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A regularization method for solving the Cauchy problem for the Helmholtz equation ☆

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ABSTRACT

In this paper, we investigate a Cauchy problem associated with Helmholtz-type equation in an infinite "strip". This problem is well known to be severely ill-posed. The optimal error bound for the problem with only nonhomogeneous Neumann data is deduced, which is independent of the selected regularization methods. A framework of a modified Tikhonov regularization in conjunction with the Morozov's discrepancy principle is proposed, it may be useful to the other linear ill-posed problems and helpful for the other regularization methods. Some sharp error estimates between the exact solutions and their regularization approximation are given. Numerical tests are also provided to show that the modified Tikhonov method works well.

1. Introduction

The Cauchy problem of an elliptic equation is well known to be ill-posed in the sense of Hadamard. Some conditional stability results were given by some papers [1–4], these results are based on the exact given data. However, in practice, the given data is polluted for a variety of reasons such as measurement error, round-off error in machine representations. Because of these reasons, regularization strategies are necessary in order to compute such a solution in some stable way. Recently, a lot of regularization methods have been provided. For computational aspects, the readers can consult Hào and Lesnic [5], Reinhardt et al. [6], Cheng and Yamamoto [7] and Hon and Wei [8]. For theoretical aspects, the readers can refer to Xiong [9], Xiong and Fu [10] and Qian et al. [11].

The Helmholtz equation is a special kind of elliptic equation and is especially important in some practical physical applications. It is often used to describe the vibration of a structure [12], the acoustic cavity problem [13], the radiation wave [14], the scattering of a wave [15], the problem of heat conduction in fins [16], the Debye–Hückel theory [17], the linearization of the Poisson–Boltzmann equation [18], etc. In the last decade, there were many researches on the Cauchy problem of Helmholtz equations, e.g. [19–30,10,31,32] are related to the analytical solutions, and [33–43] are about the numerical solutions. For more information about the Cauchy problem of Helmholtz equations, one can refer to [43,26].

In this paper, we consider the following Cauchy problem for the Helmholtz equation in a "strip" domain:

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$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}^n, \quad n \geqslant 1, \\ u(0,y) = \varphi_1(y), & y \in \mathbb{R}^n, \\ u_x(0,y) = \varphi_2(y), & y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2}$ is an $n + 1$ dimensional Laplace operator and the constant $k > 0$ is the number of wave. The solution $u(x,y)$ for $0 < x \leqslant 1$ will be determined from the noisy data ${\varphi_1}^\delta(y)$ and ${\varphi_2}^\delta(y)$ which satisfy:

$$\|\varphi_1^\delta - \varphi_1\|_{L^2(\mathbb{R}^n)} \leqslant \delta, \quad \|\varphi_2^\delta - \varphi_2\|_{L^2(\mathbb{R}^n)} \leqslant \delta. \tag{1.2}$$

This model and its many applications are introduced by Regińska and Regiński [26]. They used a Fourier method to solve (1.1) by decomposing it into a well-posed problem and an ill-posed problem. Here we divide (1.1) into the following two ill-posed problems:

$$\begin{cases} \Delta u_1(x,y) + k^2 u_1(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}^n, \quad n \geqslant 1, \\ u_1(0,y) = \varphi_1(y), & y \in \mathbb{R}^n, \\ (u_1)_x(0,y) = 0, & y \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \Delta u_2(x,y) + k^2 u_2(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}^n, \quad n \geqslant 1, \\ u_2(0,y) = 0, & y \in \mathbb{R}^n, \\ (u_2)_x(0,y) = \varphi_2(y), & y \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

According to the linearity of the problem (1.1), $u = u_1 + u_2$ is the solution of problem (1.1). Therefore we only need to solve problems (1.3) and (1.4), respectively. From the analysis in [26], we know that both of them are severely ill-posed and some regularization methods are necessary for stable reconstruction of the solutions. Here we prefer to use a modified Tikhonov method to consider them. In fact, this paper is devoted to three aspects: (1) The optimal error bound for ill-posed problem (1.4) will be provided, which is independent of the selected regularization methods; (2) The *a priori* strategy for choosing the parameter α and the corresponding error estimate for problem (1.4) will be given; (3) A framework for the error estimate by using *a posteriori* strategy in the Morozov's discrepancy principle will be proposed, which will be used for problems (1.3) and (1.4). It is worth pointing out here that Qin et al. [23] and Xiong and Fu [10] have also applied the modified Tikhonov method to the Cauchy problem for the Helmholtz equation. However, they only considered problem (1.3) and did not study the above three aspects. And the techniques in this paper can be also applied to the Cauchy problems for the modified Helmholtz equation (i.e., the Yukawa equation) and even more generalized linear ill-posed problems. Moreover, this method can also treat general domain which will be explained by Remark 5.9.

For solving many ill-posed problems, the Tikhonov regularization techniques are famous, widely applicable and very effective. However it is quite difficult to obtain an explicit error estimate for some complicated problems with parametric variable. In this paper we will derive some inequalities in order to use a modified Tikhonov method for solving problem (1.1). The idea of modified Tikhonov method was firstly proposed by Carasso [44].

The paper is organized as follows. In Section 2, we give some auxiliary results. In Section 3, we discuss the optimal error bound for problem (1.4). In Section 4, the *a priori* parameter choice rule for problem (1.4) is suggested and the corresponding error estimate is obtained. In Section 5, we propose a framework of the *a posteriori* parameter choice rule in the Morozov's discrepancy principle. We also apply this framework to problems (1.3) and (1.4) and obtain the corresponding error estimates. In Section 6, the numerical results are presented. Finally, a short conclusion in Section 7 summarizes the content of this paper.

2. Preliminaries

In this section, we give some auxiliary results for using the modified Tikhonov method later.

For $g(y) \in L(\mathbb{R}^n)$, $\hat{g}(\xi)$ denotes the Fourier transform, which is defined by

$$\hat{g}(\xi) = \frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} g(y) dy, \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{2.1}$$

Let $\|\cdot\|$ denote the norm in $L^2(\mathbb{R}^n)$. Then the Parseval formula is

$$\|g\| = \|\widehat{g}\|. \tag{2.2}$$

Applying the Fourier transform to problems (1.3) and (1.4) with respect to the variable $y \in \mathbb{R}^n$, respectively, it is easy to obtain that:

$$\widehat{u_1}(x,\xi) = \cosh\left(x\sqrt{|\xi|^2 - k^2}\right)\widehat{\varphi_1}(\xi) =: \kappa_1(x,\xi)\widehat{\varphi_1}(\xi), \tag{2.3}$$

and

$$\widehat{u_2}(x,\xi) = \frac{\sinh\left(x\sqrt{|\xi|^2-k^2}\right)}{\sqrt{|\xi|^2-k^2}}\,\widehat{\varphi_2}(\xi) =: \kappa_2(x,\xi)\widehat{\varphi_2}(\xi). \tag{2.4}$$

Suppose that the *a priori* bounds are:

$$\|u_1(1,\cdot)\| \leqslant E \quad \text{and} \quad \|u_2(1,\cdot)\| \leqslant E. \tag{2.5}$$

From the solutions (2.3) and (2.4), we can see that these Cauchy problems are both severely ill-posed, please see [26,10] in detail. To numerically solve such ill-posed problems from the noisy data ${\varphi_1}^\delta$, ${\varphi_2}^\delta$ in a stable way, some regularization methods should be applied. In this paper, we propose the following modified Tikhonov regularization method:

$$\widehat{u_j}^{\delta}_{\alpha_j,MTik}(x,\xi) = \begin{cases} \frac{\kappa_j(x,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi_j^\delta}(\xi), & |\xi|>k,\\ \kappa_j(x,\xi)\widehat{\varphi_j^\delta}(\xi), & |\xi|\leqslant k, \end{cases} \tag{2.6}$$

for $j = 1,2$, and $0 < x < 1$.

For the convenience of the discussion later, we give some auxiliary lemmas.

Lemma 2.1. *The following results are obvious:*

(a) $\kappa_1(x,\xi) = \begin{cases} \cosh(x\sqrt{|\xi|^2-k^2}), & |\xi|\geqslant k,\\ \cos(x\sqrt{k^2-|\xi|^2}), & |\xi|<k; \end{cases}$

(b) $\frac{e^s}{2} \leqslant \cosh(s) \leqslant e^s$ *for* $s \geqslant 0$.

Lemma 2.2. *For $s > 0$, $0 < \alpha < 1$ and $0 < x < 1$, the following inequalities hold.*

(a) $\frac{\sinh(xs)}{s} \leqslant e^{xs}$;

(b) $\frac{\sinh(xs)}{\sinh(s)} \leqslant e^{(x-1)s}$;

(c) *For* $g_1(s) := \frac{\frac{\sinh(xs)}{s}}{1+\alpha_2(\frac{\sinh(s)}{s})^2}$, *there holds* $g_1(s) \leqslant \left(\frac{1}{2\sqrt{\alpha}}\right)^x$;

(d) *For* $g_2(s) := \frac{\alpha\frac{\sinh(s)\sinh(xs)}{s\ \ s}}{1+\alpha(\frac{\sinh(s)}{s})^2}$, *there holds* $g_2(s) \leqslant (\frac{2}{\sqrt{\alpha}})^{x-1}$.

Proof.

(a) Based on the Taylor expansion of sinh (xs), there holds:

$$\frac{\sinh(xs)}{s} = \frac{1}{s}\sum_{n=0}^{\infty}\frac{(xs)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(s)^{2n}x^{2n+1}}{(2n+1)!} \leqslant \sum_{n=0}^{\infty}\frac{(xs)^{2n}}{(2n)!} = \cosh(xs) \leqslant e^{xs}.$$

(b) It is straightforward.

(c) If $s \geqslant \ln\frac{1}{2\sqrt{\alpha}}$, according to (b) $\frac{\sinh(xs)}{\sinh(s)} \leqslant e^{(x-1)s}$, we obtain:

$$g_1(s) \leqslant \frac{\frac{\sinh(xs)}{s}}{2\sqrt{\alpha\left(\frac{\sinh(s)}{s}\right)^2}} = \frac{\sinh(xs)}{2\sqrt{\alpha}\sinh(s)} \leqslant \frac{e^{(x-1)s}}{2\sqrt{\alpha}} \leqslant \left(\frac{1}{2\sqrt{\alpha}}\right)^x.$$

If $0 < s \leqslant \ln\frac{1}{2\sqrt{\alpha}}$, by virtue of (a) $\frac{\sinh(xs)}{s} \leqslant e^{xs}$, we obtain:

$$g_1(s) \leqslant \frac{\sinh(xs)}{s} \leqslant e^{xs} \leqslant \left(\frac{1}{2\sqrt{\alpha}}\right)^x.$$

(d) For $s \geqslant \ln\frac{2}{\alpha}$, there holds:

$$g_2(s) \leqslant \frac{\sinh(xs)}{\sinh(s)} \leqslant e^{(x-1)s} \leqslant \left(\frac{2}{\sqrt{\alpha}}\right)^{x-1}.$$

For $0 < s \leqslant \ln\frac{2}{\alpha}$, the estimate can be obtained:

$$g_2(s) = \frac{\alpha \frac{\sinh(xs)}{s}}{\frac{1}{\frac{\sinh(s)}{s}} + \alpha \frac{\sinh(s)}{s}} \leqslant \frac{\alpha \frac{\sinh(xs)}{s}}{2\sqrt{\alpha}} \leqslant \frac{\sqrt{\alpha}}{2} e^{xs} \leqslant \left(\frac{2}{\sqrt{\alpha}}\right)^{x-1}.$$

The lemma is proved. □

Lemma 2.3. *For s > 0 and 0 < x < 1, the following monotonicities hold.*

(a) *For $f_1(s) := \sinh(s) - s\cosh(s)$, it is strictly monotonically decreasing, and $f_1(s) < 0$.*
(b) *For $f_2(s) := \frac{s^2}{\sinh^2(xs)}$, it is strictly monotonically decreasing.*
(c) *For $f_3(s) := s\coth(s)$, it is strictly monotonically increasing.*
(d) *For $f_4(s) := 4s\sinh(s)\cosh(s) - s^2\sinh^2(s) - 3\sinh^2(s) - s^2\cosh^2(s)$, it is strictly monotonically decreasing.*
(e) *For $f_5(x) := \left(\frac{\sinh(xs)}{s}\right)^{\frac{1}{x}}$, it is strictly monotonically increasing.*

Proof. Simple computations show that the results (a)–(d) are straightforward. Now we only consider (e).

Note that $\ln f_5(x) = \frac{1}{x}\ln\frac{\sinh(xs)}{s}$, we have $\frac{f_5'(x)}{f_5(x)} = -\frac{1}{x^2}\ln\frac{\sinh(xs)}{s} + \frac{s\cosh(xs)}{x\sinh(xs)}$, which implies that $f_5'(x) = f_5(x)(-\frac{1}{x^2}\ln\frac{\sinh(xs)}{s} + \frac{s\cosh(xs)}{x\sinh(xs)})$. From $f_5(x) > 0, 0 < x < 1$ and $-\frac{1}{x^2}\ln\frac{\sinh(xs)}{s} + \frac{s\cosh(xs)}{x\sinh(xs)} = \frac{1}{x}(s\frac{\cosh(xs)}{\sinh(xs)} - \frac{1}{x}\ln\frac{\sinh(xs)}{s})$ we obtain that $f_5'(x) > 0$ is equivalent to $s\frac{\cosh(xs)}{\sinh(xs)} - \frac{1}{x}\ln\frac{\sinh(xs)}{s} > 0$.

If $\ln\frac{\sinh(xs)}{s} < 0, s\frac{\cosh(xs)}{\sinh(xs)} - \frac{1}{x}\ln\frac{\sinh(xs)}{s} > 0$ is obvious;
If $\ln\frac{\sinh(xs)}{s} > 0$, since $\frac{\sinh(xs)}{s} \leqslant e^{xs}$ (see Lemma 2.2 (a)), then:

$$s\frac{\cosh(xs)}{\sinh(xs)} - \frac{1}{x}\ln\frac{\sinh(xs)}{s} \geqslant s\frac{\cosh(xs)}{\sinh(xs)} - \frac{1}{x}\ln e^{xs} = s\frac{\cosh(xs)}{\sinh(xs)} - s > 0.$$

Consequently, $f_5(x)$ is strictly monotonically increasing. □

Lemma 2.4. *The following properties of $\kappa_j(x,\xi)$ are obvious:*

(a) $\kappa_2(x,\xi) = \begin{cases} \frac{\sinh\left(x\sqrt{|\xi|^2-k^2}\right)}{\sqrt{|\xi|^2-k^2}}, & |\xi| \geqslant k, \\ \frac{\sin\left(x\sqrt{k^2-|\xi|^2}\right)}{\sqrt{k^2-|\xi|^2}}, & |\xi| < k; \end{cases}$
(b) $|\kappa_2(x,\xi)| \leqslant x$ *for* $|\xi| \leqslant k$;
(c) $\lim_{|\xi|\to\infty} \kappa_j(x,\xi) = \infty$;
(d) $\frac{\alpha_j|\kappa_j(1,\xi)|^2}{1+\alpha_j|\kappa_j(1,\xi)|^2} \leqslant 1$;
(e) $\frac{\alpha_j|\kappa_j(1,\xi)|}{1+\alpha_j|\kappa_j(1,\xi)|^2} \leqslant \frac{\sqrt{\alpha_j}}{2}$.

3. Optimal error bound

Consider an ill-posed operator equation [45–49]:

$$Ax = y, \tag{3.1}$$

where $A : X \to Y$ is a linear bounded operator between infinite dimensional Hilbert spaces X and Y with non-closed range in Y. Any operator $R : Y \to X$ can be considered as a special method for approximately solving (3.1) and the approximate solution is denoted by Ry^δ, where the noisy data $y^\delta \in Y$ satisfy:

$$\|y - y^\delta\| \leqslant \delta. \tag{3.2}$$

Let $M \subset X$ be a bounded set. The worst case error $\Delta(\delta, R)$ on the set M for identifying x from y^δ is

$$\Delta(\delta,R) := \sup\left\{\|Ry^\delta - x\| \,|\, x \in M, y^\delta \in Y, \|Ax - y^\delta\| \leqslant \delta\right\}. \tag{3.3}$$

The best possible error bound (or optimal error bound) is defined as the infimum over all mappings $R : Y \to X$:

$$\omega(\delta) := \inf_R \Delta(\delta, R). \tag{3.4}$$

Now we review an optimality result for the source set $M = M_{\varphi,E}$ which is given by [49]:

$$M_{\varphi,E} = \left\{x \in X | x = [\varphi(A^*A)]^{\frac{1}{2}} v, \|v\| \leqslant E\right\}. \tag{3.5}$$

The operator function $\varphi(A^*A)$ is well defined via the representation $\varphi(A^*A) = \int_0^a \varphi(\lambda)dE_\lambda$, where $A^*A = \int_0^a \lambda dE_\lambda$ is the spectral decomposition of A^*A, $\{E_\lambda\}$ denotes the spectral family of the operator A^*A, and a is a constant such that $\|A^*A\| \leqslant a$.

In order to derive an explicit (best possible) optimal error bound for the worst case error $\Delta(\delta,R)$ defined in (3.3), the following assumption is given in [48,49]:

Assumption 3.1. The function $\varphi(\lambda)$ in (3.5): $(0,a] \to (0,\infty)$, where a is a constant with $\|A^*A\| \leqslant a$, is continuous and satisfies:

(i) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$;
(ii) φ is strictly monotonically increasing on $(0,a]$;
(iii) $\rho(\lambda) := \lambda\varphi^{-1}(\lambda)$: $(0,\varphi(a)] \to (0,a\varphi(a)]$ is convex.

Rewrite Eq. (2.4) as an operator equation:

$$A_x\widehat{u_2}(x,\xi) = \widehat{\varphi_2}(\xi), \tag{3.6}$$

where A_x is a multiplication operator with parametric variable x as the follows:

$$A_x^* = A_x = \begin{cases} \frac{\sqrt{|\xi|^2-k^2}}{\sinh(x\sqrt{|\xi|^2-k^2})}, & |\xi| \geqslant k, \\ \frac{\sqrt{k^2-|\xi|^2}}{\sin(x\sqrt{k^2-|\xi|^2})}, & |\xi| < k; \end{cases} \tag{3.7}$$

and

$$A_x^{-1} = \begin{cases} \frac{\sinh(x\sqrt{|\xi|^2-k^2})}{\sqrt{|\xi|^2-k^2}}, & |\xi| \geqslant k, \\ \frac{\sin(x\sqrt{k^2-|\xi|^2})}{\sqrt{k^2-|\xi|^2}}, & |\xi| < k. \end{cases} \tag{3.8}$$

For treating the ill-posed part, we need to transform the *a priori* bound (2.5) into an equivalent form in the frequency domain:

$$M_{\varphi,E} = \left\{ \widehat{u_2}(x,\cdot) \in L^2(\mathbb{R}^n) | \widehat{u_2}(x,\xi) = \frac{\sinh(x\sqrt{|\xi|^2-k^2})}{\sinh(\sqrt{|\xi|^2-k^2})}\widehat{u_2}(1,\xi), \|\widehat{u_2}(1,\xi)\| \leqslant E \right\}. \tag{3.9}$$

So, if we set $\eta := \sqrt{|\xi|^2-k^2}$, the function $\varphi(\lambda)$ in (3.5) possesses the parameter representation:

$$\begin{cases} \lambda(\eta) = \frac{\eta^2}{\sinh^2(x\eta)}, \\ \varphi(\eta) = \frac{\sinh^2(x\eta)}{\sinh^2(\eta)}. \end{cases} \tag{3.10}$$

From Lemma 2.4(b), (c) and (3.8), we know that problem (1.4) is ill-posed for $|\xi| \geqslant k$. Therefore we only consider the case $\eta > 0$. Corresponding to Assumption 3.1, $\varphi(\lambda)$ should possess the following properties.

Lemma 3.2. *For $0 < x < 1$ and $\eta > 0$, the function $\varphi(\lambda)$ is continuous and satisfies the properties:*

(i) *$\lim_{\lambda\to 0}\varphi(\lambda) = 0$.*
(ii) *$\varphi(\lambda)$ is strictly monotonically increasing.*
(iii) *$\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$ is strictly monotonically increasing and possesses the parameter representation:*

$$\begin{cases} \lambda(\eta) = \frac{\sinh^2(x\eta)}{\sinh^2(\eta)}, \\ \rho(\eta) = \frac{\eta^2}{\sinh^2(\eta)}. \end{cases} \tag{3.11}$$

(iv) *$\rho^{-1}(\lambda)$ is strictly monotonically increasing and possesses the parameter representation:*

$$\begin{cases} \lambda(\eta) = \frac{\eta^2}{\sinh^2(\eta)}, \\ \rho^{-1}(\eta) = \frac{\sinh^2(x\eta)}{\sinh^2(\eta)}. \end{cases} \tag{3.12}$$

(v) *For any fixed $x \in (0,1)$, the inverse function ρ^{-1} of ρ satisfies:*

$$\rho^{-1}(\lambda) = \left(\frac{\lambda}{4}\right)^{1-x}\left(\ln\frac{1}{\sqrt{\lambda}}\right)^{2(x-1)}(1+o(1)), \quad \text{for } \lambda \to 0. \tag{3.13}$$

(vi) *The function $\rho(\lambda)$ given by (3.11) is strictly convex.*

Proof. A similar result is outlined in [48] but without proof. For completeness, we give a complete proof here:

(i) Consider $\lambda(\eta)$ given by (3.10), from Lemma 2.3 (b) we realize that $\lambda(\eta)$ is strictly monotonically decreasing with $\lim_{\eta\to\infty}\lambda(\eta) = 0$. Therefore,

$$\lim_{\lambda\to 0}\varphi(\lambda) = \lim_{\eta\to\infty}\frac{\sinh^2(x\eta)}{\sinh^2(\eta)} = 0, \quad \text{as } 0 < x < 1.$$

(ii) Noting that $s\coth(s)$ is strictly monotonically increasing (see Lemma 2.3 (c)):

$$\dot{\varphi}(\eta) = \frac{2\sinh^2(x\eta)}{\eta\sinh^2(\eta)}(x\eta\coth(x\eta) - \eta\coth(\eta)) < 0.$$

Combining with $\dot{\lambda} < 0$, it is easy to see that $\varphi\prime(\lambda) = \frac{\dot{\varphi}(\eta)}{\dot{\lambda}(\eta)} > 0$.

(iii) and (iv) are obvious.

(v) We only need to prove that $\lim_{\lambda\to 0}F(\lambda) = 1$, where

$$F(\lambda) := \rho^{-1}(\lambda)\Bigg/\left(\left(\frac{\lambda}{4}\right)^{1-x}\left(\ln\frac{1}{\sqrt{\lambda}}\right)^{2(x-1)}\right).$$

Using (3.12) and noting $\lambda(\eta)$ is strictly monotonically decreasing with $\lim_{\eta\to\infty}\lambda(\eta) = 0$, we have:

$$\lim_{\lambda\to 0}F(\lambda) = \lim_{\eta\to\infty}\frac{\sinh^2(x\eta)}{\sinh^2(\eta)}\Bigg/\left[\left(\frac{\eta^2}{4\sinh^2(\eta)}\right)^{1-x}\left(\ln\frac{\sinh(\eta)}{\eta}\right)^{2(x-1)}\right]$$
$$= \lim_{\eta\to\infty}\frac{4^{1-x}\sinh^2(x\eta)\sinh^{2(1-x)}(\eta)}{\sinh^2(\eta)}\Bigg/\left[\eta^{2(1-x)}\left(\ln\frac{\cosh(\eta)}{1}\right)^{2(x-1)}\right] = 1.$$

(vi) From $\rho''(\lambda) = \frac{\ddot{\rho}\dot{\lambda} - \dot{\rho}\ddot{\lambda}}{\dot{\lambda}^3}$ and $\dot{\lambda} < 0$ we obtain that $\rho'' > 0$ is equivalent to $\ddot{\rho}\dot{\lambda} < \dot{\rho}\ddot{\lambda}$. Note that $\lambda(\eta) = \rho(\eta)r(\eta)$ with $r(\eta) = \frac{\sinh^2(x\eta)}{\eta^2}$. From Lemma 2.3(a), we know that $\dot{r}(\eta) = \frac{2\sinh(x\eta)}{\eta^3}(x\eta\cosh(x\eta) - \sinh(x\eta)) > 0$. Hence the inequality $\rho''(\lambda) > 0$ is equivalent to the inequality:

$$\rho\ddot{\rho} - 2\dot{\rho}^2 < \rho\dot{\rho}\frac{\ddot{r}}{\dot{r}}. \tag{3.14}$$

After simply computing, there are:

$$\dot{\rho} = \frac{2\eta(\sinh(\eta) - \eta\cosh(\eta))}{\sinh^3(\eta)},$$

$$\ddot{\rho} = \frac{2\sinh^2(\eta) - 8\eta\sinh(\eta)\cosh(\eta) + 6\eta^2\cosh^2(\eta) - 2\eta^2\sinh^2(\eta)}{\sinh^4(\eta)},$$

$$\ddot{r} = \frac{2x^2\eta^2\cosh^2(x\eta) + 2x^2\eta^2\sinh^2(x\eta) - 8x\eta\sinh(x\eta)\cosh(x\eta) + 6\sinh^2(x\eta)}{\eta^4}.$$

Therefore, (3.14) is equivalent to:

$$\frac{4\eta\sinh(\eta)\cosh(\eta) - \eta^2\sinh^2(\eta) - 3\sinh^2(\eta) - \eta^2\cosh^2(\eta)}{\sinh(\eta)(\eta\cosh(\eta) - \sinh(\eta))}$$
$$< \frac{4x\eta\sinh(x\eta)\cosh(x\eta) - (x\eta)^2\sinh^2(x\eta) - 3\sinh^2(x\eta) - (x\eta)^2\cosh^2(x\eta)}{\sinh(x\eta)(x\eta\cosh(x\eta) - \sinh(x\eta))}. \tag{3.15}$$

Lemma 2.3(a) and (d) show that (3.15) holds. □

Under Assumption 3.1, the next theorem gives a general formula for the optimal error bound.

Theorem 3.3 [49]. *Let $M_{\varphi_1,E}$ be given by (3.5) and Assumption 3.1 be satisfied. Moreover, let $\frac{\delta^2}{E^2} \in \sigma(A^*A\varphi_1(A^*A))$, where $\sigma(A^*A)$ denotes the spectrum of operator A^*A. Then there holds:*

$$\omega(\delta,E)=E\sqrt{\rho^{-1}\left(\frac{\delta^2}{E^2}\right)}. \tag{3.16}$$

Based on Lemma 3.2 and Theorem 3.3, we can obtain the optimal error bound for problem (1.4).

Theorem 3.4. *Suppose conditions (1.2) and (2.5) hold. Then the optimal error bound for solving problem (1.4) is:*

$$\omega(\delta,E)=E^x\left(\frac{\delta}{2}\right)^{1-x}\left(\ln\frac{E}{\delta}\right)^{x-1}(1+o(1)),\quad \text{for } \delta\to 0, 0<x<1. \tag{3.17}$$

Proof. For $|\xi|>k$, based on Lemma 3.2 and Theorem 3.3, we know that for $\delta\to 0$:

$$\omega_1(\delta,E)=E\sqrt{\left(\frac{\delta^2}{4E^2}\right)^{1-x}\left(\ln\frac{1}{\sqrt{\delta^2/E^2}}\right)^{2(x-1)}(1+o(1))}=E^x\left(\frac{\delta}{2}\right)^{1-x}\left(\ln\frac{E}{\delta}\right)^{x-1}(1+o(1)).$$

For $|\xi|\leqslant k$, from Lemma 2.4(a), (b) and (1.2), we obtain:

$$\omega_2(\delta,E)\leqslant\int_{|\xi|\leqslant k}\left(\kappa_2(x,\xi)\widehat{\varphi_2^\delta}(\xi)-\kappa_2(x,\xi)\widehat{\varphi_2}(\xi)\right)d\xi\leqslant x\delta\leqslant\delta.$$

Then the optimal error bound for solving problem (1.4) is

$$\omega(\delta,E)=\omega_1(\delta,E)+\omega_2(\delta,E)=E^x\left(\frac{\delta}{2}\right)^{1-x}\left(\ln\frac{E}{\delta}\right)^{x-1}(1+o(1)),\quad \text{for } \delta\to 0, 0<x<1. \qquad \square$$

Remark 3.5. Xiong and Fu [10] have proved that under the assumption (2.5) problem (1.3) has the optimal error bound:

$$\omega(\delta,E)=E^x\delta^{1-x}(1+o(1))\quad \text{for } \delta\to 0,$$

which is not right and should be (see e.g. [48])

$$\omega(\delta,E)=E^x\left(\frac{\delta}{2}\right)^{1-x}(1+o(1))\quad \text{for } \delta\to 0. \tag{3.18}$$

Remark 3.6. Comparing (3.17) with (3.18), we know that the optimal error bound for problem (1.4) is "better" than that of problem (1.3), which implies that the ill-posedness of problem (1.4) is not stronger than that of problem (1.3).

4. The *a priori* parameter choice

This section is devoted to the *a priori* parameter choices of the modified Tikhonov regularization method (2.6).

The *a priori* parameter choice of (2.6) for $j = 1$ is similar to what was presented in Theorem 3.4 in [10]. Here we omit it and are only interested in the *a priori* parameter choice of problem (1.4).

Using the Parseval formula and the triangle inequality, we know that:

$$\left\|u^\delta_{2_{\alpha_2,MTik}}(x,\cdot)-u_2(x,\cdot)\right\|=\left\|\widehat{u_2}^\delta_{\alpha_2,MTik}(x,\cdot)-\widehat{u_2}(x,\cdot)\right\|\leqslant\left\|\widehat{u_2}^\delta_{\alpha_2,MTik}(x,\cdot)-\widehat{u_2}_{\alpha_2,MTik}(x,\cdot)\right\|+\left\|\widehat{u_2}_{\alpha_2,MTik}(x,\cdot)-\widehat{u_2}(x,\cdot)\right\|. \tag{4.1}$$

For the first part of the right hand side above, applying Lemma 2.2(c) and Lemma 2.4(a), (b), we have:

$$\begin{aligned}\left\|\widehat{u_2}^\delta_{\alpha_2,MTik}(x,\cdot)-\widehat{u_2}_{\alpha_2,MTik}(x,\cdot)\right\|&\leqslant\left(\int_{|\xi|>k}\left|\frac{\kappa_2(x,\xi)}{1+\alpha_2|\kappa_2(1,\xi)|^2}\left(\widehat{\varphi_2}^\delta(\xi)-\widehat{\varphi_2}(\xi)\right)\right|^2d\xi\right)^{\frac{1}{2}}\\&\quad+\left(\int_{|\xi|\leqslant k}\left|\kappa_2(x,\xi)\left(\widehat{\varphi_2}^\delta(\xi)-\widehat{\varphi_2}(\xi)\right)\right|^2d\xi\right)^{\frac{1}{2}}\\&\leqslant\frac{\delta}{(2\sqrt{\alpha_2})^x}+\delta.\end{aligned} \tag{4.2}$$

For the second term of the right hand side in (4.1), according to Lemma 2.2 (d) and (2.5), there holds:

$$\|\widehat{u_2}_{\alpha_2,MTik}(x,\cdot)-\widehat{u_2}(x,\cdot)\| = \left(\int_{|\xi|>k}\left|\frac{\kappa_2(x,\xi)}{1+\alpha_2|\kappa_2(1,\xi)|^2}\widehat{\varphi_2}(\xi)-\kappa_2(x,\xi)\widehat{\varphi_2}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}$$
$$= \left(\int_{|\xi|>k}\left|\frac{\alpha_2|\kappa_2(1,\xi)|\kappa_2(x,\xi)}{1+\alpha_2|\kappa_2(1,\xi)|^2}\widehat{u_2}(1,\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \leqslant \left(\frac{2}{\sqrt{\alpha_2}}\right)^{x-1}E \leqslant \left(\frac{1}{2\sqrt{\alpha_2}}\right)^{x-1}E. \tag{4.3}$$

Inserting (4.2) and (4.3) into (4.1), we obtain the following main theorem:

Theorem 4.1. *Let $u^{\delta}_{2_{\alpha_2},MTik}(x,y)$ defined by (2.6) be the modified Tikhonov regularization solution and $u_2(x,y)$ be the exact solution of problem (1.4). If conditions (1.2) and (2.5) hold and we choose:*

$$\alpha_2 = \left(\frac{\delta}{2E}\right)^2, \tag{4.4}$$

then there holds error estimate:

$$\left\|u^{\delta}_{2_{\alpha_2},MTik}(x,\cdot)-u_2(x,\cdot)\right\| \leqslant 2\delta^{1-x}E^x+\delta, \quad 0<x<1. \tag{4.5}$$

5. The *a posteriori* parameter choice

In this section, we consider the *a posteriori* regularization parameter choice in the Morozov's discrepancy principle. A framework of the modified Tikhonov regularization method proposed in (2.6) is given and applied to the ill-posed problems (1.3) and (1.4).

Lemma 5.1. *Set $\varrho_1(\alpha_1)=\|u^{\delta}_{1_{\alpha_1},MTik}(0,\cdot)-\varphi_1^{\delta}(\cdot)\|$ and $\varrho_2(\alpha_2)=\|(u^{\delta}_{2_{\alpha_2},MTik})_x(0,\cdot)-\varphi_2^{\delta}(\cdot)\|$. If $0<\delta<\|\varphi_j^{\delta}\|$ for j = 1, 2, then there hold:*

(a) *$\varrho_j(\alpha_j)$ is a continuous function;*
(b) *$\lim_{\alpha_j\to 0^+}\varrho_j(\alpha_j)=0$;*
(c) *$\lim_{\alpha_j\to+\infty}\varrho_j(\alpha_j)=\|\varphi_j^{\delta}\|$;*
(d) *$\varrho_j(\alpha_j)$ is a strictly increasing function.*

Proof. According to the Parseval formula (2.2), the above results are straightforward. □

Lemma 5.2. *For $0<x<1$, there hold:*

$$\begin{cases} |\kappa_1(x,\xi)|^{\frac{1}{x}} \leqslant 2|\kappa_1(1,\xi)|, & \text{for } |\xi|>k,\\ |\kappa_1(x,\xi)| \leqslant 1, & \text{for } |\xi|\leqslant k, \end{cases} \tag{5.1}$$

and

$$\begin{cases} |\kappa_2(x,\xi)|^{\frac{1}{x}} \leqslant |\kappa_2(1,\xi)|, & \text{for } |\xi|>k,\\ |\kappa_2(x,\xi)| \leqslant 1, & \text{for } |\xi|\leqslant k. \end{cases} \tag{5.2}$$

Proof. From Lemma 2.1, we know that for $|\xi|>k$:

$$|\kappa_1(x,\xi)|^{\frac{1}{x}} = \left(\cosh\left(x\sqrt{|\xi|^2-k^2}\right)\right)^{\frac{1}{x}} \leqslant \left(\exp\left(x\sqrt{|\xi|^2-k^2}\right)\right)^{\frac{1}{x}} = \exp\left(\sqrt{|\xi|^2-k^2}\right) \leqslant 2\left(\cosh\left(\sqrt{|\xi|^2-k^2}\right)\right)$$
$$= 2|\kappa_1(x,\xi)|,$$

for $|\xi|\leqslant k$, $|\kappa_1(x,\xi)| = |\cos(x\sqrt{k^2-|\xi|^2})| \leqslant 1$.

The result (5.2) can be easily deduced by Lemma 2.3(e) and Lemma 2.4(a), (b). □

Lemma 5.3. *Set $z_j(x,y)=u_j(x,y)-u^{\delta}_{j_{\alpha_j},MTik}(x,y)$ for j = 1, 2. For $0<x<1$, there holds:*

$$\begin{cases} \|z_1(x,\cdot)\| \leqslant 2^x \|z_1(1,\cdot)\|^x \|z_1(0,\cdot)\|^{1-x} + \delta, \\ \|z_2(x,\cdot)\| \leqslant \|z_2(1,\cdot)\|^x \|(z_2)_x(0,\cdot)\|^{1-x} + \delta. \end{cases} \tag{5.3}$$

Proof. Since $z_j(x,y) = u_j(x,y) - u^{\delta}_{j_{\alpha_j,MTik}}(x,y)$, we have:

$$\begin{aligned} \|z_j(x,\cdot)\|^2 &= \|\hat{z}_j(x,\cdot)\|^2 = \left\|\widehat{u_j}(x,\cdot) - \widehat{u_j}^{\delta}_{\alpha_j,MTik}(x,\cdot)\right\|^2 \\ &= \int_{|\xi|>k} |\kappa_j(x,\xi)|^2 \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^2 d\xi + \int_{|\xi|\leqslant k} |\kappa_j(x,\xi)|^2 \left|\widehat{\varphi_j}(\xi) - \widehat{\varphi_j^{\delta}}(\xi)\right|^2 d\xi. \end{aligned} \tag{5.4}$$

For the first term on the righthand side of (5.4), from the Hölder inequality and Lemma 5.2, we have:

$$\begin{aligned} \int_{|\xi|>k} |\kappa_j(x,\xi)|^2 \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^2 d\xi &= \int_{|\xi|>k} |\kappa_j(x,\xi)|^2 \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^{2x} \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^{2(1-x)} d\xi \\ &\leqslant \left(\int_{|\xi|>k} \left(|\kappa_j(x,\xi)|^2 \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^{2x}\right)^{\frac{1}{x}} d\xi\right)^x \\ &\quad \cdot \left(\int_{|\xi|>k} \left(\left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^{2(1-x)}\right)^{\frac{1}{1-x}} d\xi\right)^{1-x} \\ &= \left(\int_{|\xi|>k} |\kappa_j(x,\xi)|^{\frac{2}{x}} \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^2 d\xi\right)^x \left(\int_{|\xi|>k} \left|\widehat{\varphi_j}(\xi) - \frac{\widehat{\varphi_j^{\delta}}(\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\right|^2 d\xi\right)^{1-x} \\ &\leqslant \begin{cases} \left(\int_{|\xi|>k} |2\kappa_1(1,\xi)|^2 \left|\widehat{\varphi_1}(\xi) - \frac{\widehat{\varphi_1^{\delta}}(\xi)}{1+\alpha_1|\kappa_1(1,\xi)|^2}\right|^2 d\xi\right)^x \left(\int_{|\xi|>k} \left|\widehat{\varphi_1}(\xi) - \frac{\widehat{\varphi_1^{\delta}}(\xi)}{1+\alpha_1|\kappa_1(1,\xi)|^2}\right|^2 d\xi\right)^{1-x}, \\ \left(\int_{|\xi|>k} |\kappa_2(1,\xi)|^2 \left|\widehat{\varphi_2}(\xi) - \frac{\widehat{\varphi_2^{\delta}}(\xi)}{1+\alpha_2|\kappa_2(1,\xi)|^2}\right|^2 d\xi\right)^x \left(\int_{|\xi|>k} \left|\widehat{\varphi_2}(\xi) - \frac{\widehat{\varphi_2^{\delta}}(\xi)}{1+\alpha_2|\kappa_2(1,\xi)|^2}\right|^2 d\xi\right)^{1-x}, \end{cases} \end{aligned} \tag{5.5}$$

which deduces:

$$\begin{cases} \int_{|\xi|>k} |\kappa_1(x,\xi)|^2 \left|\widehat{\varphi_1}(\xi) - \frac{\widehat{\varphi_1}^{\delta}(\xi)}{1+\alpha_1|\kappa_1(1,\xi)|^2}\right|^2 d\xi \leqslant 2^{2x}\|z_1(1,\cdot)\|^{2x}\|z_1(0,\cdot)\|^{2(1-x)}, \\ \int_{|\xi|>k} |\kappa_2(x,\xi)|^2 \left|\widehat{\varphi_2}(\xi) - \frac{\widehat{\varphi_2^{\delta}}(\xi)}{1+\alpha_2|\kappa_2(1,\xi)|^2}\right|^2 d\xi \leqslant \|z_2(1,\cdot)\|^{2x}\|(z_2)_x(0,\cdot)\|^{2(1-x)}. \end{cases} \tag{5.6}$$

For the second term on the righthand side of (5.4), according to Lemma 5.2, there is

$$\int_{|\xi|\leqslant k} |\kappa_j(x,\xi)|^2 \left|\widehat{\varphi_j}(\xi) - \widehat{\varphi_j^{\delta}}(\xi)\right|^2 d\xi \leqslant \int_{|\xi|\leqslant k} \left|\widehat{\varphi_j}(\xi) - \widehat{\varphi_j^{\delta}}(\xi)\right|^2 d\xi \leqslant \delta^2. \tag{5.7}$$

Thus we get the desired result (5.3) from (5.4)–(5.7). □

Lemma 5.4. *Choose $\tau > 1$ such that $0 < \tau\delta < \|\varphi_j^{\delta}\|$ for j = 1,2. Then there exists the unique regularization parameter $\alpha_{j\delta} > 0$ such that:*

$$\begin{cases} \left\|u^{\delta}_{1_{\alpha_1,MTik}}(0,\cdot) - \varphi_1^{\delta}(\cdot)\right\| = \tau\delta, \\ \left\|(u^{\delta}_{2_{\alpha_2,MTik}})_x(0,\cdot) - \varphi_2^{\delta}(\cdot)\right\| = \tau\delta. \end{cases} \tag{5.8}$$

Furthermore, the following inequality holds:

$$\|z_j(1,\cdot)\| \leqslant \left(1+\sqrt{\frac{1+4(\tau-1)^2}{2(\tau-1)^2}}\right)\|u_j(1,\cdot)\|. \tag{5.9}$$

Proof. Lemma 5.1 implies that we can find the unique number $\alpha_{j\delta} > 0$ such that (5.8) holds.

It is easy to see that:

$$\|z_j(1,\cdot)\| = \left\|u_j(1,\cdot) - u^\delta_{j_{\alpha_j,MTik}}(1,\cdot)\right\| \leqslant \|u_j(1,\cdot)\| + \left\|u^\delta_{j_{\alpha_j,MTik}}(1,\cdot)\right\|, \tag{5.10}$$

and

$$\left\|u^\delta_{j_{\alpha_j},MTik}(1,\cdot)\right\|^2 = \left\|\widehat{u}^\delta_{j_{\alpha_j,MTik}}(1,\cdot)\right\|^2 = \int_{|\xi|>k}\left|\frac{\kappa_j(1,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi^\delta_j}(\xi)\right|^2 d\xi + \int_{|\xi|\leqslant k}\left|\kappa_j(1,\xi)\widehat{\varphi^\delta_j}(\xi)\right|^2 d\xi. \tag{5.11}$$

From Lemma 2.4 (e) it follows that:

$$\begin{aligned}
\int_{|\xi|>k}\left|\frac{\kappa_j(1,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi^\delta_j}(\xi)\right|^2 d\xi &= \int_{|\xi|>k}\left|\frac{\kappa_j(1,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}(\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi)+\widehat{\varphi_j}(\xi))\right|^2 d\xi \\
&\leqslant 2\left(\int_{|\xi|>k}\left|\frac{\kappa_j(1,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}(\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi))\right|^2 d\xi + \int_{|\xi|>k}\left|\frac{\kappa_j(1,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi_j}(\xi)\right|^2 d\xi\right) \\
&\leqslant 2\left(\frac{1}{4\alpha_j}\int_{|\xi|>k}\left|\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi)\right|^2 d\xi + \int_{|\xi|>k}\left|\frac{\kappa_j(1,\xi)}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi_j}(\xi)\right|^2 d\xi\right) \\
&\leqslant 2\left(\frac{1}{\alpha_j}\int_{|\xi|>k}\left|\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi)\right|^2 d\xi + \int_{|\xi|>k}\left|\kappa_j(1,\xi)\widehat{\varphi_j}(\xi)\right|^2 d\xi\right),
\end{aligned} \tag{5.12}$$

and from Lemma 5.2 there holds:

$$\begin{aligned}
\int_{|\xi|\leqslant k}\left|\kappa_j(1,\xi)\widehat{\varphi^\delta_j}(\xi)\right|^2 d\xi &\leqslant 2\left(\int_{|\xi|\leqslant k}\left|\kappa_j(1,\xi)(\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi))\right|^2 d\xi + \int_{|\xi|\leqslant k}\left|\kappa_j(1,\xi)\widehat{\varphi_j}(\xi)\right|^2 d\xi\right) \\
&\leqslant 2\left(\int_{|\xi|\leqslant k}\left|\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi)\right|^2 d\xi + \int_{|\xi|\leqslant k}\left|\kappa_j(1,\xi)\widehat{\varphi_j}(\xi)\right|^2 d\xi\right), \\
&\leqslant 2\left(\frac{1}{\alpha_j}\int_{|\xi|\leqslant k}\left|\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi)\right|^2 d\xi + \int_{|\xi|\leqslant k}\left|\kappa_j(1,\xi)\widehat{\varphi_j}(\xi)\right|^2 d\xi\right), \quad \text{as} \quad 0<\alpha_j<1.
\end{aligned} \tag{5.13}$$

Due to (5.11)–(5.13) and (1.2), we have:

$$\left\|u^\delta_{j_{\alpha_j},MTik}(1,\cdot)\right\|^2 \leqslant 2\left(\frac{\delta^2}{\alpha_j} + \|u_j(1,\cdot)\|^2\right). \tag{5.14}$$

According to (5.8) and Lemma 2.4 (d) and (e), there is

$$\begin{aligned}
\tau\delta &= \left(\int_{|\xi|>k}\left|\frac{\alpha_j|\kappa_j(1,\xi)|^2}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi^\delta_j}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\leqslant \left(\int_{|\xi|>k}\left|\frac{\alpha_j|\kappa_j(1,\xi)|^2}{1+\alpha_j|\kappa_j(1,\xi)|^2}(\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi))\right|^2 d\xi\right)^{1/2} + \left(\int_{|\xi|>k}\left|\frac{\alpha_j|\kappa_j(1,\xi)|^2}{1+\alpha_j|\kappa_j(1,\xi)|^2}\widehat{\varphi_j}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\leqslant \left(\int_{|\xi|>k}\left|\widehat{\varphi^\delta_j}(\xi)-\widehat{\varphi_j}(\xi)\right|^2 d\xi\right)^{1/2} + \left(\frac{\alpha_j}{4}\int_{|\xi|>k}\left|\kappa_j(1,\xi)\widehat{\varphi_j}(\xi)\right|^2 d\xi\right)^{1/2} \leqslant \delta + \frac{\sqrt{\alpha_j}}{2}\|u_j(1,\cdot)\|,
\end{aligned}$$

which implies $(\tau-1)\delta \leqslant \frac{\sqrt{\alpha_j}}{2}\|u_j(1,\cdot)\|$, i.e.,

$$\frac{\delta^2}{\alpha_j} \leqslant \frac{1}{4(\tau-1)^2}\|u_j(1,\cdot)\|^2. \tag{5.15}$$

From (5.10) (5.14) and (5.15), we conclude that (5.9) is proved. □

Recall from (5.8), (1.2) and the definition of $z_j(x,y)$ that:

$$\begin{aligned}\|z_1(0,\cdot)\| &= \|u_1(0,\cdot) - u^\delta_{1_{\alpha_1,MTik}}(0,\cdot)\| = \|\varphi_1(\cdot) - u^\delta_{1_{\alpha_1,MTik}}(0,\cdot)\| \leqslant \|\varphi_1(\cdot) - \varphi_1^\delta(\cdot)\| + \|\varphi_1^\delta(\cdot) - u^\delta_{1_{\alpha_1,MTik}}(0,\cdot)\| \\ &\leqslant (1+\tau)\delta,\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}\|(z_2)_x(0,\cdot)\| &= \|(u_2)_x(0,\cdot) - (u^\delta_{2_{\alpha_2,MTik}})_x(0,\cdot)\| = \|\varphi_2(\cdot) - (u^\delta_{2_{\alpha_2,MTik}})_x(0,\cdot)\| \\ &\leqslant \|\varphi_2(\cdot) - \varphi_2^\delta(\cdot)\| + \|\varphi_2^\delta(\cdot) - (u^\delta_{2_{\alpha_2,MTik}})_x(0,\cdot)\| \leqslant (1+\tau)\delta.\end{aligned} \tag{5.17}$$

Owing to (5.16), (5.17) and lemmas 5.3, 5.4, the main result of this section can be formulated as follows:

Theorem 5.5. *Suppose that the a priori bounds (2.5) and condition (5.8) hold, and there exists $\tau > 1$ such that $0 < \tau\delta < \|\varphi_j^\delta\|$. Then for $0 < x < 1$, there hold:*

$$\begin{cases}\left\|u_1(x,\cdot) - u^\delta_{1_{\alpha_1,MTik}}(x,\cdot)\right\| \leqslant 2^x\left(1 + \sqrt{\frac{1+4(\tau-1)^2}{2(\tau-1)^2}}\right)(1+\tau)^{1-x}E^x\delta^{1-x} + \delta, \\ \left\|u_2(x,\cdot) - u^\delta_{2_{\alpha_2,MTik}}(x,\cdot)\right\| \leqslant \left(1 + \sqrt{\frac{1+4(\tau-1)^2}{2(\tau-1)^2}}\right)(1+\tau)^{1-x}E^x\delta^{1-x} + \delta.\end{cases} \tag{5.18}$$

Remark 5.6. Theorem 5.5 implies that:

$$\|u_j(x,\cdot) - u^\delta_{j_{\alpha_j,MTik}}(x,\cdot)\| \leqslant C_j(x,\tau)E^x\delta^{1-x}(1+o(1)), \quad \text{for } 0 < x < 1, \tag{5.19}$$

where $C_1(x,\tau) := 2^x(1 + \sqrt{\frac{1+4(\tau-1)^2}{2(\tau-1)^2}})(1+\tau)^{1-x}, C_2(x,\tau) := \left(1 + \sqrt{\frac{1+4(\tau-1)^2}{2(\tau-1)^2}}\right)(1+\tau)^{1-x}$. Compared with (3.18) and (3.17), it is easy to know that our *a posteriori* method is order optimal under the *a priori* bound (2.5) for problem (1.3) but not for problem (1.4).

Remark 5.7. From Theorems 4.1 and 5.5, the *a priori* parameter choice rule gives the same convergence rate as the *a posteriori* parameter choice. However, such *a priori* information is rarely available in practice. We can not obtain the *a priori* bound since we do not know the exact solution in practice. This drawback is overcome by the *a posteriori* parameter choice.

Remark 5.8. From the proof of the above theorems, we can see that if the symbol $\kappa(x,\xi)$ of a linear ill-posed problem satisfies the following similar condition:

$$\begin{cases}|\kappa(x,\xi)|^{\frac{1}{x}} \leqslant c|\kappa(x,\xi)|, & \text{for } |\xi| > k, \\ |\kappa(x,\xi)| \leqslant d, & \text{for } |\xi| \leqslant k,\end{cases} \tag{5.20}$$

then our Modified Tikhonov method can be used for it. In addition, maybe this framework of the *a posteriori* parameter choice presented in this paper is helpful for the other regularization methods.

Remark 5.9. In fact, this method can treat general domain. If domain $\Omega \subset \mathbb{R}^n$ is open, connected and bounded, λ_n are the eigenvalues of the operator $-\triangle_y$ and $w_n(y)$ are the corresponding eigenfunctions. Then for the following problem:

$$\begin{cases}\Delta u(x,y) + k^2u(x,y) = 0, & x \in (0,1), \quad y \in \Omega \subset \mathbb{R}^n, \quad n \geqslant 1, \\ u(x,y) = 0, & x \in (0,1), \quad y \in \partial\Omega, \\ u(0,y) = \varphi_1(y), & y \in \Omega, \\ u_x(0,y) = \varphi_2(y), & y \in \Omega,\end{cases} \tag{5.21}$$

we can also divide it into two ill-posed problems:

$$\begin{cases}\Delta u_1(x,y) + k^2u_1(x,y) = 0, & x \in (0,1), \quad y \in \Omega \subset \mathbb{R}^n, \quad n \geqslant 1, \\ u_1(x,y) = 0, & x \in (0,1), \quad y \in \partial\Omega, \\ u_1(0,y) = \varphi_1(y), & y \in \Omega, \\ (u_1)_x(0,y) = 0, & y \in \Omega,\end{cases}$$

and

$$\begin{cases} \Delta u_2(x,y) + k^2 u_2(x,y) = 0, & x \in (0,1), \quad y \in \Omega \subset \mathbb{R}^n, \quad n \geqslant 1, \\ u_2(x,y) = 0, & x \in (0,1), \quad y \in \partial\Omega, \\ u_2(0,y) = 0, & y \in \Omega, \\ (u_2)_x(0,y) = \varphi_2(y), & y \in \Omega. \end{cases}$$

Using the properties of the eigenvalues λ_n and the eigenfunctions $w_n(y)$, we can obtain:

$$u_1(x,y) = \sum_{n=1}^{\infty} \cosh\left(x\sqrt{\lambda_n - k^2}\right)(\varphi_1, w_n)w_n,$$

and

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{\sinh\left(x\sqrt{\lambda_n - k^2}\right)}{\sqrt{\lambda_n - k^2}}(\varphi_2, w_n)w_n,$$

which are similar to in a "strip" domain. Then it is easy to use our method to treat problem (5.21). For the nonhomogeneous case:

$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = f(x,y), & x \in (0,1), \quad y \in \Omega \subset \mathbb{R}^n, \quad n \geqslant 1, \\ u(x,y) = g(x,y), & x \in (0,1), \quad y \in \partial\Omega, \\ u(0,y) = \varphi_1(y), & y \in \Omega, \\ u_x(0,y) = \varphi_2(y), & y \in \Omega, \end{cases} \tag{5.22}$$

according to the linear property, one can also divide it into one well-posed problem and problem (5.21). For the numerical aspect of this general domain, please refer to the paper [50].

6. Numerical implementation

In this section, we present the numerical implementation of the modified Tikhonov regularization method using the *a posteriori* parameter choice rule. We briefly describe the numerical implementation for the case $y \in \mathbb{R}^1$, although similar arguments apply for higher dimensions. Suppose that the vectors Φ and Ψ represent samples from the functions $\varphi_1(y)$ and $\varphi_2(y)$, then some normally distributed noises of variance ϵ are added to Φ and Ψ, and then we obtain the perturbation data Φ^δ and Ψ^δ, respectively. The following steps summarize the modified Tikhonov method using the *a posteriori* parameter choice rule in detail.

Step 1. Take the fast fourier transform (FFT) for the vector Φ^δ (or Ψ^δ).
Step 2. Choose $\tau = 1.1$ suggested by Hanke and Hansen [51] and Hanke [52], and use the bisection method (see e.g. [53]) to obtain the regularization parameters α_j according to criterion (5.8).
Step 3. Compute $\widehat{u_j}^\delta_{\alpha_j,MTik}(x,\xi)$ by (2.6).
Step 4. Take the inverse FFT for $\widehat{u_j}^\delta_{\alpha_j,MTik}(x,\xi)$ to get $u^\delta_{j_{\alpha_j,MTik}}(x,y)$.

Please refer to [54] for detailed instructions on the FFT technique. In order to investigate the convergence of the algorithm, we use R_j^δ to indicate the regularized solution of u_j and evaluate the absolute error $e_a(u_j)$ and the relative error $e_r(u_j)$ defined by

$$e_a(u_j) := \|R_j^\delta(x,\cdot) - u_j(x,\cdot)\|_{l^2},$$
$$e_r(u_j) := \frac{\|R_j^\delta(x,\cdot) - u_j(x,\cdot)\|_{l^2}}{\|u_j(x,\cdot)\|_{l^2}},$$

respectively. Here the $\|\cdot\|_{l^2}$ norm can be understood as

$$\|F\|_{l^2} := \sqrt{\frac{1}{M+1}\sum_{n=1}^{M+1}(F(n))^2}.$$

Example. The problem with analytical solution $u(x,y) = e^{-|y|}(\cos(\sqrt{\beta}x) + \sin(\sqrt{\beta}x))$, $\beta = k^2+1$.

$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}, \\ u(0,y) = e^{-|y|}, & y \in \mathbb{R}, \\ u_x(0,y) = \sqrt{\beta}e^{-|y|}, & y \in \mathbb{R}. \end{cases} \tag{6.1}$$

According to the analysis in Section 1, we only need to solve the following two problems:

$$\begin{cases} \Delta u_1(x,y) + k^2 u_1(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}, \\ u_1(0,y) = e^{-|y|}, & y \in \mathbb{R}, \\ (u_1)_x(0,y) = 0, & y \in \mathbb{R}, \end{cases} \tag{6.2}$$

$$\begin{cases} \Delta u_2(x,y) + k^2 u_2(x,y) = 0, & x \in (0,1), \quad y \in \mathbb{R}, \\ u_2(0,y) = 0, & y \in \mathbb{R}, \\ (u_2)_x(0,y) = \sqrt{\beta}e^{-|y|}, & y \in \mathbb{R}. \end{cases} \tag{6.3}$$

Table 1
The errors between the exact and approximate solutions of (6.1) with $k = 1$, $x = 0.5$ for different ϵ.

ϵ	α_1	$e_a(u_1)$	$e_r(u_1)$	α_2	$e_a(u_2)$	$e_r(u_2)$
1e−3	3.6380e−011	0.0065	0.0283	0.0156	0.0160	0.1083
1e−2	0.0625	0.0415	0.2417	0.0156	0.0162	0.1095
1e−1	0.0313	0.0679	0.3955	0.0313	0.0335	0.2268

Table 2
The errors between the exact and approximate solutions in (6.1) with $\epsilon = 10^{-3}$, $x = 0.5$ for different k.

k	α_1	$e_a(u_1)$	$e_r(u_1)$	α_2	$e_a(u_2)$	$e_r(u_2)$
0.5	3.6380e−011	0.0065	0.0282	0.0156	0.0124	0.1029
1.0	3.6380e−011	0.0065	0.0283	0.0156	0.0160	0.1083
1.5	4.3656e−011	0.0063	0.0275	0.0625	0.0199	0.1118
2.0	4.3656e−011	0.0065	0.0285	0.0625	0.0243	0.1191
3.0	8.7311e−011	0.0063	0.0286	0.1250	0.0316	0.1391
10	1.0997	0.0242	0.3491	0.8750	0.0101	0.0469
20	1.1000	0.0050	0.0264	0.9998	0.0122	0.0966

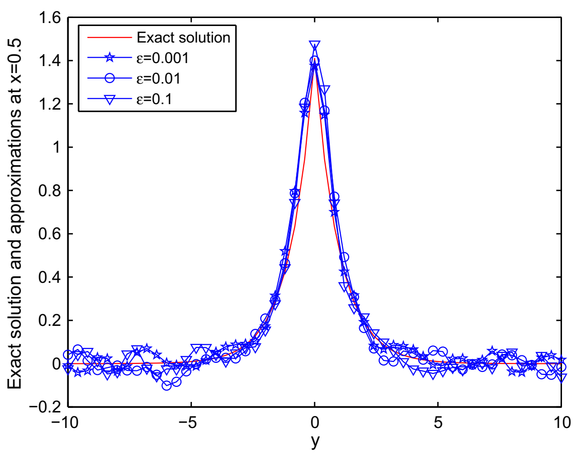

Fig. 1. Problem (6.1) at $x = 0.5$ with $k = 1$: the comparisons of the exact and regularization solutions with different noisy levels.

We fix the domain $\{(x,y)|0 < x \leqslant 1, |y| \leqslant 10\}$. To observe the effect on different noisy levels ϵ, we consider the case of $k = 1$ at $x = 0.5$. Table 1 gives the comparisons of the errors between the exact and regularization solutions for different ϵ, from which we can see that the smaller the ϵ is, the better the computed approximation is.

Table 2 compares the errors between the exact solutions and the regularization solutions for different k with $x = 0.5$, $\epsilon = 10^{-3}$ in (6.1). From Table 2, we can see that the approximative effect is well even for large k.

Fig. 1 illustrates the comparisons between the exact and regularization solutions with three different levels of noise added into both Dirichlet and Neumann (see problem (6.1)) data. From both Table 1 and Fig. 1 it can be seen that as the magnitude of noise decreases, the numerical solutions converge to the corresponding exact solutions.

7. Conclusion

In this paper, we considere a Cauchy problem of the Helmholtz equation in a ‘‘strip’’ domain. For this severely ill-posed problem, we deduce the optimal error bound with only nonhomogeneous Neumann data. According to this optimal error bound, one can judge if a regularization method is OK or not.

About the regularization strategy, we propose a modified Tikhonov method. For the choice of regularization parameter, we give not only the *a priori* but also the *a posteriori* rules. Moreover, about the *a posteriori* rule in the Morozov's discrepancy principle, we suggest a framework of the error estimate.

About our numerical experiments, we use the fast Fourier transform technique. Although we considere a ‘‘strip’’ domain, the solution of our example is almost zero for $|y| > 10$. Since $e^{-|y|}(\cos(\sqrt{\beta}x) + \sin(\sqrt{\beta}x)) < 2e^{-10} = \mathbf{o}(10^{-5})$, for $|y| > 10$, it is reasonable to consider the numerical experiments in a finite rectangular domain $\{(x,y)|0 < x \leqslant 1, |y| \leqslant 10\}$ instead of a ‘‘strip’’ domain. The numerical results show that the method works well.

From Remark 5.9, we know that the modified Tiknonov method can also be used for the general domain. The numerical implementation for the general domain is more interesting but not easy, which will be considered in the future.

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