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Torsion of cracked components using collocation techniques

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Abstract The use of collocation-based (strong form) techniques for the analysis of cracked cross-sections subjected to torsional moments is addressed. Three techniques, namely, the Trefftz collocation method, the asymmetric or Kansa's collocation method and the method of fundamental solutions, are, briefly, described and compared. As the problem exhibits singularities specific techniques have to be used. The basic strategy is to enlarge or to enrich the approximation of the variable of interest with terms that take into account the singularity. Other approach relies on domain decomposition. In this work, the advantages and disadvantages of the different approaches are discussed.

Key words: Trefftz method; collocation; radial basis functions; torsion

INTRODUCTION

In the numerical solution of systems of partial differential equations two main approaches are used: either the residue (or error) is minimized on an average, weighted, way over the whole or part of the domain, or the residue is enforced to be zero (or as small as numerically possible) at a given set of points.
This last form is what is known as collocation or strong form satisfaction of the PDE.

There are various forms of collocation, there are various methods, with different names and not all of them referring to collocation, that rely on the collocation approach.
In the context of PDE, and in a very synthetic manner, collocation is a two-step approach. The first of these two steps is the construction of an approximation for the variable of interest (of the actual physical and mathematical problem being solved) and the second step is the enforcement of the field and boundary conditions of the problem at a set of selected boundary and/or domain points.

The differences between the various forms of collocation are, especially, due to the different ways of constructing the approximations.
If the building blocks, that is, the members of the family of basis functions used for the approximation, are, simply, polynomials without any particular feature, chances are that only very rough approximations may be obtained and collocation will be necessary at a large number of boundary and domain points.
If, on the other hand, the building blocks are chosen in such a way that, for example, they are actual solutions of the homogeneous differential equation then all that remains to do is to collocate at only a small number of boundary points.

Distinct manners of defining the collocation points may also be used to distinguish between forms of collocation. The emphasis in this work is on collocation techniques that do not require the definition of a structured grid of points, that is, techniques that may fall into the larger family of meshles or meshfree methods. And this is the case for the asymmetric or Kansa's RBF collocation technique, the Trefftz collocation method and the method of fundamental solutions.

As for the different ways of building the approximations two of the techniques which are being revisited here, namely the Trefftz method and the method of fundamental solutions, use actual solutions of the homogeneous differential equation and, therefore, only require boundary collocation.
The other technique, RBF collocation, uses radial basis functions of a much more general type than those above at the expense of having to collocate at both the boundary and the domain.

Basically, when using collocation, one must choose between restrictions to the type of functions used and the number and location of the points where to collocate.

Before moving on to the brief description of the techniques it should be pointed out that the main reason for using collocation is its inherent simplicity but it should also be pointed out that there are numerical issues to take into consideration when dealing with collocation techniques, namely the ill conditioning and, sometimes, the reduction of the numerical stability.

In the following sections a brief description of the three collocation approaches is made. Then the problem of torsion is formulated and applied to the analysis of torsion of uncracked and cracked circular cross-sections.

RBF COLLOCATION

Radial basis functions are essential ingredients of the techniques generally known as "meshless methods". In a way or another all of the meshless (or mesh reduction) techniques attempt to solve boundary-value problems without resorting to the definition of a "mesh" (at least not in the form meshes are usually defined in the context of finite or boundary element methods) thus requiring some sort of radial function to measure the influence of a given location on another part of the domain.

The use of radial basis functions (RBF) followed by collocation, a technique first proposed by Kansa [1], [2], after the work of Hardy [3] on multivariate approximation, is now becoming an established approach and various applications to problems of structures and fluids have been made in recent years.

Examples of application of the above technique (as well as the symmetric approach proposed by Fasshauer [4] may be found, for example, in the works of the author (and co-authors), [5,6,7]. The range of problems analysed include: Kirchhoff plate bending, plate stretching, non-linear damage models for reinforced concrete, one-dimensional stability and one-dimensional free vibration problems.

Kansa's method (or asymmetric collocation) starts by building an approximation to the field of interest (normally displacement components) from the superposition of radial basis functions (globally or compactly supported) conveniently placed at points in the domain (and, or, at the boundary).

The unknowns (which are the coefficients of each RBF) are obtained from the (approximate) enforcement of the boundary conditions as well as the governing equations by means of collocation. Usually, this approximation only considers regular radial basis functions (such as the globally supported multiquadrics or the compactly supported Wendland [8] functions).

Radial basis functions (RBFs) are all that exhibit radial symmetry, that is, may be seen to depend only (apart from some known parameters) on the distance between the center of the function and a generic point.

Amongst the infinite set of RBFs the Multiquadrics (MQ), $\sqrt{(\mathbf{x}-\mathbf{x}_j)^2+c_j^2}$ is one of the most popular and is the one used in this work. The c_j is a shape parameter that strongly influences the quality of the approximation.

In a very brief manner, the solution of PDE with the RBF asymmetric collocation (or Kansa's) approach is obtained by applying the corresponding differential operators (denoting, for example, by *LI* the interior or domain differential operator and by *LB* the boundary differential operator) to the radial basis functions and then to use collocation at an appropriate set of *M* boundary collocation points and *N* domain collocation points.

From this, a system of linear equations of the following type may be obtained:

$$LIu_h(\mathbf{x}_i) = \sum_{j=M+1}^{N+M} \alpha_j LI\phi(\|\mathbf{x}_i - \mathbf{x}_j\|) + \sum_{j=M+1}^{N+M} \beta_j LIp_k(\mathbf{x}_i)$$

$$LBu_h(\mathbf{x}_i) = \sum_{j=1}^{M} \alpha_j LB\phi(\|\mathbf{x}_i - \mathbf{x}_j\|) + \sum_{j=1}^{M} \beta_j LBp_k(\mathbf{x}_i) \qquad (1)$$

subject to the conditions

$$\sum_{j=1}^{N+M} \alpha_j p_k(\mathbf{x}_j) = 0$$

where $\phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ is the radial basis function (MQ in this case), the $p_k(x)$ are polynomial terms (required, mathematically speaking, for the sake of uniqueness of the approximation, in practical terms with a very limited effect, if any, in the majority of cases), α_j and β_j unknowns are determined from the satisfaction of the domain and boundary constraints at the collocation points.
In Fig.1 a distribution of collocation and RBF centers on a square for a plate bending problem. RBF collocation is quite insensitive to the lack of regularity of the distribution of points.

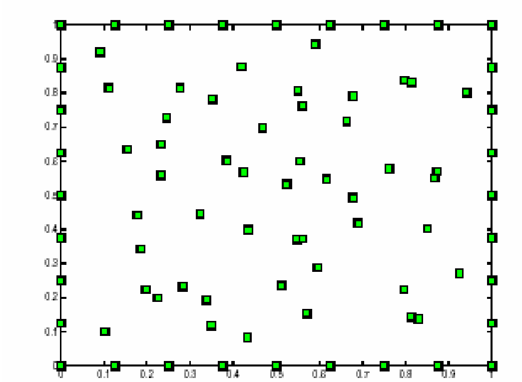

Fig. 1 Example of a distribution of collocation and RBF centers on a square

TREFFTZ COLLOCATION

The classical Trefftz method [9] consists in the solution of a partial differential equation by the superposition of a number of functions, which are themselves solutions of the homogeneous governing equation, appropriately scaled by a number of unknown parameters. These unknowns are then obtained from the approximate satisfaction of the boundary conditions by means of collocation or in an weighted residual sense.

The main characteristic of the Trefftz methods is, thus, the use of trial functions that satisfy, in a certain region (locally), all the governing differential equations of the problem.
Complete sets of solutions of such (homogeneous) partial differential equations are then required so that completeness and convergence can be guaranteed.

Although systems of functions, for which completeness cannot be guaranteed, may be found to yield accurate results it is always best (if such a general solution exists for the particular problem being analysed) to use T(Trefftz)-complete systems, see Herrera [10].

It should be pointed out that it is always necessary to truncate, in an adequate way, the T-complete functions. Because of that and also to avoid (bad) numerical conditioning it is often better to partition the domain being analysed in a suitable manner so that in all of the subregions a smaller number of terms of the T-complete system is used. The necessary continuity along the interfaces are then enforced in a pointwise or in an integral weighted residual sense.

These and other aspects of Trefftz-based formulations have been studied by several authors, Reviews on the subject may be found in Zielinski [11], Kita and Kamiya [12] and Jirousek and Wrobleski [13]. Trefftz collocation has also been applied to singular problems, [14,15], among others.

In a very brief manner and taking the two-dimensional Laplace equation for the bounded region Ω as an example,

$$\nabla^2 u = 0 \text{ in } \Omega, \tag{2}$$

subject to the potential u and flux q (defined as the derivative of the potential in the normal direction) boundary conditions:

$$\begin{aligned} u &= \bar{u} \text{ on } \Omega_u, \\ q &= \bar{q} \text{ on } \Omega_q, \end{aligned} \tag{3}$$

If an approximate solution exists in the form of a suitably truncated expansion as follows:

$$\begin{aligned} \tilde{u} &= \sum_{j=1}^{N} \alpha_j \psi_j(x) \text{ and} \\ \tilde{q} &= \sum_{j=1}^{N} \alpha_j \zeta_j(x), \end{aligned} \tag{4}$$

where all the ψ are solutions of the homogeneous equation (the ζ are normal derivatives) and of the type

$$\begin{aligned} \psi_0 &= 1 \\ \psi_j &= r^j \cos(\theta) \quad ; \quad r^j \sin(\theta) \end{aligned} \tag{5}$$

then a solution (the unknown α_j coefficients) may be obtained by simply collocating, that is, forcing the residue to be zero at a set of suitably selected points:

$$\begin{aligned} \rho_u(z_k) &= \tilde{u}(z_k) - \bar{u}(z_k) = 0 \text{ and} \\ \rho_q(z_k) &= \tilde{q}(z_k) - \bar{q}(z_k) = 0 \end{aligned} \tag{6}$$

MFS COLLOCATION

The Method of Fundamental Solutions (MFS), introduced in the 1960's (cf. Kupradze [16], Arantes [17]) as an alternative numerical method to the boundary integral methods, is a simple yet powerful technique that has been used to obtain highly accurate numerical approximations of PDE solutions with simple codes and small computational effort.

The application of the MFS for simply connected domains with regular boundary and its generalization to multiply connected domains is straightforward, Alves and Chen [18].

The setting of the MFS looks similar to that of Trefftz collocation but there are some noticeable differences. To start with there is now an articial boundary surrounding (this is not absolutely necessary but it is quite common to do it) the domain of interest. On this artificial boundary sources (that is, centers of fundamental solutions) are placed.

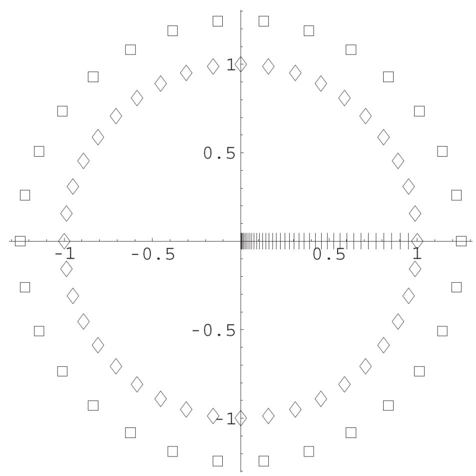

Fig. 2 Example of a distribution of collocation points (◊) and sources(□) for a circle

Then the superposition of the effect of these sources is calculated at every boundary collocation point at the real boundary.
Residues are obtained, much in the same way as for the Trefftz method, and the unknown coefficients are obtained.

So, referring to equations (2) to (5) the only visible difference is that instead of the ψ solutions of the homogeneous equation used in the Trefftz approach (which is basically an infinite series where all the terms are centered at a single point, the origin of coordinate reference system), the MFS uses a solution, a singular solution of the type $-\frac{1}{2\pi}\log(r)$, which is expressed by a single term but places this function at a set of points on an artificial boundary.

As before, from the approximate solution:

$$\tilde{u} = \sum_{j=1}^{N} \alpha_j \psi_j(z, x_j), \tag{7}$$

where $\psi_j(z, x_j)$ is the value at the collocation point z of the fundamental solutions placed at source point x_j.
A solution may be obtained by simply collocating, that is, forcing the residue to be zero at a set of suitably selected points, see equation (6).

COMMON ASPECTS OF THE THREE COLLOCATION TECHNIQUES

In very recent works the common aspects of the three techniques have been exposed in a way that may well lead, in the near future, to a common framework for the theoretical background of the methods.

The first of those works is that of Chen *et al.* [19] who claims *"In this paper, it is proved that the two approaches, known in the literature as the method of fundamental solutions (MFS) and the Trefftz method, are mathematically equivalent in spite of their essentially minor and apparent differences in formulation. In deriving the equivalence of the Trefftz method and the MFS for the Laplace and biharmonic problems, it is interesting to find that the complete set in the Trefftz method for the Laplace and biharmonic problems are embedded in the degenerate kernels of the MFS. The degenerate scale appears using the MFS when the geometrical matrix is singular."*

The second one, soon to be published Alves [20], says *"It should be stated that most common RBF domain approximations are in fact variations of the MFS boundary approximation in a higher dimension. For instance, the inverse multiquadrics (IMQ) used in 2D problems, can be seen as an MFS boundary approximation to the 3D Laplace problem. The 2D domain is to be seen as a part of a 3D boundary (with null third component, x3 = 0) and the discussion on the choice of the parameters cj is just a discussion on where to place the artificial boundary, since these parameters represent the distance between the true and the artificial boundary. When a constant parameter c is considered, then the artificial boundary is just placed in the parallel plane with x3 = c.*

In the same manner that inverse multiquadrics in 2D domains can be seen as an application of the Laplacian fundamental solution in 3D boundaries, also multiquadrics (MQ) in 2D domains may be seen to be an application of the Bilaplacian fundamental solution to 3D boundaries. Also, and now not surprisingly, the Gaussian RBF in n-D is related to the fundamental solution of the heat equation in n-D space + 1 time dimension. Note that in terms of PDE approximation, the MQ are a more natural basis in the Kansa type approach since the presence of a Laplacian, in the PDE, partially transforms the MQs into IMQs."

It should also be pointed out that acknowledging these common aspects is very important in the sense that mathematical proofs of an approach may also be valid (with the necessary adaptations) for another approach but it is far from saying that the numerical aspects, the algorithms, the strategies used by one technique may be applied to another technique.

But this is actually what happens when the three collocation techniques are applied to problems exhibiting singularities: the same superposition approach is considered in all cases and this is what is going to be explained next.

APPLICATION OF COLLOCATION TECHNIQUES TO THE ANALYSIS OF PROBLEMS EXHIBITING SINGULARITIES

As written earlier, the main difference between the methods rests with the type of approximation functions used.

RBF collocation uses very basic functions and, therefore, requires some extra information in order to model adequately the singular behaviour exhibited around cracks (or other sources of singularities). So it it is also necessary to superimpose singular functions to the regular basis used. In fact standard (that is, regular) RBF approximations tend to lead to poor solutions around cracks or corners. This motivated Platte and Driscoll [21] to consider an augmented approximation by combining regular and singular functions for eigenmodes computation.

The Trefftz method and the Method of Fundamental Solutions use, on the other hand, functions (the T-complete series and the fundamental solution) that are actual solutions of the differential equation (be it

in the homogeneous form or in the Dirac form). This fact alone leads to more accurate approximations that could well avoid the need of superpositions as the one considered above.
In fact, this is not the case. Despite the increased quality of the approximation it is still necessary to consider, for problems exhibiting singularities, the superposition of specialized functions that take that type of behaviour into account.

In brief, to the standard basis of functions used in all of the techniques previously described a new set of singular functions is added so that problems exhibiting boundary singularities (such as crack problems) may be solved.

In particular, the problem of cracked bars under torsion as in Chen *et al.* [22] is considered here. Several numerical tests were made that show high quality approximations of the singular solution, when using the enriched versions of the methods.

The singular terms added to each of the expansions (be it the one obtained by the RBF, the Trefttz or the MFS) is the same. No surprise here as the same physical behaviour (a singularity at the tip of a crack) is being reproduced.
This is what is being added:

$$u_S(x) \approx \sum_{k=1}^{\infty} \beta_k r^{(k-0.5)} \cos((k-0.5)\theta) \tag{8}$$

The singular functions added are obviously problem dependent but the good thing is that they are available for the majority of problems commonly encountered by engineers.
Details on the techniques briefly described above may be found in the following references [15,23,24].

CONCLUSIONS

This work discusses a set of collocation techniques which have received, in recente years and owing to its inherent simplicity and quite good accuracy, the attention of the engineering community.
Until recently the similarities between these methods had not been recognized and the methods tended to be presented as non related.
As referred above the situation has now changed and there are proofs (not only suspicions but real mathematical evidence) of the strong links between them. And this only confirms what had already been experienced by the author (and other colleagues) with each of the methods separately.

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