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Degenerate scale problem for plane elasticity in a multiply connected region with outer elliptic boundary

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Abstract This paper investigates the degenerate scale problem for plane elasticity in a multiply connected region with an outer elliptic boundary. Inside the elliptic boundary, there are many voids with arbitrary configurations. The problem is studied on the relevant homogenous boundary integral equation. The suggested solution is derived from a solution of a relevant problem. It is found that the degenerate scale and the non-trivial solution along the elliptic boundary in the problem are same as in the case of a single elliptic contour without voids. The present study mainly depends on integrations of several integrals, which can be integrated in a closed form.

Keywords Boundary integral equation · Degenerate scale · Plane elasticity · Multiply connected region

1 Introduction

Many researchers were attracted by the boundary integral equation (BIE). Some pioneering works for BIE were initiated in [1,2]. The basic theory for BIE could be found from [3,4] The development of the boundary element method was summarized in [5].

In earlier times, Christiansen pointed out that some integral equations of the first kind with logarithmic kernel are known to have a non-unique solution in some exceptional cases [6]. This means that the relevant homogenous equation has a non-trivial solution. Alternatively speaking, the well-known BIE formulation, particularly, for the exterior Dirichlet problem, does not work when some particular scales are encountered. The most typical example is as follows. One considers the 2D-Laplace equation $\nabla^2 u(x, y) = 0$ for an exterior circular boundary, and gets a solution $u = \ln r$. If the radius of the inner circle takes $a = 1$, the boundary value at the inner circle will be vanishing, or equals zero. In this exceptional case, a zero boundary value (or $u(x, y) = 0$ on the boundary) corresponds to a non-vanishing solution (or $u = \ln r$ in the exterior problem bounded by a circular hole). This is the degenerate scale problem. This result is universal. However, if the configuration of the void is arbitrary, it is more difficult to get: (1) the degenerate scale and (2) the relevant non-trivial solution.

The degenerate scale problem in BIE is a particular boundary value problem in plane elasticity. The problem typically arises from the exterior Dirichlet problem in plane elasticity. In fact, once the degenerate scale is reached, the relevant homogenous equation for the boundary tractions has a non-trivial solution. Alternatively speaking, the non-homogeneous equation has non-unique solution, or multiple solutions. Clearly, the degenerate scale represents an illness condition. Therefore, one must avoid meeting illogical solution caused by occurrence of the degenerate scale.

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The degenerate scale problems were studied by many researchers using a variety of methods. The degenerate scale can be evaluated by solving some BIE in a normal scale [7–9]. Numerical procedure was developed to evaluate the degenerate scale directly from the zero value of a determinant [10]. Degenerate scale for multiply connected Laplace problems was solved [11].

This paper investigates degenerate scale problem for plane elasticity in a multiply connected region with an outer elliptic boundary. Inside the elliptic boundary, there are many voids with arbitrary configurations. The problem is studied on the relevant homogenous BIE. The merit of this study is to evaluate a non-trivial solution for the homogenous BIE. It is assumed that the tractions on all the inner void boundaries are equal to zero and tractions on the outer elliptic boundary are equal to some functions. Therefore, all the integrations are performed on the outer elliptic boundary only. The degenerate scale for the problem is found. It is found that the degenerate scale and the non-trivial solution along the elliptic boundary in the problem are same as in the case of a single elliptic contour without voids. The present study mainly depends on integrations of several integrals, which can be integrated in the closed forms.

2 Degenerate scale problem for plane elasticity with a single elliptic boundary

The degenerate scale problem in the multiply connected region has a close relation with the same problem in a single connected boundary. Therefore, the degenerate scale problem for plane elasticity with a single elliptic boundary is introduced.

Without losing generality, we introduce the BIE for plane elasticity for a single elliptic boundary B_1 (Fig. 1). The source point is denoted by $\xi(\xi_1, \xi_2)$, and the field point is denoted by $x(x_1, x_2)$. In the plane strain case, the BIE is as follows [4,12]

$$\frac{1}{2}u_i(\xi) + \int_{B_1} P_{ij}^*(\xi, x)u_j(x)\mathrm{d}s(x) = \int_{B_1} U_{ij}^*(\xi, x)p_j(x)\mathrm{d}s(x), \quad (i = 1, 2, \text{ for } \xi \in \Gamma) \tag{1}$$

and the integral kernels $P_{ij}^*(\xi, x)$ and $U_{ij}^*(\xi, x)$ are defined by

$$P_{ij}^*(\xi, x) = -\frac{1}{4\pi(1-\nu)}\frac{1}{r}\left\{(r_{,1}n_1 + r_{,2}n_2)((1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}) + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\right\} \tag{2}$$

$$U_{ij}^*(\xi, x) = \frac{1}{8\pi(1-\nu)G}\left\{-\kappa \ln(r)\delta_{ij} + r_{,i}r_{,j} - 0.5\delta_{ij}\right\} \tag{3}$$

where Kronecker deltas δ_{ij} is defined as, $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$, $\kappa = 3 - 4\nu$, G denotes the shear modulus of elasticity, ν is the Poisson's ratio, and the normal $n(n_1, n_2)$ at the boundary point always directs at outward side. In addition, we have

$$r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2, \quad r_{,1} = \frac{x_1 - \xi_1}{r} = \cos\alpha, \quad r_{,2} = \frac{x_2 - \xi_2}{r} = \sin\alpha \tag{4}$$

where the angles α are indicated in Fig. 1. Note that the BIE can be used for both exterior and interior boundary value problems.

Substituting $u_i(\xi) = 0$ and $u_j(x) = 0$ into the left-hand side of Eq. (1), we can obtain the following homogeneous equation

$$\int_{B_1} U_{ij}^*(\xi, x)p_j(x)\mathrm{d}s(x) = 0 \quad (i = 1, 2, \text{ for } \xi \in \Gamma) \tag{5}$$

Equation (5) reveals that regardless of the exterior and the interior problems same homogeneous equation is obtained. Based on Eq. (5), the degenerate scale problem can be formulated as follows. One wants to find a particular scale such that the integral equation (5) has a non-trivial solution $p_j(x) \neq 0$ (for $j = 1, 2,\ x \in B_1$). Here, $p_j(x) = 0$ $(j = 1, 2)$ is a trivial solution.

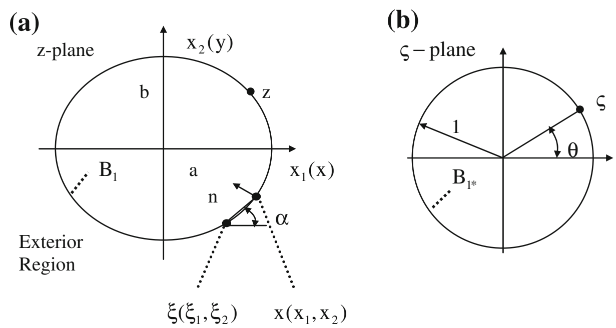

Fig. 1 **a** An exterior region to the elliptic boundary, **b** a mapping on ς − plane

The following analysis depends on the complex variable function method in plane elasticity [13]. In this method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of two complex potentials $\phi_1(z)$ and $\psi_1(z)$ such that

$$\sigma_x + \sigma_y = 4Re\phi_1'(z)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\phi_1''(z) + \psi_1'(z)] \tag{6}$$

$$f = -Y + iX = \phi_1(z) + z\overline{\phi_1'(z)} + \overline{\psi_1(z)} \tag{7}$$

$$2G(u + iv) = \kappa\phi_1(z) - z\overline{\phi_1'(z)} - \overline{\psi_1(z)} \tag{8}$$

where $z = x + iy$ denotes complex variable, G is the shear modulus of elasticity, $\kappa = (3 - \nu)/(1 + \nu)$ is for the plane stress problems, $\kappa = 3 - 4\nu$ is for the plane strain problems, and ν is the Poisson's ratio. In the present study, the plane strain condition is assumed thoroughly. In the following, we occasionally rewrite the displacements "u", "v" as $u_1, u_2, \sigma_x, \sigma_y, \sigma_{xy}$ as $\sigma_{11}, \sigma_{22}, \sigma_{12}$, and "$x$", "$y$" as x_1, x_2, respectively.

Assume that there is an ellipse with two half-axis a and b. The following mapping function is introduced (Fig. 1)

$$z = \omega(\varsigma) = R\left(\varsigma + \frac{m}{\varsigma}\right), \quad (\text{with } 0 \leqslant m < 1) \tag{9}$$

which maps the elliptical contour and its exterior region (in z-plane) into a unit circle and its exterior region (in ς-plane) [13]. From Eq. (9), we have $R(1+m)$, $b = R(1-m)$, $R = (a+b)/2$ and $m = (a-b)/(a+b)$.

After using the conformal mapping, the following functions are introduced

$$\phi(\varsigma) = \phi_1(z)\mid_{z=\omega(\zeta)}, \quad \psi(\varsigma) = \psi_1(z)\mid_{z=\omega(\zeta)}, \quad \phi_1'(z) = \phi'(\varsigma)/\omega'(\varsigma)$$

$$\phi_1''(z) = \frac{\phi''(\varsigma)\omega'(\varsigma) - \phi'(\varsigma)\omega''(\varsigma)}{(\omega'(\varsigma))^3}, \quad \psi_1'(z) = \psi'(\varsigma)/\omega'(\varsigma) \tag{10}$$

Therefore, the stresses, resultant forces and displacements can be expressed as

$$\sigma_x + \sigma_y = 4Re\left(\frac{\phi'(\varsigma)}{\omega'(\varsigma)}\right)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2\left(\frac{\overline{\omega(\varsigma)}[\phi''(\varsigma)\omega'(\varsigma) - \phi'(\varsigma)\omega''(\varsigma)]}{(\omega'(\varsigma))^3} + \frac{\psi'(\varsigma)}{\omega'(\varsigma)}\right) \tag{11}$$

$$f = -Y + iX = \phi(\varsigma) + \frac{\omega(\varsigma)\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}} + \overline{\psi(\varsigma)} \tag{12}$$

$$2G(u+iv)=\kappa\phi(\varsigma)-\frac{\omega(\varsigma)\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}-\overline{\psi(\varsigma)} \tag{13}$$

After using the conformal mapping technique, two solutions for the homogeneous equation (5) have been obtained previously. In the first solution, the relevant complex potentials and the degenerate scale are as follows [12]

$$\phi(\varsigma)=(\ln\varsigma+\ln\ R),\quad \psi(\varsigma)=-\kappa(\ln\ \varsigma+\ln\ R)-\frac{1+m^2}{\varsigma^2-m} \tag{14}$$

$$R_{\text{cr1}}=\exp(m/2\kappa) \tag{15}$$

In Eq. (15), the subscript "cr1" means that it is the first critical value for "R", or R_{cr1} is the first degenerate scale.

The merit of Eqs. (14) and (15) is as follows. If one substitutes:

(1) the $p_j(x)$ derived from the complex potentials shown by Eq. (14), and
(2) the "R" value shown by Eq. (15), or $R=R_{\text{cr1}}=\exp(m/2\kappa)$

into Eq. (5), the homogenous equation (5) is satisfied.

Similarly, in the second solution, the relevant complex potentials and the degenerate scale are as follows:

$$\phi(\varsigma)=i(\ln\varsigma+\ln R),\quad \psi(\varsigma)=i\left(\kappa(\ln\varsigma+\ln R)-\frac{1+m^2}{\varsigma^2-m}\right) \tag{16}$$

$$R_{\text{cr2}}=\exp(-m/2\kappa) \tag{17}$$

The merit of Eqs. (16) and (17) is similar to the first case. The degenerate scales shown by Eqs. (15) and (17) were obtained by Chen in an earlier time [14].

In the literature, instead of Eq. (3), the following kernel was suggested [4]

$$U_{ij}^{**}(\xi,x)=\frac{1}{8\pi(1-v)G}\left\{-\kappa\ \ln(r)\delta_{ij}+r_{,i}r_{,j}\right\} \tag{18}$$

which is different from Eq. (3) by a constant. It was proved that the kernel could be used to the interior problem, since the tractions applied on the boundary of finite region must be in equilibrium. However, this kernel is derived from a fundamental solution that is not expressed in a pure deformable form. Thus, this kernel cannot be used to the exterior boundary value problem with applied loadings not in equilibrium [15].

3 Formulation and solution of the degenerate scale problem for a multiply connected region with outer elliptic boundary

3.1 Formulation

Without losing generality, we introduce the BIE for plane elasticity for a multiple connected region with outer elliptic boundary B_1. Inside the elliptic boundary B_1, there are many voids $B_k(k=2,3,N)$ with arbitrary configurations (Fig. 2). The source point is denoted by $\xi(\xi_1,\xi_2)$, and the field point is denoted by $x(x_1,x_2)$. In this case, the BIE is as follows

$$\frac{1}{2}u_i(\xi)+\int_B P_{ij}^*(\xi,x)u_j(x)\mathrm{d}s(x)=\int_B U_{ij}^*(\xi,x)p_j(x)\mathrm{d}s(x),$$
$$(i=1,2,\ \text{for}\ \xi\in B,B=B_1+B_2+\cdots+B_N) \tag{19}$$

where the integral kernels $P_{ij}^*(\xi,x)$ and $U_{ij}^*(\xi,x)$ have been defined by Eqs. (2) and (3).

As before, after substituting $u_i(\xi)=0$ and $u_j(x)=0$ into left-hand side of Eq. (19), the following homogeneous equation is obtainable

$$\int_B U_{ij}^*(\xi,x)p_j(x)\mathrm{d}s(x)=0,\quad (i=1,2,\ \text{for}\ \xi\in B,B=B_1+B_2+\cdots+B_N) \tag{20}$$

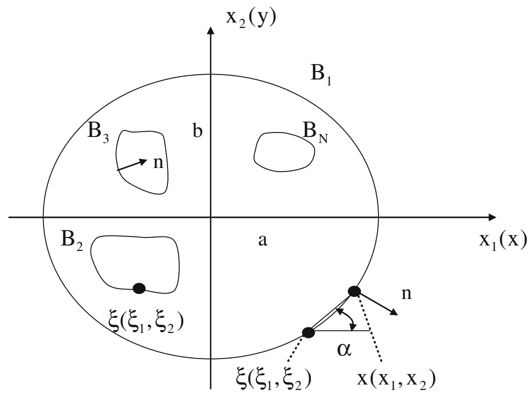

Fig. 2 A multiply connected region with an outer elliptic boundary

In the degenerate scale problem, one needs to find same particular size such that Eq. (20) has a non-trivial solution for $p_j(x)$ $(j = 1, 2)$, or $p_j(x) \neq 0$ $(j = 1, 2)$ (for $x \in B$, $B = B_1 + B_2 + \cdots + B_N$). This is the highest demand for the formulation. However, we can propose a lower demand for the formulation. For example, the lower demand is as follows: $p_j(x) \neq 0$ $(j = 1, 2)$ for $x \in B_1$ and $p_j(x) = 0$ $(j = 1, 2)$ for $x \in B_k$, $k = 2, 3, \ldots, N$.

Clearly, it is not easy to obtain a non-trivial solution from Eq. (20) directly. However, it can be obtained from the previous result for a single elliptic contour case [12].

3.2 Evaluation of some integrals in the closed forms

In order to solve the degenerate scale problem addressed, one must evaluate some integrals in advance. In the derivation, the mapping function shown by Eq. (9) is still used (Fig. 1). For a point $\varsigma = e^{i\theta}$ on the unit circle, we have

$$d\theta = \frac{d\varsigma}{i\varsigma} \tag{21}$$

In the meantime, we can let

$$z - t = e^{i\alpha} \tag{22}$$

In the first group of evaluation, three integrals K_1, K_2 and K_3 are defined as follows (Fig. 3)

$$K_1 = \int_{B_{1*}} \ln|z - t|\, d\theta = Re \int_{B_{1*}} \ln(z - t) d\theta = \mathrm{Im} \int_{B_{1*}} \ln(z - t) \frac{d\varsigma}{\varsigma}, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{23}$$

$$K_2 - iK_3 = \int_{B_{1*}} \frac{\bar{z} - \bar{t}}{z - t} d\theta = \int_{B_{1*}} e^{-2i\alpha} d\theta \text{ or } K_2 = \int_{B_{1*}} \cos 2\alpha d\theta,$$
$$K_3 = \int_{B_{1*}} \sin 2\alpha d\theta, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{24}$$

where Γ denotes the elliptic contour (Fig. 3). Note that, in Eqs. (23) and (24), the integration $d\theta$ is performed for argument θ in $\varsigma = e^{i\theta}$. In addition, the subscript B_{1*} denotes the unit circle.

After some manipulations, from Eqs. (a11) and (a12) in Appendix A, we will find the following results:

$$K_1 = \int_{B_{1*}} \ln|z - t|\, d\theta = \mathrm{Im} \int_{B_{1*}} \ln(z - t) \frac{d\varsigma}{\varsigma} = 2\pi \ln R, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{25}$$

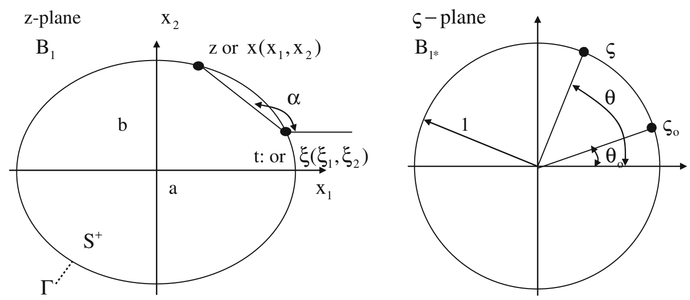

Fig. 3 Mapping relation for "z" to "ς" and "t" to "ς_o" with both "z" and "t" on the elliptic contour Γ

$$K_2 = \int_{B_{1*}} \cos 2\alpha \mathrm{d}\theta = 2\pi m, \quad K_3 = \int_{B_{1*}} \sin 2\alpha \mathrm{d}\theta = 0, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{26}$$

In the second group of evaluation for three integrals, the points "z" (for integration) is on the elliptic contour Γ, or $z \in \Gamma$. However, the point "t" now is an inner point to the elliptic contour, or $t \in S^+$, where S^+ denotes the finite elliptic region (Fig. 4). In this case, the three integrals L_1, L_2 and L_3 are defined as follows (Fig. 4):

$$L_1 = \int_{B_{1*}} \ln|z-t| \,\mathrm{d}\theta = Re \int_{B_{1*}} \ln(z-t)\mathrm{d}\theta = \mathrm{Im} \int_{B_{1*}} \ln(z-t)\frac{d\varsigma}{\varsigma}, \quad (\text{for } z \in \Gamma, t \in S^+) \tag{27}$$

$$L_2 - iL_3 = \int_{B_{1*}} \frac{\bar{z}-\bar{t}}{z-t}\mathrm{d}\theta = \int_{B_{1*}} \mathrm{e}^{-2i\alpha}\mathrm{d}\theta \text{ or } L_2 = \int_{B_{1*}} \cos 2\alpha \mathrm{d}\theta,$$
$$L_3 = \int_{B_{1*}} \sin 2\alpha \mathrm{d}\theta, \ (\text{for } z \in \Gamma, t \in S^+) \tag{28}$$

After some manipulations, from Eqs. (a22) and (a23) in Appendix A, we will find the following results

$$L_1 = \int_{B_{1*}} \ln|z-t| \,\mathrm{d}\theta = 2\pi \ln R, \quad (\text{for } z \in \Gamma, t \in S^+) \tag{29}$$

$$L_2 = \int_{B_{1*}} \cos 2\alpha \mathrm{d}\theta = 2\pi m, \quad L_3 = \int_{B_{1*}} \sin 2\alpha \mathrm{d}\theta = 0, \quad (\text{for } z \in \Gamma, t \in S^+) \tag{30}$$

It is interesting to point out that the integrals K_1, K_2 and K_3 shown by Eqs. (25) and (26) and the integrals L_1, L_2 and L_3 shown by Eqs. (29) and (30) take the same value. However, they have different definitions.

3.3 Solution

As claimed above, the non-trivial solution cannot be obtained from Eq. (20) directly. However, it can be obtained from previous result for a single elliptic contour case [12].

We can prove that, the homogeneous equation (20) can be satisfied if the following conditions are fulfilled:

(1) $p_j(x) = 0$ $(j = 1, 2)$ for $x \in B_k$, $k = 2, 3, \ldots, N$.
(2) $p_j(x) \neq 0$ $(j = 1, 2)$ for $x \in B_1$, $p_j(x)$ are derived from the complex potentials shown by Eq. (14), or by Eq. (16).
(3) the "R" value is equal to $R_{\mathrm{cr1}} = \exp(m/2\kappa)$ shown by Eq. (15), or $R_{\mathrm{cr2}} = \exp(-m/2\kappa)$ shown by Eq. (17).

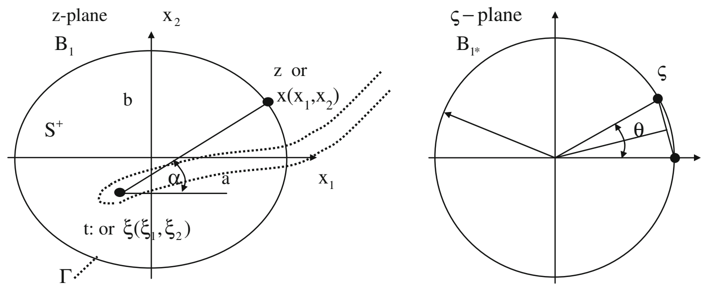

Fig. 4 Mapping relation for "z" to "ς" with "z" on the elliptic contour ($z \in \Gamma$) and "t" in the elliptic contour ($t \in S^+$)

Clearly, after using the first condition, or $p_j(x) = 0$ $(j = 1, 2)$ for $x \in B_k$, $k = 2, 3, \ldots, N$, Eq. (20) can be reduced to

$$\int_{B_1} U^*_{ij}(\xi, x)p_j(x)\mathrm{d}s(x) = 0, \quad (i = 1, 2, \text{ for } \xi \in B, B = B_1 + B_2 + \cdots + B_N) \tag{31}$$

Equation (31) can be rewritten as

$$\int_{B_1} U^*_{ij}(\xi, x)p_j(x)\mathrm{d}s(x) = 0, \quad (i = 1, 2, \text{ for } \xi \in B_1) \tag{32}$$

$$\int_{B_1} U^*_{ij}(\xi, x)p_j(x)\mathrm{d}s(x) = 0, \quad (i = 1, 2, \text{ for } \xi \in B_k, k = 2, 3, \ldots, N) \tag{33}$$

First, the homogenous equation (32) is studied. For convenience in derivation, Eq. (32) is rewritten in the following form

$$I_1 = 0, \text{ with } I_1 = \int_{B_1} \left(U^*_{11}(\xi, x)p_1(x)\mathrm{d}s(x) + U^*_{12}(\xi, x)p_2(x)\mathrm{d}s(x)\right) \quad (\text{for } \xi \in B_1) \tag{34}$$

$$I_2 = 0, \text{ with } I_2 = \int_{B_1} \left(U^*_{21}(\xi, x)p_1(x)\mathrm{d}s(x) + U^*_{22}(\xi, x)p_2(x)\mathrm{d}s(x)\right) \quad (\text{for } \xi \in B_1) \tag{35}$$

Clearly, in Eqs. (34) and (35), $p_1(x)\mathrm{d}s(x)$ corresponds to $\mathrm{d}X$ and $p_2(x)\mathrm{d}s(x)$ corresponds to $\mathrm{d}Y$ in Eq. (12). Thus, we have

$$p_1(x)\mathrm{d}s(x) = \mathrm{Im}\mathrm{d}\left\{\phi(\varsigma) + \frac{\omega(\varsigma)\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}} + \overline{\psi(\varsigma)}\right\} \tag{36}$$

$$p_2(x)\mathrm{d}s(x) = -Re\mathrm{d}\left\{\phi(\varsigma) + \frac{\omega(\varsigma)\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}} + \overline{\psi(\varsigma)}\right\} \tag{37}$$

For a point $\varsigma = \mathrm{e}^{i\theta}$ on the unit circle, we have

$$\bar{\varsigma} = 1/\varsigma, \quad \mathrm{d}\theta = \frac{\mathrm{d}\varsigma}{i\varsigma} \quad (\text{for } \varsigma = \mathrm{e}^{i\theta} \text{on the unit circle}) \tag{38}$$

Substituting Eq. (14) into Eqs. (36) and (37) and using Eq. (38) yields

$$p_1(x)\mathrm{d}s(x) = (\kappa + 1)\mathrm{d}\theta \tag{39}$$
$$p_2(x)\mathrm{d}s(x) = 0 \tag{40}$$

Note that in the integral kernel $U^*_{ij}(\xi, x)$, there are $r_{,1} = \cos\alpha$, $r_{,2} = \sin\alpha$, $2r_{,1}r_{,1} - 1 = \cos 2\alpha$, $2r_{,1}r_{,2} = \sin 2\alpha$ (Fig. 3). Therefore, after using Eqs. (39) and (40), the integral I_1 in Eq. (34) can be rewritten as

$$I_1 = \frac{1}{16\pi(1-\upsilon)G}(-2\kappa I_{11} + I_{12}) \quad (\text{for } \xi \in B_1) \tag{41}$$

where

$$I_{11} = \int_{B_{1*}} \ln r(x, \xi) \mathrm{d}\theta, \quad I_{12} = \int_{B_{1*}} \cos 2\alpha \mathrm{d}\theta \tag{42}$$

Since both points $x(x_1, x_2)$ and $\xi(\xi_1, \xi_2)$ are located on the elliptic contour (Fig. 3) I_{11} is equal to K_1 shown by Eq. (25), and I_{12} is equal to K_2 shown by Eq. (26). Therefore, we have

$$I_{11} = \int_{B_{1*}} \ln r(x, \xi) \mathrm{d}\theta = 2\pi \ln R, \quad I_{12} = \int_{B_{1*}} \cos 2\alpha \mathrm{d}\theta = 2\pi m \tag{43}$$

Substituting Eq. (43) into Eq. (41), we will find

$$I_1 = \frac{1}{8(1-\upsilon)G}(-2\kappa \ln R + m) \tag{44}$$

Finally, substituting the third condition, or $R = R_{\mathrm{cr1}} = \exp(m/2\kappa)$ into Eq. (44), we will find

$$I_1 \mid_{R=R_{\mathrm{cr1}}} = 0 \tag{45}$$

Similarly, the integral I_2 in Eq. (35) can be rewritten as

$$I_2 = \int_{B_{1*}} \sin 2\alpha \mathrm{d}\theta \quad (\text{for } \xi \in B_1) \tag{46}$$

In fact, the integral I_2 is equal to K_3 shown by Eq. (26). Therefore, we have

$$I_2 = \int_{B_{1*}} \sin 2\alpha \mathrm{d}\theta = 0 \quad (\text{for } \xi \in B_1) \tag{47}$$

Finally, the above-mentioned assertion is proved. Alternatively speaking, under the above-mentioned three conditions, the homogeneous equation (32) is satisfied.

As shown by Eqs. (25), (26), (29) and (30), if "t" (or $\xi(\xi_1, \xi_2)$) is located in the elliptic contour, or $t \in S^+$, the three integrals $\int_{B_1} \ln r(x, \xi)\mathrm{d}\theta$, $\int_{B_1} \cos 2\alpha \mathrm{d}\theta$ and $\int_{B_1} \sin 2\alpha \mathrm{d}\theta$ take the same values as in the case of $t \in \Gamma$ (Figs. 3, 4). Therefore, under the above-mentioned three conditions, the homogeneous equation (33) is satisfied.

Alternatively, we can conclude the obtained result as follows. When the degenerate scale is reached, or $R = R_{\mathrm{cr1}} = \exp(m/2\kappa)$, the homogeneous equations (20) has a non-trivial solution which is shown by first and second conditions mentioned above, or (1) $p_j(x) = 0$ $(j = 1, 2)$ for $x \in B_k$, $k = 2, 3, \ldots, N$, (2) $p_j(x) \neq 0$ $(j = 1, 2$, for $x \in B_1)$ are derived from the complex potentials shown by Eq. (14).

Similarly, from the non-trivial solution and the degenerate scale shown by Eqs. (16) and (17), similar result can be found.

From above-mentioned analysis, we see that under above-mentioned three conditions, the final solution is obtained. After using the first condition, or $p_j(x) = 0$ $(j = 1, 2)$ for $x \in B_k$, $k = 2, 3, \ldots, N$, the integral equation can be simplified in the form of Eq. (31). In this case, the unknown functions are reduced to two, or $p_j(x) = 0$ $(j = 1, 2)$ for $x \in B_1$. In Eq. (31), or Eqs. (32) and (33), all integrations are performed on the elliptic boundary $x \in B_1$. However, the point ξ in Eq. (31) is now defined by $\xi \in B$, $B = B_1 + B_2 + \cdots + B_N$. The Eq. (31) is satisfied for $\xi \in B_1$ under conditions (2) and (3), and this result is from a previous study [12]. In this paper, we prove that the Eq. (31) is also satisfied for $\xi \in B_k$, $(k = 2, 3, \ldots, N)$ under conditions (2) and (3). In fact, the kernel $U^*_{ij}(\xi, x)$ defined in Eq. (31) only depends on the position of ξ and x. Thus, if the Eq. (31) is satisfied for one point of ξ, the equation is satisfied for $\xi \in B_k$ $(k = 2, 3, \ldots, N)$ under conditions (2) and (3). After some manipulations, the assertion is proved.

Table 1 Computed degenerate scales for a doubly connected region $a_{d,1} = f_1(b/a, d/b)$ and $a_{d,2} = f_2(b/a, d/b)$ (see Fig. 5; Eq. (48))

d/b	b/a									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$a_{d,1} = f_1(b/a, d/b)$										
−0.4	2.2836	2.0070	1.7878	1.6102	1.4636	1.3407	1.2364	1.1467	1.0688	1.0006
0.0	2.2836	2.0070	1.7878	1.6102	1.4636	1.3407	1.2364	1.1467	1.0688	1.0006
0.4	2.2836	2.0070	1.7878	1.6102	1.4636	1.3407	1.2364	1.1467	1.0688	1.0006
Exact[a]	2.2821	2.0057	1.7867	1.6092	1.4627	1.3399	1.2356	1.1459	1.0681	1.0000
$a_{d,2} = f_2(b/a, d/b)$										
−0.4	1.4495	1.3858	1.3256	1.2690	1.2162	1.1669	1.1209	1.0780	1.0380	1.0006
0.0	1.4495	1.3858	1.3256	1.2690	1.2162	1.1669	1.1209	1.0780	1.0380	1.0006
0.4	1.4495	1.3858	1.3256	1.2690	1.2162	1.1669	1.1209	1.0780	1.0380	1.0006
Exact[b]	1.4486	1.3849	1.3247	1.2682	1.2154	1.1661	1.1202	1.0773	1.0374	1.0000

[a] Exact from Eq. (15)
[b] Exact from Eq. (17)

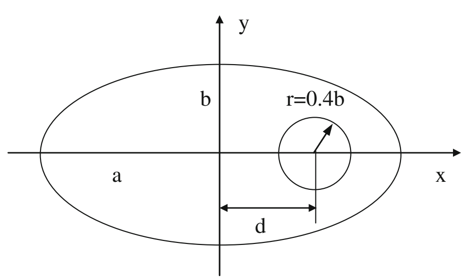

Fig. 5 Doubly connected region with the outer ellipse and inner circle

4 Numerical illustration

To examine the theoretical result mentioned above, a numerical example is presented below. It is found that coordinate transform method for finding the degenerate scale is effective [7,9]. This method has now been used to the case of a doubly connected infinite region [16]. Using this method, a numerical examination is presented below.

The merit of the coordinate transform method is as follows [7,9,16]. After using the coordinate transform, the original homogenous BIE in a degenerate scale can be reduced to a non-homogenous BIE in the normal scale. The reduced non-homogenous BIE in the normal scale can be solved with unique solution. This is the basic idea in the method. However, the detailed derivation is a little bit complicated than the mentioned basic idea.

In the example, an ellipse with two half-axes (a, b) contains a circular hole with a radius $r = 0.4b$. The center of the circular hole is shifted by a distance "d" (Fig. 5). In computation, 80 divisions are used for the discretization of the ellipse, and 30 divisions for the circle. $\nu = 0.3$ is used in computation. Thus, an algebraic equation with 220 unknowns was formulated. For $a/b = 0.1, 0.2, \ldots, 1.0$ and $d/b = -0.4, 0.0$ and 0.4, the computed degenerate scales for size "a" can be expressed as

$$a_{d,1} = f_1(b/a, d/b) \quad \text{and} \quad a_{d,2} = f_2(b/a, d/b) \tag{48}$$

The computed results for $a_{d,1}$ and $a_{d,2}$ are listed in Table 1. In Table 1, the exact results for a single ellipse are also attached, which is derived from Eqs. (15) and (17). It is seen from tabulated results that; (1) there is no influence from the position of the inner circular hole, (2) the deviations from the computed results to the exact solutions for the simple elliptic contour case are very small.

5 Conclusions

In this paper, it is proved that degenerate scale for the outer elliptic contour does not depend on the voids involved in the contour. This result is proved exactly.

As claimed previously, the degenerate scale will cause illness solution for BIE. It is assumed that the ellipse has a ratio $b/a = 1/3$ with $m = 0.5$ and $\kappa = 1.8$. In this case, from Eq. (15) the first solution is $R = R_{cr1} = \exp(m/2\kappa) = 1.14900$, $a = R(1+m) = 1.72349$ and $b = R(1-m) = 0.57450$. From Eq. (17), the second solution is $R = R_{cr2} = \exp(-m/2\kappa) = 0.87032$, $a = R(1+m) = 1.30549$ and $b = R(1-m) = 0.43516$. Therefore, one should avoid using the mentioned sizes in computation, particularly, in the Dirichlet boundary value problem.

Previously, a numerical examination for degenerate scale problem for ellipse-shaped ring region was carried out [10]. The degenerate scale for half-axis is denoted by a_{cr}. Computed result proved that if one takes the real scale $a = 0.9a_{cr}$ or $a = 1.1a_{cr}$, the illness solution disappears.

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Appendix A: Evaluation of some integrals in the closed forms

In order to solve the degenerate scale problem, one must evaluate some integrals in advance. In the derivation, the mapping function shown by Eq. (9) is still used here (Fig. 1). For a point $\varsigma = e^{i\theta}$ on the unit circle, we have

$$d\theta = \frac{d\varsigma}{i\varsigma} \tag{a1}$$

In the meantime, we can let

$$z - t = e^{i\alpha} \tag{a2}$$

In the first group of evaluation, three integrals K_1, K_2 and K_3 are defined as follows (Fig. 3)

$$K_1 = \int_{B_{1*}} \ln|z-t|\, d\theta = Re \int_{B_{1*}} \ln(z-t) d\theta = \text{Im} \int_{B_{1*}} \ln(z-t)\frac{d\varsigma}{\varsigma}, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{a3}$$

$$K_2 - iK_3 = \int_{B_{1*}} \frac{\bar{z}-\bar{t}}{z-t} d\theta = \int_{B_{1*}} e^{-2i\alpha} d\theta \quad \text{or} \quad K_2 = \int_{B_{1*}} \cos 2\alpha d\theta, \quad K_3 = \int_{B_{1*}} \sin 2\alpha d\theta, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{a4}$$

where Γ denotes the elliptic contour. Note that, in Eqs. (a3) and (a4), the integration $d\theta$ is performed for argument θ in $\varsigma = e^{i\theta}$. In addition, the subscript B_{1*} denotes the unit circle.

Since both "z" and "t" are located on the ellipse, or $z \in \Gamma$, $t \in \Gamma$, we can let

$$z = R\left(\varsigma + \frac{m}{\varsigma}\right), \quad t = R\left(\varsigma_o + \frac{m}{\varsigma_o}\right), \quad z - t = R(\varsigma - \varsigma_o)\left(1 - \frac{m}{\varsigma_o \varsigma}\right) \tag{a5}$$

Therefore, from Eqs. (a3) and (a5) we have

$$K_1 = K_{11} + K_{12} + K_{13} \tag{a6}$$

$$K_{11} = \text{Im} \int_{B_{1*}} \ln R \frac{d\varsigma}{\varsigma} = 2\pi \ln R \tag{a7}$$

$$K_{12} = \text{Im} \int_{B_{1*}} \ln(\varsigma - \varsigma_o) \frac{d\varsigma}{\varsigma} \tag{a8}$$

$$K_{13} = \text{Im} \int_{B_{1*}} \ln\left(1 - \frac{m}{\varsigma_o \varsigma}\right) \frac{d\varsigma}{\varsigma} = 0 \tag{a9}$$

Clearly, in Eq. (a9), the function $\ln(1 - \frac{m}{\varsigma_o \varsigma})$ can be considered as an analytic function in the region outside of the unit circle, since $|m/(\varsigma_o \varsigma)| < 1$, and the result is obtained.

For the integral K_{12}, we have (see Fig. 3)

$$\begin{aligned} K_{12} &= \text{Im} \int_{B_{1*}} \ln(\varsigma - \varsigma_o) \frac{d\varsigma}{\varsigma} = Re \int_{B_{1*}} \ln(\varsigma - \varsigma_o) d\theta = \int_0^{2\pi} \ln\left|2 \sin \frac{\theta - \theta_o}{2}\right| d\theta \\ &= \int_{\theta_o}^{2\pi + \theta_o} \ln\left|2 \sin \frac{\theta - \theta_o}{2}\right| d\theta = \int_0^{2\pi} \ln\left(2 \sin \frac{u}{2}\right) du = 4 \int_0^{\pi/2} \ln(2 \sin v)\, dv = 0 \end{aligned} \tag{a10}$$

Finally, from Eqs. (a6) to (a10), we have

$$K_1 = \int_{B_{1*}} \ln|z - t|\, d\theta = \text{Im} \int_{B_{1*}} \ln(z - t) \frac{d\varsigma}{\varsigma} = 2\pi \ln R, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{a11}$$

For the points ς and ς_o on the unit circle, there are $\bar{\varsigma} = 1/\varsigma$ and $\bar{\varsigma}_o = 1/\varsigma_o$. Therefore, the integral $K_2 - iK_3$ can be rewritten and evaluated immediately

$$K_2 - iK_3 = \int_{B_{1*}} e^{-2i\alpha} d\theta = \int_{B_{1*}} \frac{\bar{z} - \bar{t}}{z - t} d\theta = \int_{B_{1*}} \frac{m\varsigma_o \varsigma - 1}{\varsigma_o \varsigma - m} d\theta = \int_{B_{1*}} \frac{m\varsigma_o \varsigma - 1}{\varsigma_o \varsigma - m} \frac{d\varsigma}{i\varsigma} = 2\pi m, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{a12}$$

In Eq. (a12), function $\frac{m\varsigma_o\varsigma - 1}{\varsigma_o\varsigma - m}$ can be considered as an analytic function with the principal portion $G_\infty(\varsigma) = m$ at infinity [13].

Clearly, Eq. (a12) can be rewritten as

$$K_2 = \int_{B_{1*}} \cos 2\alpha d\theta = 2\pi m, \quad K_3 = \int_{B_{1*}} \sin 2\alpha d\theta = 0, \quad (\text{for } z \in \Gamma, t \in \Gamma) \tag{a13}$$

In the second group of evaluation for three integrals, the points "z" (for integration) is on the elliptic contour Γ, or $z \in \Gamma$. However, the point "t" now is an inner point to the elliptic contour, or $t \in S^+$, where S^+ denotes the finite elliptic region (Fig. 4). In this case, the three integrals L_1, L_2 and L_3 are defined as follows (Fig. 4)

$$L_1 = \int_{B_{1*}} \ln|z - t|\, d\theta = Re \int_{B_{1*}} \ln(z - t) d\theta = \text{Im} \int_{B_{1*}} \ln(z - t) \frac{d\varsigma}{\varsigma}, \quad (\text{for } z \in \Gamma, t \in S^+) \tag{a14}$$

$$L_2 - iL_3 = \int_{B_{1*}} \frac{\bar{z} - \bar{t}}{z - t} d\theta = \int_{B_{1*}} e^{-2i\alpha} d\theta, \quad \text{or } L_2 = \int_{B_{1*}} \cos 2\alpha d\theta,\ L_3 = \int_{B_{1*}} \sin 2\alpha d\theta, \quad (\text{for } z \in \Gamma, t \in S^+) \tag{a15}$$

Clearly, $\ln|z - t|$ is a single-valued function. However, $\ln(z - t)$ is a multiple-valued function. In this case, one may define a branch cut line from point "t" to $z = a$ to infinity (Fig. 4). However, from the integrand $\ln(z - t)d\theta$ in Eq. (a14) we see that, the final result for L_1 does not depend on the taken branch. In Eq. (a14), the term $\ln(z - t)$ can be written in the form

$$\ln(z - t) = \ln z + \ln\left(1 - \frac{t}{z}\right) = \ln R + \ln \varsigma + \ln\left(1 + \frac{m}{\varsigma^2}\right) + \ln\left(1 - \frac{t}{R(\varsigma + m/\varsigma)}\right) \tag{a16}$$

Note that the function $\ln(1-\frac{t}{z})=\ln(z-t)-\ln z$ (here $t\in S^{+}$) is an analytic single-valued function defined in the region outside the elliptic contour. This property is simply because both functions $\ln(z-t)$ and $\ln z$ have same increment $2\pi i$ when z goes around the elliptic contour. This property does not change after conformal mapping. Thus, $\ln(1-\frac{t}{R(\varsigma+m/\varsigma)})=\ln(1-\frac{t}{z})$ is also an analytic single-valued function defined in the region outside the unit circle. Clearly, $\ln(1+\frac{m}{\varsigma^2})$ is also a single-valued function in the region outside the unit circle. From Eqs. (a14) and (a16), we have

$$L_1=L_{11}+L_{12}+L_{13}+L_{14} \tag{a17}$$

$$L_{11}=\mathrm{Im}\int\limits_{B_{1*}}\ln R\frac{\mathrm{d}\varsigma}{\varsigma}=2\pi\ln R \tag{a18}$$

$$L_{12}=\mathrm{Im}\int\limits_{B_{1*}}\ln\varsigma\frac{\mathrm{d}\varsigma}{\varsigma}=\mathrm{Im}\int\limits_{\Gamma_*}\ln\varsigma\,\mathrm{d}(\ln\varsigma)=\mathrm{Im}\frac{(\ln\varsigma)^2}{2}\Big|_{\ln\varsigma=0}^{\ln\varsigma=2\pi i}=\mathrm{Im}(-2\pi^2)=0 \tag{a19}$$

$$L_{13}=\mathrm{Im}\int\limits_{B_{1*}}\ln\left(1+\frac{m}{\varsigma^2}\right)\frac{\mathrm{d}\varsigma}{\varsigma}=0 \tag{a20}$$

$$L_{14}=\mathrm{Im}\int\limits_{B_{1*}}\ln\left(1-\frac{t\varsigma}{R(\varsigma^2+m)}\right)\frac{\mathrm{d}\varsigma}{\varsigma}=0 \tag{a21}$$

In Eqs. (a20) and (a21), two functions $\ln(1+\frac{m}{\varsigma^2})$ and $\ln(1-\frac{t\varsigma}{R(\varsigma^2+m)})$ can be considered as a single-valued function defined in the region outside of unit circle, as mentioned above. Finally, we have

$$L_1=\int\limits_{B_{1*}}\ln|z-t|\,\mathrm{d}\theta=2\pi\ln R,\quad (\text{for } z\in\Gamma, t\in S^{+}) \tag{a22}$$

It is interesting to point out that the integral K_1 shown by Eq. (a11) and the integral L_1 shown by Eq. (22) take the same value. However, they have different definitions.

Similar to Eq. (a12), the integral for L_2-iL_3 is defined and evaluated as

$$\begin{aligned}L_2-iL_3&=\int\limits_{B_{1*}}\mathrm{e}^{-2i\alpha}\mathrm{d}\theta=\int\limits_{B_{1*}}\frac{\bar z-\bar t}{z-t}\mathrm{d}\theta=\int\limits_{B_{1*}}\frac{R(m\varsigma^2+1)-\bar t\varsigma}{R(\varsigma^2+m)-t\varsigma}\mathrm{d}\theta\\&=Re\int\limits_{B_{1*}}\frac{R(m\varsigma^2+1)-\bar t\varsigma}{R(\varsigma^2+m)-t\varsigma}\frac{\mathrm{d}\varsigma}{i\varsigma}=\mathrm{Im}\int\limits_{B_{1*}}\frac{R(m\varsigma^2+1)-\bar t\varsigma}{R(\varsigma^2+m)-t\varsigma}\frac{\mathrm{d}\varsigma}{\varsigma}\\&=\mathrm{Im}(2\pi im)=2\pi m,\quad (\text{for } z\in\Gamma, t\in S^{+})\end{aligned} \tag{a23}$$

Note that, the function $R(\varsigma^2+m)-t\varsigma$ has no zero for ς at the region outside of the unit circle. Second, the $\frac{R(m\varsigma^2+1)-\bar t\varsigma}{R(\varsigma^2+m)-t\varsigma}$ has its principal portion at infinity $G_\infty(\varsigma)=m$. Thus, the result shown by Eq. (a23) is obtained. Note that in Eq. (a23), the relation $\bar\varsigma=1/\varsigma$ is used for the point ς on the unit circle.

Clearly, Eq. (a23) can be rewritten as

$$L_2=\int\limits_{B_{1*}}\cos 2\alpha\mathrm{d}\theta=2\pi m,\quad L_3=\int\limits_{B_{1*}}\sin 2\alpha\mathrm{d}\theta=0,\quad (\text{for } z\in\Gamma, t\in S^{+}) \tag{a24}$$

It is interesting to point out that the integral K_2-iK_3 shown by Eq. (a12) and the integral L_2-iL_3 shown by Eq. (a23) take the same value. However, they have different definitions.

References

1. Rizzo, F.J.: An integral equation approach to boundary value problems in classical electrostatics. Q. J. Appl. Math. **25**, 83–95 (1967)
2. Cruse, T.A.: Numerical solutions in three-dimensional electrostatics. Int. J. Solids Struct. **5**, 1259–1274 (1969)
3. Jaswon, M.A., Symm, G.T.: Integral Equation Methods in Potential Theory and Elastostatics. Academic Press, London (1977)
4. Brebbia, C.A., Telles, T.C.F., Wrobel, L.C.: Boundary Element Techniques—Theory and Applications in Engineering. Springer, Heidelberg (1984)
5. Cheng, A.H.D., Cheng, D.S.: Heritage and early history of the boundary element method. Eng. Anal. Boundary Elements **29**, 286–302 (2005)
6. Christiansen, S.: On two methods for elimination of non-unique solutions of an integral equation with logarithmic kernel. Appl. Anal. **13**, 1–18 (1982)
7. Chen, J.T., Lin, S.R., Chen, K.H.: Degenerate scale problem when solving Laplace's equation by BEM and its treatment. Int. J. Numer. Mech. Eng. **62**, 233–261 (2005)
8. Vodicka, R., Mantic, V.: On invertibility of elastic single-layer potential operator. J. Elast. **74**, 147–173 (2004)
9. Vodicka, R., Mantic, V.: On solvability of a boundary integral equation of the first kind for Dirichlet boundary value problems in plane elasticity. Comput. Mech. **41**, 817–826 (2008)
10. Chen, Y.Z., Wang, Z.X., Lin, Z.X.: Numerical examination for degenerate scale problem for ellipse-shaped ring region. Int. J. Numer. Methods Eng. **71**, 1208–1230 (2007)
11. Chen, J.T., Shen, W.C.: Degenerate scale for multiply connected Laplace problems. Mech. Res. Commun. **34**, 69–77 (2007)
12. Chen, Y.Z., Lin, Z.X., Wang, Z.X.: Evaluation of the degenerate scale for BIE in plane elasticity and antiplane elasticity by using conformal mapping. Eng. Anal. Boundary Elements **33**, 147–159 (2009)
13. Muskhelishvili, N.I.: Some Basic Problems of Mathematical Theory of Elasticity. Noordhoff, Groningen (1953)
14. Chen, J.T., Kuo, S.R., Lin, J.H.: Analytical study and numerical experiments for degenerate scale problems in the boundary element method of two-dimensional elasticity. Int. J. Numer. Methods Eng. **54**, 1669–1681 (2002)
15. Chen, Y.Z., Wang, Z.X., Lin, Z.X.: A new kernel in BIE and the exterior boundary value problem in plane elasticity. Acta Mech. **206**, 207–224 (2009)
16. Chen, Y.Z., Lin, Z.X., Wang, Z.X.: Numerical solution for degenerate scale problem for exterior multiply connected region. Eng. Anal. Boundary Elements (2009, in press)