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Multiple scattering and modified Green's functions

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Abstract

Exterior boundary-value problems for the Helmholtz equation can be reduced to boundary integral equations. It is known that the simplest of these fail to be uniquely solvable at certain 'irregular frequencies.' For a single smooth scatterer, it is also known that irregular frequencies can be eliminated by using a modified fundamental solution, one that has additional singularities inside the scatterer. This approach is extended to treat the three-dimensional exterior Neumann problem for any finite number of disjoint smooth scatterers, using a fundamental solution that has additional singularities inside every scatterer.

Keywords: Boundary integral equations; Acoustics; Modified fundamental solutions

1. Introduction

Exterior boundary-value problems for the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0,$$

can be reduced to boundary integral equations in various ways [2,8]. If the goal is to obtain a Fredholm integral equation of the second kind, as is traditional,

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then there are two basic methods, known as the *indirect* and *direct* methods. Both methods use a fundamental solution, usually taken as the free-space Green's function; in three dimensions, this is

$$G(P, Q) = -\exp(ikR)/(2\pi R), \tag{1}$$

where R is the distance between the two points, P and Q.

Consider the exterior Neumann problem, where the normal derivative of u is prescribed on the boundary S. Then, in the indirect method, one starts with an assumed representation for u as a single-layer potential,

$$u(P) = \int_S \mu(q)\, G(P, q)\, ds_q, \tag{2}$$

and then derives a Fredholm integral equation of the second kind for the source density $\mu(q)$, where $q \in S$. In the direct method, one starts with the Helmholtz integral representation (obtained by applying Green's theorem to u and G) and then obtains a Fredholm integral equation of the second kind for the unknown boundary values of u.

Now, there is a well-known difficulty associated with the two methods sketched above: they both lead to integral equations that are not uniquely solvable when k^2 coincides with an eigenvalue of the corresponding interior Dirichlet problem—these are called the *irregular values* of k^2, or the *irregular frequencies*.

The indirect and direct methods can be modified in various ways so as to eliminate irregular frequencies. The main ideas are: modify the fundamental solution; modify the integral representation; combine two different integral equations; or augment one integral equation with some constraints. For reviews of such modifications, see [2, §3.6] or [14]. Here, we focus on the first of these ideas, where the free-space Green's function is replaced by a different fundamental solution. This idea was developed by Ursell [18,19], Jones [7] and Kleinman and Roach [9,10].

For multiple-scattering problems, the boundary S is not connected: physically, we may be interested in the scattering of waves by a collection of N obstacles. The standard theory assumes implicitly that $N = 1$, although much of it does extend to multiple-scattering problems without difficulty. One exception to this is the use of modified fundamental solutions for such problems. Nevertheless, we shall obtain a Fredholm integral equation of the second kind which we prove is *always* uniquely solvable: irregular frequencies do not occur. This is an extension to scattering by N three-dimensional obstacles of some work by Jones, Ursell, Kleinman and Roach, cited above. We use a modified fundamental solution which has additional singularities inside each scatterer. Our proof of unique solvability makes essential use of the addition theorems for outgoing and regular spherical wavefunctions.

2. Formulation

Suppose that we have N bounded, simply connected scatterers, B_i, $i = 1, 2, \dots, N$. The boundary of B_i is S_i, assumed to be smooth. We define

$$B = \bigcup_{i=1}^{N} B_i \quad \text{and} \quad S = \bigcup_{i=1}^{N} S_i,$$

so that B is the collection of all the interiors of the N scatterers and S is all their boundaries. The unbounded connected exterior is denoted by B_{e}.

We consider the following boundary-value problem.

Exterior Neumann Problem. Find a function $u(P)$ for $P \in B_{\mathrm{e}}$, where

$$(\nabla^2 + k^2)u = 0 \quad \text{in } B_{\mathrm{e}}, \tag{3}$$

$$u \text{ satisfies the Sommerfeld radiation condition at infinity} \tag{4}$$

and

$$\partial u/\partial n = f \quad \text{on } S. \tag{5}$$

Here, $f(q)$ is a given function, defined for $q \in S$, and $\partial/\partial n$ denotes normal differentiation in the direction from S towards B_{e}.

A standard argument [2, Theorem 3.13] using Rellich's lemma shows that the exterior Neumann problem has at most one solution: $f \equiv 0$ on S implies that $u \equiv 0$ in B_{e}.

The *exterior Dirichlet problem* is formulated in the same way, except (5) is replaced by $u = g$ on S, where $g(q)$ is a given function, defined for $q \in S$.

In what follows, we limit ourselves to the exterior Neumann problem. However, our analysis can be adapted to the exterior Dirichlet problem.

3. Integral equations: indirect method

Let us look for a solution of the exterior Neumann problem in the form of a single-layer potential; thus, we write $u(P)$ as (2) for $P \in B_{\mathrm{e}}$, where the density μ is to be found. For any reasonable μ, (3) and (4) are satisfied. It remains to satisfy the boundary condition (5). Imposing this, using the jump condition for the single-layer potential, we obtain

$$\mu(p) + \int_S \mu(q) \frac{\partial}{\partial n_p} G(p, q)\, ds_q = f(p), \quad p \in S. \tag{6}$$

If we can solve this integral equation for μ, we will have solved the exterior Neumann problem for u.

It turns out that (6) is uniquely solvable for μ, for any f, except when k^2 is an irregular value. At these irregular frequencies, the following boundary-value problem has a non-trivial solution.

Interior Dirichlet Problem. Find a function $\psi(P)$ for $P \in B$, where

$$(\nabla^2 + k^2)\psi = 0 \quad \text{in } B, \quad \text{and} \quad \psi = 0 \quad \text{on } S.$$

Let us denote the set of irregular values by IV(S). It is clear that

$$\mathrm{IV}(S) = \bigcup_{i=1}^{N} \mathrm{IV}(S_i), \tag{7}$$

because we can obtain a non-trivial solution of the interior Dirichlet problem for S by taking $\psi(P)$ to be an eigenfunction of the interior Dirichlet problem for S_j, say, with $\psi(P) \equiv 0$ for $P \in B_i$, $i = 1, 2, \ldots, N$, $i \neq j$; see [3, Chapter VI, §1.3].

The fact (7) is unfortunate, because it means that, in general, there are N times as many irregular frequencies as there are for a single scatterer. Thus, unless the scatterers are identical, the integral equation will have many irregular values, a countable set for each scatterer.

Note that the irregular values do not depend on the relative location or orientation of the scatterers, merely on their shape.

The integral equation (6) is an example of an *indirect boundary integral equation*, so called because the unknown density function does not have a clear physical interpretation. For multiple-scattering problems, it has been used by Isaacson [6], Sorensen [17] and Radlinski [15].

4. Integral equations based on Green's theorem: direct method

For the exterior Neumann problem, the integral representation obtained from an application of Green's theorem is

$$2u(P) = \int_S \left\{ G(P,q)\, f(q) - u(q) \frac{\partial}{\partial n_q} G(P,q) \right\} ds_q, \quad P \in B_{\mathrm{e}}. \tag{8}$$

This formula gives u at P in terms of $u(q)$, $q \in S$. To find $u(q)$, we let $P \to p \in S$, and obtain

$$u(p) + \int_S u(q) \frac{\partial}{\partial n_q} G(p,q)\, ds_q = \int_S G(p,q) f(q)\, ds_q, \quad p \in S. \tag{9}$$

This is another Fredholm integral equation of the second kind. As (9) is the Hermitian adjoint of (6), it has the same irregular values, namely IV(S).

The unknown u occurring in (9) has physical significance, and may even be the desired physical quantity. For this reason, integral equations based on Green's theorem are often known as *direct boundary integral equations*.

For multiple-scattering problems, direct boundary integral equations have been used by several people, including Millar [13], Andreasen [1] and Seybert et al. [16].

In what follows, we concentrate on the indirect method. Similar results can be obtained for the direct method.

5. Modified fundamental solutions: one scatterer

We have described the standard indirect method in Section 3, using the free-space Green's function G. However, there is no need to use G; one may use

$$G_1(P;Q) = G(P,Q) + H(P;Q),$$

where H has the following properties: for every $P \in B_\mathrm{e}$, $H(P;Q)$ satisfies the Helmholtz equation for *all* $Q \in B_\mathrm{e}$, and the radiation condition with respect to Q; $H(P;Q)$ must have some singularities at $P = Q$ for some $Q \in B$. (If a function v satisfies the Helmholtz equation everywhere in space and the radiation condition, one can prove that v must vanish everywhere; for a proof, see [4, p. 317].)

So, let us modify the fundamental solution with specific choices for H. We do this first for one three-dimensional scatterer ($N = 1$), so as to review what is known.

Choose the origin O at a point in $B \equiv B_1$, the interior of $S \equiv S_1$. Let B_ρ denote a ball of radius ρ, centred at O, with $B_\rho \subset B$. Let $\boldsymbol{r}_P$ and $\boldsymbol{r}_Q$ denote the position vectors of P and Q, respectively, with respect to O. Then, we replace the free-space Green's function $G(P,Q) \equiv G(\boldsymbol{r}_P, \boldsymbol{r}_Q)$ by

$$G_1(P,Q) = G(\boldsymbol{r}_P, \boldsymbol{r}_Q) - 2ik \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^m a_{\ell m} \psi_\ell^m(\boldsymbol{r}_P) \psi_\ell^{-m}(\boldsymbol{r}_Q), \quad (10)$$

where ψ_ℓ^m is a radiating spherical wavefunction, defined by (A.1); note that $\psi_\ell^m(\boldsymbol{r})$ is singular at $\boldsymbol{r} = \mathbf{0}$. The factors $\{-2ik(-1)^m\}$ are inserted for later convenience and also render the coefficients $a_{\ell m}$ dimensionless. These coefficients will be chosen later; for now, we merely impose the conditions that the infinite series in (10) be uniformly convergent for P and Q outside B_ρ, and that it can be differentiated twice, term by term.

So, we look for a solution of the exterior Neumann problem in the form

$$u(P) = \int_S \mu(q) G_1(P,q)\, ds_q \quad (11)$$

whence $\mu(q)$ satisfies

$$\mu(p) + \int_S \mu(q) \frac{\partial}{\partial n_p} G_1(p,q)\, ds_q = f(p), \quad p \in S. \tag{12}$$

The solvability of this integral equation is governed by the solvability of the corresponding homogeneous equation, namely

$$\mu(p) + \int_S \mu(q) \frac{\partial}{\partial n_p} G_1(p,q)\, ds_q = 0, \quad p \in S. \tag{13}$$

Theorem 5.1. *Suppose that the homogeneous integral equation* (13) *has a non-trivial solution* $\mu(q)$. *Then, the interior wavefunction*

$$U(P) = \int_S \mu(q)\, G_1(P,q)\, ds_q, \quad P \in B, \tag{14}$$

vanishes on S.

Proof [18, pp. 120, 123]**.** Define $U(P)$ for $P \in B_\mathrm{e}$ by (14); $\partial U/\partial n$ vanishes on S by (13). The uniqueness theorem for the exterior Neumann problem then asserts that $U \equiv 0$ in B_e. The result follows by noting that U is continuous across the source distribution on S [2, Theorem 2.12]. □

If we can show that $U \equiv 0$ in B, it will follow that (13) has only the trivial solution (because μ is proportional to the discontinuity in $\partial U/\partial n$ across S) and hence that the inhomogeneous equation (12) is uniquely solvable for any f. This can be achieved with some restrictions on the coefficients $a_{\ell m}$.

Theorem 5.2. *Suppose that*

$$\left| a_{\ell m} + \frac{1}{2} \right| > \frac{1}{2} \quad \textit{for } \ell = 0, 1, 2, \ldots \textit{ and } m = -\ell, \ldots, \ell, \tag{15}$$

or

$$\left| a_{\ell m} + \frac{1}{2} \right| < \frac{1}{2} \quad \textit{for } \ell = 0, 1, 2, \ldots \textit{ and } m = -\ell, \ldots, \ell. \tag{16}$$

Then, every solution of the homogeneous integral equation (13) *is a solution of*

$$\mu(p) + \int_S \mu(q) \frac{\partial}{\partial n_p} G(p,q)\, ds_q = 0, \quad p \in S, \tag{17}$$

which also satisfies

$$A_{\ell m} \equiv -2ik(-1)^m \int_S \mu(q) \psi_\ell^{-m}(\boldsymbol{r}_q)\, ds_q = 0, \tag{18}$$

for $\ell = 0, 1, 2, \ldots$ *and* $m = -\ell, \ldots, \ell$.

Proof. For $P \in B$ with $P \neq O$, we have

$$U(P) = \int_S \mu(q)\, G(P, q)\, ds_q + \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} A_{\ell m} \psi_\ell^m(\boldsymbol{r}_P),$$

where μ is a solution of (13). If we restrict P to lie in B_ρ, we can use the addition theorem (bilinear expansion),

$$G(\boldsymbol{r}_P, \boldsymbol{r}_Q) = -2ik \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^m \hat{\psi}_\ell^m(\boldsymbol{r}_P) \psi_\ell^{-m}(\boldsymbol{r}_Q), \tag{19}$$

which is valid for $r_P < r_Q$, to give

$$U(\boldsymbol{r}_P) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ A_{\ell m} \hat{\psi}_\ell^m(\boldsymbol{r}_P) + a_{\ell m} A_{\ell m} \psi_\ell^m(\boldsymbol{r}_P) \right\},$$
$$P \in B_\rho,\ P \neq O. \tag{20}$$

Here, $\hat{\psi}_\ell^m$ is a regular spherical wavefunction, defined by (A.1). Note that if we can show (18), then we can infer from (20) that $U \equiv 0$ in B_ρ and then, by analytic continuation, in B.

Next, following Ursell [19] and Colton and Kress [2, Theorem 3.35], we consider the integral

$$I \equiv \int_{\Omega_\rho} \left(U \frac{\partial \overline{U}}{\partial n} - \overline{U} \frac{\partial U}{\partial n} \right) ds,$$

where Ω_ρ denotes the spherical boundary of the ball $B_\rho \subset B$ and the overbar denotes complex conjugation. Using Green's theorem and Theorem 5.1, we see that

$$I = \int_S \left(U \frac{\partial \overline{U}}{\partial n} - \overline{U} \frac{\partial U}{\partial n} \right) ds = 0.$$

We can also evaluate I directly using the following lemma.

Lemma 5.3. *Suppose that* $U(\boldsymbol{r}_P)$, $P \in B_\rho$, *has an expansion*

$$U(\boldsymbol{r}_P) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ A_{\ell m} \hat{\psi}_\ell^m(\boldsymbol{r}_P) + B_{\ell m} \psi_\ell^m(\boldsymbol{r}_P) \right\}.$$

Then

$$\int_{\Omega_\rho} \left(U \frac{\partial \overline{U}}{\partial n} - \overline{U} \frac{\partial U}{\partial n} \right) ds = -\frac{2i}{k\rho} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(|B_{\ell m}|^2 + \operatorname{Re}\left\{ A_{\ell m} \overline{B_{\ell m}} \right\} \right).$$

Proof. Let

$$[u,v]=\int_{\Omega_\rho}\left(u\frac{\partial\bar v}{\partial n}-\bar v\frac{\partial u}{\partial n}\right)ds$$

so that $I=[U,U]$. Substituting for U gives

$$[U,U]=\sum_{\ell,m}\sum_{L,M}\bigl\{A_{\ell m}\overline{A_{LM}}\bigl[\hat\psi_\ell^m,\hat\psi_L^M\bigr]+B_{\ell m}\overline{B_{LM}}\bigl[\psi_\ell^m,\psi_L^M\bigr]\bigr\}$$
$$+2i\ \mathrm{Im}\sum_{\ell,m}\sum_{L,M}A_{\ell m}\overline{B_{LM}}\bigl[\hat\psi_\ell^m,\psi_L^M\bigr].$$

As both $\hat\psi_\ell^m$ and $\hat\psi_L^M$ are regular wavefunctions in B_ρ, $[\hat\psi_\ell^m,\hat\psi_L^M]=0$. Next,

$$\bigl[\hat\psi_\ell^m,\psi_L^M\bigr]=k\rho\bigl\{j_\ell(k\rho)\,\overline{h_L'(k\rho)}-j_\ell'(k\rho)\,\overline{h_L(k\rho)}\bigr\}\int_\Omega Y_\ell^m\overline{Y_L^M}\,d\Omega$$
$$=-i(k\rho)^{-1}\delta_{\ell L}\delta_{mM},$$

using the orthogonality of the spherical harmonics Y_ℓ^m over the unit sphere Ω, (A.2), and the Wronskian for spherical Bessel functions, $j_n(w)y_n'(w)-j_n'(w)y_n(w)=w^{-2}$. Similarly

$$\bigl[\psi_\ell^m,\psi_L^M\bigr]=-2i(k\rho)^{-1}\delta_{\ell L}\delta_{mM},$$

and then the result follows. □

Thus, returning to the proof of Theorem 5.2, we find that

$$0=I=-\frac{2i}{k\rho}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}|A_{\ell m}|^2\bigl\{\mathrm{Re}(a_{\ell m})+|a_{\ell m}|^2\bigr\},\tag{21}$$

using Lemma 5.3 with $B_{\ell m}=a_{\ell m}A_{\ell m}$. Since $a_{\ell m}$ satisfies the inequalities (15) or (16), it follows that (21) can only be satisfied if $A_{\ell m}=0$ for $\ell=0,1,2,\ldots$ and $m=-\ell,\ldots,\ell$. Also, substituting (10) in (13) shows that μ satisfies (17). □

This completes our review of scattering by a single obstacle.

6. Modified fundamental solutions: several scatterers

Consider the exterior Neumann problem for N three-dimensional scatterers. Choose an origin $O^n\in B_n$, the interior of S_n, and let $\boldsymbol{r}_P^n$ denote the position vector of a point P with respect to O^n. Let B_ρ^n denote a ball of radius ρ, centred at O^n; we choose ρ sufficiently small so that $B_\rho^n\subset B_n$ for $n=1,2,\ldots,N$. Let

$$G_1(P,Q) = G(\boldsymbol{r}_P, \boldsymbol{r}_Q) - 2ik \sum_{n=1}^{N} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^m a_{\ell m}^n \psi_\ell^m\big(\boldsymbol{r}_P^n\big) \psi_\ell^{-m}\big(\boldsymbol{r}_Q^n\big), \tag{22}$$

where $a_{\ell m}^n$ are constants. This fundamental solution is singular at every origin O^n, $n = 1, 2, \ldots, N$ (recall that $\psi_\ell^m(\boldsymbol{r}_P^n)$ is singular at O^n). Note that it is essential that our fundamental solution has this property; if we chose a fundamental solution that was not singular inside B_j, say, then we could not eliminate those irregular frequencies associated with S_j.

We look for a solution of our problem in the form (11), whence the source density $\mu(q)$ satisfies the integral equation (12). Moreover, the same arguments as before show that Theorem 5.1 is true (in the current notation).

Let us now investigate the solvability of the integral equation (12) and look for an analogue of Theorem 5.2. Suppose that $\mu(q)$ is any solution of the homogeneous integral equation (13). Consider the interior wavefunction $U(P)$, defined by (14), for $P \in B^j$ and some j. We restrict P to lie in $B_\rho^j \subset B_j$, and find that

$$U\big(\boldsymbol{r}_P^j\big) = \sum_{\ell,m} A_{\ell m}^j \hat{\psi}_\ell^m\big(\boldsymbol{r}_P^j\big) + \sum_{n=1}^{N} \sum_{\ell,m} a_{\ell m}^n A_{\ell m}^n \psi_\ell^m\big(\boldsymbol{r}_P^n\big), \tag{23}$$

for $P \in B_\rho^j$, where

$$A_{\ell m}^n = -2ik(-1)^m \int_S \mu(q) \psi_\ell^{-m}\big(\boldsymbol{r}_q^n\big)\, ds_q. \tag{24}$$

In order to use Lemma 5.3, we need the expansion of U to be in terms of functions centred on O^j, that is we need the addition theorem for outgoing spherical wavefunctions, (A.5). This gives the expansion

$$\psi_\ell^m\big(\boldsymbol{r}_P^n\big) = \sum_{L=0}^{\infty} \sum_{M=-L}^{L} S_{\ell L}^{mM}(\boldsymbol{b}^{nj}) \hat{\psi}_M^L\big(\boldsymbol{r}_P^j\big),$$

where $\boldsymbol{r}_P^n = \boldsymbol{b}^{nj} + \boldsymbol{r}_P^j$. This expansion is valid for $|\boldsymbol{r}_P^j| < |\boldsymbol{b}^{nj}|$, which is always true in our application. Note that $\boldsymbol{b}^{nj}$ is the position vector of O^j with respect to O^n, whence $\boldsymbol{b}^{nj} = -\boldsymbol{b}^{jn}$. The separation matrix $S_{\ell L}^{mM}$ is defined by (A.6). Hence, (23) becomes

$$U\big(\boldsymbol{r}_P^j\big) = \sum_{\ell,m} \hat{\psi}_\ell^m\big(\boldsymbol{r}_P^j\big) \left\{ A_{\ell m}^j + \sum_{\substack{n=1 \\ n \neq j}}^{N} \sum_{L,M} a_{LM}^n A_{LM}^n S_{L\ell}^{Mm}(\boldsymbol{b}^{nj}) \right\} + \sum_{\ell,m} \psi_\ell^m\big(\boldsymbol{r}_P^j\big) a_{\ell m}^j A_{\ell m}^j, \quad P \in B_\rho^j. \tag{25}$$

Using Green's theorem, Lemma 5.3 (for B_ρ^j) and the fact that $U(\boldsymbol{r}_P^j)$ vanishes on S_j, we obtain

$$0=\sum_{\ell,m}\left(\left|\mathcal{A}_{\ell m}^j\right|^2+\operatorname{Re}\left\{a_{\ell m}^j\right\}\left|A_{\ell m}^j\right|^2\right)$$
$$+\operatorname{Re}\sum_{\ell,m}\overline{\mathcal{A}_{\ell m}^j}\sum_{\substack{n=1\\n\neq j}}^{N}\sum_{L,M}\mathcal{A}_{LM}^n S_{L\ell}^{Mm}(\boldsymbol{b}^{nj}),\tag{26}$$

where $\mathcal{A}_{\ell m}^n=a_{\ell m}^n A_{\ell m}^n$. Eq. (26) holds for $j=1,2,\ldots,N$.

By comparison with the proof of Theorem 5.2, we expect to be able to deduce from (26) that $A_{\ell m}^n=0$. To do this, we sum over j and obtain

$$0=\sum_{j=1}^{N}\sum_{\ell,m}\operatorname{Re}\left(a_{\ell m}^j\right)\left|A_{\ell m}^j\right|^2+K_N,\tag{27}$$

where

$$K_N=\sum_{j=1}^{N}\sum_{\ell,m}\left|\mathcal{A}_{\ell m}^j\right|^2+\operatorname{Re}\sum_{j=1}^{N}\sum_{\ell,m}\overline{\mathcal{A}_{\ell m}^j}\sum_{\substack{n=1\\n\neq j}}^{N}\sum_{L,M}\mathcal{A}_{LM}^n S_{L\ell}^{Mm}(\boldsymbol{b}^{nj})$$
$$=\sum_{j=1}^{N}\sum_{\ell,m}\left|\mathcal{A}_{\ell m}^j\right|^2$$
$$+\frac{1}{2}\sum_{j=1}^{N}\sum_{\ell,m}\overline{\mathcal{A}_{\ell m}^j}\sum_{\substack{n=1\\n\neq j}}^{N}\sum_{L,M}\mathcal{A}_{LM}^n\left\{S_{L\ell}^{Mm}(\boldsymbol{b}^{nj})+\overline{S_{\ell L}^{mM}(\boldsymbol{b}^{jn})}\right\}.\tag{28}$$

But, from the definition of $S_{L\ell}^{Mn}$, (A.6), we can show that

$$S_{L\ell}^{Mm}(\boldsymbol{b}^{nj})+\overline{S_{\ell L}^{mM}(-\boldsymbol{b}^{nj})}=2\,\widehat{S}_{L\ell}^{Mm}(\boldsymbol{b}^{nj}),$$

where $\widehat{S}_{L\ell}^{Mm}$, defined by (A.4), is the separation matrix that occurs in the addition theorem for regular spherical wavefunctions:

$$\hat{\psi}_\ell^m\left(\boldsymbol{r}_P^n\right)=\sum_{L=0}^{\infty}\sum_{M=-L}^{L}\widehat{S}_{\ell L}^{mM}(\boldsymbol{b}^{nj})\hat{\psi}_M^L\left(\boldsymbol{r}_P^j\right).\tag{29}$$

Thus, (28) becomes

$$K_N=\sum_{j=1}^{N}\sum_{\ell,m}\left|\mathcal{A}_{\ell m}^j\right|^2+\sum_{j=1}^{N}\sum_{\ell,m}\overline{\mathcal{A}_{\ell m}^j}\sum_{\substack{n=1\\n\neq j}}^{N}\mathcal{C}_{\ell m}^{nj}$$
$$=\sum_{j=1}^{N}\sum_{\ell,m}\left\{\left|\mathcal{A}_{\ell m}^j\right|^2+\frac{1}{2}\sum_{\substack{n=1\\n\neq j}}^{N}\left(\overline{\mathcal{A}_{\ell m}^j}\mathcal{C}_{\ell m}^{nj}+\mathcal{A}_{\ell m}^j\overline{\mathcal{C}_{\ell m}^{nj}}\right)\right\},$$

where

$$\mathcal{C}_{\ell m}^{nj} = \sum_{L,M} \mathcal{A}_{LM}^{n} \widehat{S}_{L\ell}^{Mm}(\boldsymbol{b}^{nj}), \quad n \neq j,$$

and we have noted that K_N is real. Now, for complex quantities A and C, we have

$$\overline{A}C + A\overline{C} = |A + C|^2 - |A|^2 - |C|^2 \tag{30}$$

so that

$$K_N = \sum_{j=1}^{N} \sum_{\ell,m} \left\{ \left|\mathcal{A}_{\ell m}^{j}\right|^2 - \frac{1}{2} \sum_{\substack{n=1 \\ n \neq j}}^{N} \left(\left|\mathcal{A}_{\ell m}^{j}\right|^2 + \left|\mathcal{C}_{\ell m}^{nj}\right|^2 \right) \right\} + K_N',$$

where

$$K_N' = \frac{1}{2} \sum_{j=1}^{N} \sum_{\ell,m} \sum_{\substack{n=1 \\ n \neq j}}^{N} \left|\mathcal{A}_{\ell m}^{j} + \mathcal{C}_{\ell m}^{nj}\right|^2 \geqslant 0.$$

Further simplifications can be made because the separation matrix $\widehat{S}_{L\ell}^{Mm}$ is a unitary matrix; explicitly, we have

$$\sum_{\lambda=0}^{\infty} \sum_{\nu=-\lambda}^{\lambda} \widehat{S}_{L\lambda}^{M\nu}(\boldsymbol{b}) \overline{\widehat{S}_{\ell\lambda}^{m\nu}(\boldsymbol{b})} = \delta_{L\ell}\delta_{Mm}.$$

(This can be proved by using (29) twice.) It follows that

$$\sum_{\ell,m} \left|\mathcal{C}_{\ell m}^{nj}\right|^2 = \sum_{\ell,m} \left|\mathcal{A}_{\ell m}^{n}\right|^2, \quad n \neq j,$$

whence

$$K_N = (2 - N) \sum_{j=1}^{N} \sum_{\ell,m} \left|\mathcal{A}_{\ell m}^{j}\right|^2 + K_N'.$$

Thus, (27) becomes

$$0 = \sum_{j=1}^{N} \sum_{\ell,m} \left|A_{\ell m}^{j}\right|^2 \left\{ \mathrm{Re}\left(a_{\ell m}^{j}\right) - (N-2)\left|a_{\ell m}^{j}\right|^2 \right\} + K_N'.$$

Hence, we can deduce that $A_{\ell m}^{j} = 0$ provided all the coefficients $a_{\ell m}^{j}$ are such that

$$\mathrm{Re}\left(a_{\ell m}^{j}\right) - (N-2)\left|a_{\ell m}^{j}\right|^2 > 0.$$

Alternatively, if we use the identity

$$\overline{A}C + A\overline{C} = -|A - C|^2 + |A|^2 + |C|^2$$

instead of (30), we obtain

$$0 = \sum_{j=1}^{N} \sum_{\ell,m} \left|A_{\ell m}^{j}\right|^2 \left\{\mathrm{Re}\left(a_{\ell m}^{j}\right) + N\left|a_{\ell m}^{j}\right|^2\right\} + K_N'',$$

where $K_N'' \leqslant 0$. Hence, we deduce that $A_{\ell m}^{j} = 0$ provided that

$$\mathrm{Re}\left(a_{\ell m}^{j}\right) + N\left|a_{\ell m}^{j}\right|^2 < 0.$$

Summarising, we have proved the following result.

Theorem 6.1. *Suppose that*

$$\mathrm{Re}\left(a_{\ell m}^{j}\right) - (N-2)\left|a_{\ell m}^{j}\right|^2 > 0 \quad \text{for } \ell \geqslant 0,\ |m| \leqslant \ell \text{ and } j = 1, 2, \ldots, N$$

or

$$\mathrm{Re}\left(a_{\ell m}^{j}\right) + N\left|a_{\ell m}^{j}\right|^2 < 0 \quad \text{for } \ell \geqslant 0,\ |m| \leqslant \ell \text{ and } j = 1, 2, \ldots, N,$$

where N is the number of disjoint scatterers. Then, every solution of the homogeneous integral equation (13) *is a solution of* (17) *which also satisfies*

$$A_{\ell m}^{j} = 0 \quad \text{for } \ell \geqslant 0,\ |m| \leqslant \ell \text{ and } j = 1, 2, \ldots, N,$$

where $A_{\ell m}^{j}$ is defined by (24).

If the conditions on $a_{\ell m}^{j}$ are satisfied, this theorem implies unique solvability of the integral equation (12), for all wavenumbers k and for any f. This is an elegant theoretical result, because it yields solvability of the exterior Neumann problem without introducing non-compact operators.

It is noteworthy that Theorem 6.1 reduces formally to Theorem 5.2 when $N = 1$.

When $N = 2$, we obtain uniqueness when $\mathrm{Re}(a_{\ell m}^{j}) > 0$. This result was obtained previously by Martin [11] in two dimensions. The results for more scatterers, $N > 2$, are new; note that the conditions in Theorem 6.1 do depend on N.

7. Discussion

Theorem 6.1 is convenient analytically, because it shows how to eliminate *all* irregular frequencies. In actual computations, of course, the infinite summation in (22) would have to be truncated. For a single scatterer ($N = 1$), Jones [7] showed that one could eliminate a finite number of irregular frequencies with a truncated series in (10). It would be nice to have such a result for $N > 1$. However, our analysis does not extend readily to these cases. Previously [11], we examined the two-dimensional case with $N = 2$, and found that we could prove a partial generalization of Jones's result: we used a finite number of additional singularities inside one scatterer but an infinite number inside the other. Examples were given [11] to show that it may be difficult to obtain a complete generalization (no infinite series in (22)), even when $N = 2$. As far as we know, this situation has not changed.

Appendix A. Spherical wavefunctions

Let (r, θ, ϕ) be spherical polar coordinates of a point with position vector $\boldsymbol{r}$. Let $\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ so that $\boldsymbol{r} = r\hat{\boldsymbol{r}}$. Define

$$\psi_n^m(\boldsymbol{r}) = h_n(kr)Y_n^m(\hat{\boldsymbol{r}}) \quad \text{and} \quad \hat{\psi}_n^m(\boldsymbol{r}) = j_n(kr)Y_n^m(\hat{\boldsymbol{r}}), \tag{A.1}$$

where j_n is a spherical Bessel function and $h_n \equiv h_n^{(1)}$ is a spherical Hankel function. Y_n^m is a spherical harmonic, defined by

$$Y_n^m(\hat{\boldsymbol{r}}) = A_n^m P_n^m(\cos\theta)e^{im\phi},$$

where P_n^m is an associated Legendre function and the normalization constants A_n^m are chosen so that we have the orthogonality relation

$$\int_\Omega Y_n^m \overline{Y_\nu^\mu}\, d\Omega = \delta_{n\nu}\delta_{m\mu}; \tag{A.2}$$

here, Ω is the surface of the unit sphere, $r = 1$. $\hat{\psi}_n^m(\boldsymbol{r})$ is a *regular spherical wavefunction*. $\psi_n^m(\boldsymbol{r})$ is an *outgoing spherical wavefunction*; it is singular at $r = 0$ and satisfies the Sommerfeld radiation condition.

Let $\boldsymbol{r}_2 = \boldsymbol{r}_1 + \boldsymbol{b}$. Then

$$\hat{\psi}_n^m(\boldsymbol{r}_2) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} \widehat{S}_{n\nu}^{m\mu}(\boldsymbol{b})\hat{\psi}_\nu^\mu(\boldsymbol{r}_1). \tag{A.3}$$

This is the *addition theorem* for regular spherical wavefunctions. The entries in the separation matrix are given by

$$\widehat{S}_{n\nu}^{m\mu}(\boldsymbol{b}) = 4\pi i^{\nu-n}(-1)^\mu \sum_q i^q \hat{\psi}_q^{m-\mu}(\boldsymbol{b})\mathcal{G}(n, m; \nu, -\mu; q), \tag{A.4}$$

where $\mathcal{G}$ is a Gaunt coefficient, defined by

$$\mathcal{G}(n, m; \nu, \mu; q) = \int_\Omega Y_n^m Y_\nu^\mu \overline{Y_q^{m+\mu}}\, d\Omega.$$

The summation over q is finite.

For the outgoing spherical wavefunctions, we have

$$\psi_n^m(\boldsymbol{r}_2) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} S_{n\nu}^{m\mu}(\boldsymbol{b}) \hat{\psi}_\nu^\mu(\boldsymbol{r}_1) \tag{A.5}$$

for $r_1 < b$, and

$$\psi_n^m(\boldsymbol{r}_2) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \widehat{S}_{n\nu}^{m\mu}(\boldsymbol{b}) \psi_\nu^\mu(\boldsymbol{r}_1)$$

for $r_1 > b$, where

$$S_{n\nu}^{m\mu}(\boldsymbol{b}) = 4\pi i^{\nu-n}(-1)^\mu \sum_q i^q \psi_q^{m-\mu}(\boldsymbol{b}) \mathcal{G}(n, m; \nu, -\mu; q). \tag{A.6}$$

These addition theorems were obtained in the 1950's. For references and further information, see [5] and [12]. Note that the expansion (19) for G is the special case $n = 0$ of (A.5).

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