all, is insufficient for describing the phenomenon under consideration, as the time increases, specially for non-small bottom motion's amplitudes ε (Figs. 4–7).
- The interval of time following the start of the motion, during which the linear theory may approximately predict the phenomenon, decreases with the increase of the amplitude ε.
- Figs. 3e–7e in each case show that the error is of an oscillatory nature in time t and increases with the increase of the amplitude ε.
- The above remarks gives some bounds and limitations on the validity of the linear theory of wave propagation in describing the phenomenon under consideration.
- The actual time of calculations increases with the increase of the amplitude ε of the bottom's motion.

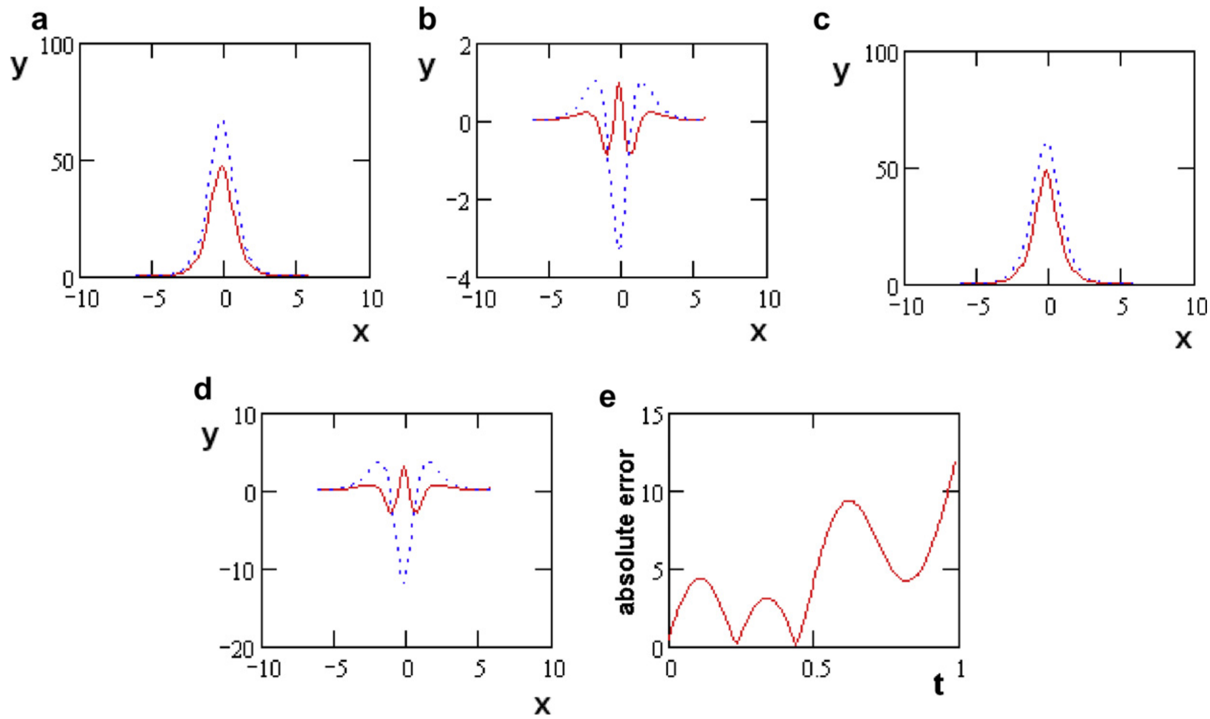

Fig. 6. The free surface elevations, magnified 1000 times, corresponding to the particular value $\varepsilon = 1.0$. The particular values of the remaining parameters are the same as those for Fig. 3.

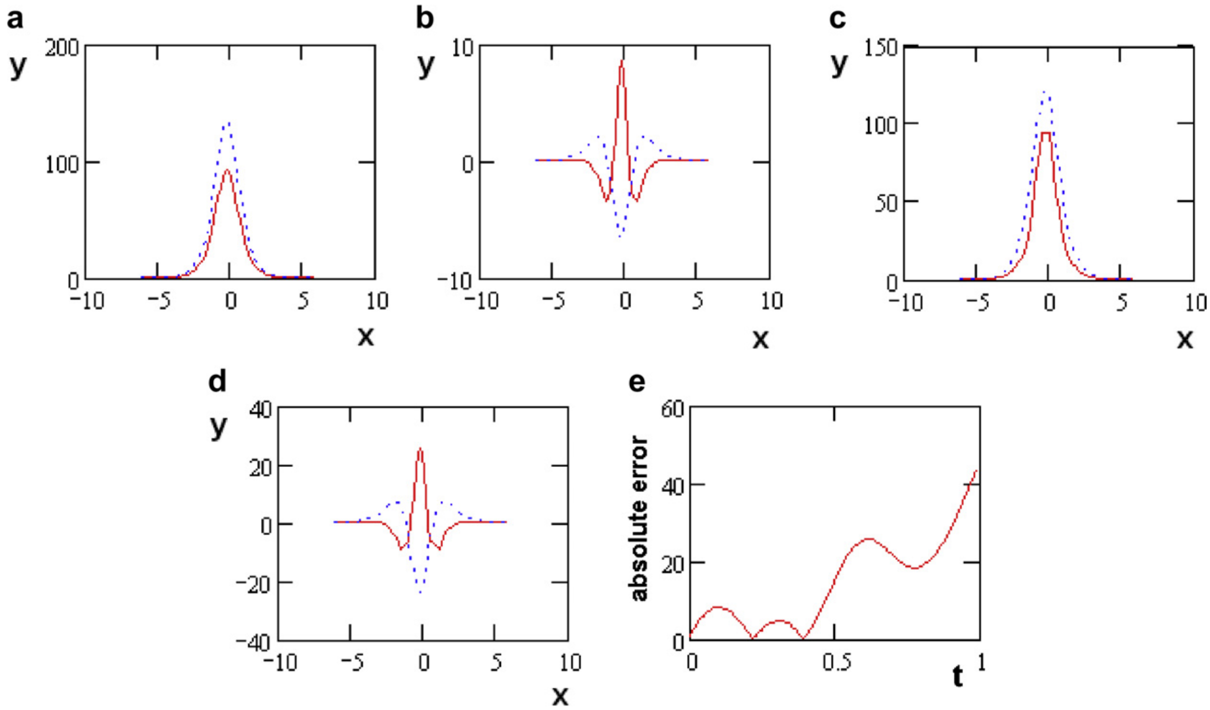

Fig. 7. The free surface elevations, magnified 1000 times, corresponding to the particular value $\varepsilon = 2.0$. The particular values of the remaining parameters are the same as those for Fig. 3.

– The instable behavior of the numerical scheme noticed by Abou-Dina and Helal [12] does not appear in the numerical experiments carried out here. This may be due to the method which they used for the numerical solution of the non-linear system of algebraic equations and also due to the presence of an erroneous minus sign in the first term of the basic Eq. (31a) and by consequence in Eqs. (32a), (32b), (35b) and (35c) along with some minor error concerning the discretization process in Eq. (36b) of their publication. These mistakes are remedied in the present paper.

10. Conclusions

The following concluding remarks are pulled out

- The mathematical model, simulating the oceanographical phenomenon of propagation of waves generated in oceans following an under-water earthquake, is a non-linear free boundary value problem subject to certain initial conditions.
- The linear theory of motion leads to acceptable predictions for this phenomenon in a narrow interval of time following the start of the motion for small bottom's displacement amplitude (ε) and is insufficient otherwise.
- The numerical procedure, based on the boundary integral technique, applied to the non-linear problem reveals the non-linear features of the phenomenon and agrees with the linear theory of motion in the narrow domain where this latter is adequate.
- The modifications and corrections introduced to the procedure permit to predict the phenomenon on a larger scale of time and bottom's motion amplitude without facing the instable behavior of the numerical scheme noticed in other publications existing in the literature.
- The method exposed above is reliable, efficient and can be used to test the different approaches proposed for the analytical study of the non-linear wave propagation problems.

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