Applications of the direct Trefftz boundary element method to the free-vibration problem of a membrane

Jiang Ren Chang[a]
Department of System Engineering and Naval Architecture, National Taiwan Ocean University, 2 Pei-Ning Road, Keelung, Taiwan 202, Republic of China

Ru Feng Liu, Weichung Yeih, and Shyh Rong Kuo
Department of Harbor and River Engineering, National Taiwan Ocean University, 2 Pei-Ning Road, Keelung, Taiwan 202, Republic of China

(Received 28 November 2001; revised 24 May 2002; accepted 28 May 2002)

In this paper, the direct Trefftz method is applied to solve the free-vibration problem of a membrane. In the direct Trefftz method, there exists no spurious eigenvalue. However, an ill-posed nature of numerical instability encountered in the direct Trefftz method requires some treatments. The Tikhonov's regularization method and generalized singular-value decomposition method are used to deal with such an ill-posed problem. Numerical results show the validity of the current approach.
© *2002 Acoustical Society of America.* [DOI: 10.1121/1.1494992]

PACS numbers: 43.40.Dx [LLT]

I. INTRODUCTION

The eigenproblem is encountered very often in both engineering practice and academic research. In the design stage of a structure system, it is well known that the engineer should avoid having the structural eigenfrequencies coincide with the driving force frequency.[1] In the linear theory of vibration analysis, it is known that the eigenvalues and corresponding eigenfunctions are used to represent arbitrary functions, which means that they construct the operator spectrum.[2] It is not surprising then that the eigenproblem analysis becomes vitally important in tackling the wonderful world of vibration.

For an arbitrarily shaped domain, the numerical method sometimes is required in analysis because the analytical solution might not be available. To date, the finite element method (FEM) and the boundary element method (BEM) have been attractive to both the academic and engineering fields because of their respective merits. Eigenproblem analysis using the boundary element method has been studied for a long time. The complex-valued singular type auxiliary functions have been adopted.[3] To avoid complex-valued computation, De Mey[4] proposed two alternatives: the real-part formulation and regular formulation. The real-part formulation basically adopts the real-part function of the complex-valued auxiliary function (the fundamental solution) as the auxiliary function. The multiple reciprocity method (MR/BEM),[5] which treats the Helmholtz equation as a Poisson's equation, has been developed for eigenproblem analysis.[6–9] Both the real-part formulation and MR/BEM result in the spurious eigenvalues reported by many researchers.[6–8] Yeih *et al.*[9] proved that the real-part formulation and MR/BEM are equivalent mathematically, and the spurious eigenvalues encountered in both formulations stem from lacking constraint equations contributed by the imaginary part of the complex-valued fundamental solution. Another approach proposed by De Mey is the regular formulation. This method adopts a nonsingular auxiliary function to construct constraint equations. Kim and Kang[10] used the wave-type base functions, one regular formulation in our opinion, to analyze the free vibration of membranes. In their paper, the wave-type base functions, which are periodic along each element and propagate into the domain of interest, were selected to construct the needed equations. They pointed out that some incorrect answers might appear, and they explained these phenomena as due to the incompleteness of the basis functions. Later, Kang *et al.*[11,12] proposed another regular formulation using the so-called nondimensional dynamic influence function. Simply speaking, their method took the response at any point inside the domain of interest as a linear combination of many nonsingular point sources located at the selected boundary nodes. They claimed that their method worked very well, and no numerical instability behaviors were reported. Recently, Chen *et al.*[13] used the circular domain and the property of circulants to theoretically examine the possibility of using the imaginary dual BEM as a solution for the Helmholtz eigenproblems. They reported that spurious eigensolutions also appear in the imaginary dual BEM; however, no numerical examples were illustrated in their paper. Kuo *et al.*[14] pointed out that the ill-posed behavior should exist in the regular BEM formulation. They also proposed a combination of the Tikhonov's regularization method and generalized singular-value decomposition to treat such an ill-posed formulation. The regular formulations Kuo *et al.* proposed were the imaginary-part dual BEM and the plane-wave method. Nevertheless, in Kuo's paper their methods failed when a multiply connected domain problem was treated.

Another nonsingular boundary-type approach is the Trefftz method, which has been widely used to deal with many types of problems[15–17] and gained considerable popularity in the BEM community.[18] The boundary-type Trefftz method basically employs a complete set of solutions satisfying the governing equation as the beginning step. To derive

[a]Electronic mail: cjr@sena.ntou.edu.tw

the boundary integral equation, either the reciprocity law (which is similar to that used in the conventional BEM) or the weight residual method can be used. A main benefit of the Trefftz method is that it does not involve singular integrals because of the properties of its solution basis functions (T-complete functions); thus, it can be categorized into the regular boundary element method. Since it can avoid the difficulties with integration over singularities in the traditional BEM and often obtains more accurate results, various formulations of the Trefftz method have been developed and further applied to the engineering problems. Two important review articles about the Trefftz method[19] and its existing formulations[20] associated with comparisons with available boundary-type solution procedures can be found. In general, the formulations of the Trefftz method can be classified into the indirect and direct ones. For the indirect Trefftz formulation, the solutions of the problem are approximated by the superposition of the T-complete functions satisfying the governing equation, while in the direct one, the T-complete functions are taken as the weight function and the integral equations are derived from the governing equations. The mathematical bases of them are fairly different.[20] Although the Trefftz method has been successfully used to solve many problems, for the eigenproblem using the Helmholtz equation few attempts[21,22] have been found in the literature, to the authors' best knowledge. The reason may come from the ill-posed behavior nature of a regular formulation as Kuo *et al.*[14] have indicated, and it leads to the inaccuracy of the numerical results.[23] Most of the researchers have been studying the indirect Trefftz formulations. As a counterpart of the indirect Trefftz method, the direct Trefftz method is relatively new from its developing history,[20] and for some problems it performs in a superior way.[24] Besides, in direct Trefftz method there exist no spurious eigenvalues for the eigenproblem analysis, and it can deal with the multiply connected domain problem within its own formulation.

Based on the advantages over the traditional BEM, in this paper we will construct the direct Trefftz formulation to solve the free-vibration problem of a membrane. We prove that the direct Trefftz method has no spurious eigenvalues but has an ill-posed nature of numerical instability. The Tikhonov's regularization method[25] and generalized singular-value decomposition[26] are used to resolve such a problem. The direct Trefftz method can yield a solution for a multiply connected domain. Numerical results are provided to show the validity of our proposed approach.

II. DERIVATION OF DIRECT TREFFTZ FORMULATION

Consider a two-dimensional finite membrane Ω enclosed by the boundary Γ, the governing equation for the free-vibration problem is written as the Helmholtz equation, i.e.,

$$(\nabla^2+k^2)u(\mathbf{x})=0, \quad \mathbf{x}\in\Omega, \tag{1}$$

where ∇^2 is the Laplacian operator, k is the wave number, $u(\mathbf{x})$ is the physical quantity at $\mathbf{x}$.

The direct Trefftz formulation is derived as follows. Let a field $W(\mathbf{x})$ satisfying the Helmholtz equation, i.e.,

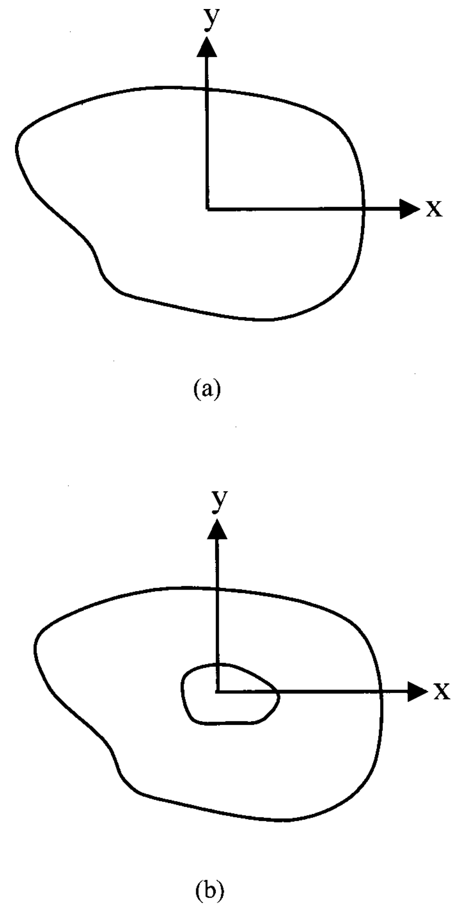


FIG. 1. (a) A simply connected domain. (b) A multiply connected domain of genus 1.

$$(\nabla^2+k^2)W(\mathbf{x})=0, \quad \mathbf{x}\in\Omega; \tag{2}$$

then, by the reciprocity theorem one can have

$$\int_\Gamma W(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial n}d\Gamma(\mathbf{x})=\int_\Gamma u(\mathbf{x})\frac{\partial W(\mathbf{x})}{\partial n}d\Gamma(\mathbf{x}), \tag{3}$$

where n denotes the out-normal direction at boundary point $\mathbf{x}$. The choice of $W(\mathbf{x})$ depends on the problem itself. A complete set of $W(\mathbf{x})$, written as $\{W_i(\mathbf{x})\}$, is chosen to give enough bases to represent all the physical quantities. This complete set is called the T-complete function set. In the mathematical language, the T-complete function set provides complete function bases to represent any physical field. For example, a simply connected domain shown in Fig. 1(a) and having the origin located inside the domain of interest, it is convenient to have the T-complete set as

$$\{J_0(kr),J_m(kr)\cos(m\theta),J_m(kr)\sin(m\theta)\}$$

for $m=1,2,3...$, in which J_m is the first kind of Bessel function of mth order, r is the distance from the origin to a domain point, and θ is the angle between the x axis and the radial vector from the origin to that domain point. For a multiply connected domain of genus 1 (i.e., a domain with one hole) and locating the origin inside the hole as shown in Fig. 1(b), the T-complete set is

$$\{J_0(kr), Y_0(kr), J_m(kr)\cos(m\theta), J_m(kr)\sin(m\theta),$$
$$Y_m(kr)\cos(m\theta), Y_m(kr)\sin(m\theta)\} \quad \text{for} \quad m=1,2,3,...,$$

where Y_m is the second kind of Bessel function of mth order.

For a boundary value problem, $\alpha_1 u+\beta_1 t=0$, where $t(\mathbf{x})\equiv[\partial u(\mathbf{x})]/\partial n$, one can assign

$$u=\beta_1\psi, \quad t=-\alpha_1\psi, \tag{4}$$

then substituting them into Eq. (3) produces

$$\int_\Gamma\left[\alpha_1 W(\mathbf{x})+\beta_1\frac{\partial W(\mathbf{x})}{\partial n}\right]\psi(\mathbf{x})d\Gamma(\mathbf{x})=0. \tag{5}$$

Changing the base functions, $W_i(\mathbf{x})$, and adopting constant element implementation for boundary unknowns ψ, one can have the following linear algebraic equation:

$$\{\alpha_1[\tilde{\mathbf{U}}]+\beta_1[\tilde{\mathbf{T}}]\}[\psi]=0, \tag{6}$$

where the components of the matrices are represented as

$$\tilde{U}_{ij}\equiv\int_{\Gamma_j} W_i(\mathbf{x})d\Gamma(\mathbf{x}), \tag{7a}$$

$$\tilde{T}_{ij}\equiv\int_{\Gamma_j}\frac{\partial W_i(\mathbf{x})}{\partial n}d\Gamma(\mathbf{x}), \tag{7b}$$

in which Γ_j is the jth element on the boundary and $W_i(\mathbf{x})$ is the ith base function.

There is something worth mentioning here; that is, the direct Trefftz method will not have spurious eigensolutions. To prove this, we need to take a look at Kuo's work.[14] Consider the original problem having boundary condition $\alpha_1 u+\beta_1 t=0$ on the boundary; the corresponding influencing matrix $\mathbf{A}_1$ is

$$\mathbf{A}_1=\alpha_1\tilde{\mathbf{U}}+\beta_1\tilde{\mathbf{T}}. \tag{8a}$$

Let us pick another complementary problem with boundary condition $\alpha_1 u+\beta_1 t=0$ on the boundary and

$$\det\begin{Vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{Vmatrix}\neq 0;$$

the influencing matrix $\mathbf{A}_2$ is

$$\mathbf{A}_2=\alpha_2\tilde{\mathbf{U}}+\beta_2\tilde{\mathbf{T}}. \tag{8b}$$

These two systems cannot have the same eigensolution. That is, at a specific wave number k, it is impossible to have the same nontrivial boundary eigensolution $\psi(\mathbf{x})$ for both systems. This theorem is proven in Kuo's paper,[14] and we adopt their results as follows for the readers' convenience.

A. Lemma 1

Given that the governing equation is a Helmholtz equation, $(\nabla^2+k^2)u(\mathbf{x})=0$, for a domain Ω enclosed by the boundary Γ, and that the overspecified homogeneous boundary conditions are $u(\mathbf{x})=0$ and $t(\mathbf{x})=0$ for $\mathbf{x}$ on the subboundary $\Gamma_1\subset\Gamma$, there exists a unique solution, $u(\mathbf{x})=0$ for $\mathbf{x}\in\Omega+\Gamma$.

B. Definition

Two sets of boundary conditions, $\alpha_1(\mathbf{x})u(\mathbf{x})+\beta_1(\mathbf{x})t(\mathbf{x})=0$ and $\alpha_2(\mathbf{x})u(\mathbf{x})+\beta_2(\mathbf{x})t(\mathbf{x})=0$, where $\alpha_1(\mathbf{x})$, $\alpha_2(\mathbf{x})$, $\beta_1(\mathbf{x})$, and $\beta_2(\mathbf{x})$ are given functions, are said to be homogeneous, linearly independent boundary conditions if and only if

$$\det\begin{Vmatrix}\alpha_1(\mathbf{x}) & \beta_1(\mathbf{x})\\ \alpha_2(\mathbf{x}) & \beta_2(\mathbf{x})\end{Vmatrix}\neq 0$$

for any $\mathbf{x}$ on the boundary.

C. Theorem 1

For the Helmholtz equation, given two systems having homogeneous, linearly independent boundary conditions on part of the boundary denoted as Γ_1, it is impossible for both systems to have the same eigensolution.

Theorem 1 supports the conclusion we mentioned above. Theorem 1 also hints that if there exists an "eigensolution" to make two systems have homogeneous, linearly independent boundary conditions degenerated at the same time, it must be the spurious eigensolution. Following this, now let us give the proof.

D. Theorem 2

For the Helmholtz equation, given a boundary condition as $\alpha_1 u+\beta_1 t=0$, the direct Trefftz formulation $\mathbf{A}_1(k)\psi=0$ cannot have a spurious eigensolution.

E. Proof

Let us pick another system with a boundary condition as $\alpha_2 u+\beta_2 t=0$ and

$$\det\begin{Vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{Vmatrix}\neq 0;$$

its corresponding eigenproblem is written as $\mathbf{A}_2(k)\psi=0$. Further, we assume that there exists a specific wave number k_c such that a nontrivial solution, $\psi(\mathbf{x})$, can satisfy $\mathbf{A}_1(k_c)\psi=0$ and $\mathbf{A}_2(k_c)\psi=0$ simultaneously. This means that $\psi(\mathbf{x})$ is a spurious eigensolution by Theorem 1. Suppose there are n constant elements on the boundary for both problems. Then, it can be said that the following linear system:

$$\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}_{2n\times n}=[\psi]_{n\times 1}\begin{bmatrix}\alpha_1\tilde{\mathbf{U}}+\beta_1\tilde{\mathbf{T}}\\ \alpha_2\tilde{\mathbf{U}}+\beta_2\tilde{\mathbf{T}}\end{bmatrix}[\psi]_{n\times 1}$$
$$=\begin{bmatrix}\alpha_1\mathbf{I} & \beta_1\mathbf{I}\\ \alpha_2\mathbf{I} & \beta_2\mathbf{I}\end{bmatrix}_{2n\times n}\begin{bmatrix}\tilde{\mathbf{U}}\\ \tilde{\mathbf{T}}\end{bmatrix}_{2n\times n}[\psi]_{n\times 1}=0, \tag{9}$$

must be linear dependent where $\mathbf{I}$ is an n by n identity matrix. It can then be said that this is possible if and only if

$$\text{rank}\left(\begin{bmatrix}\tilde{\mathbf{U}}\\ \tilde{\mathbf{T}}\end{bmatrix}\right)<n. \tag{10}$$

If n is very large and equal or unequal length element is adopted, from Eqs. (7a) and (7b), we can say that Eq. (10) is equivalent to

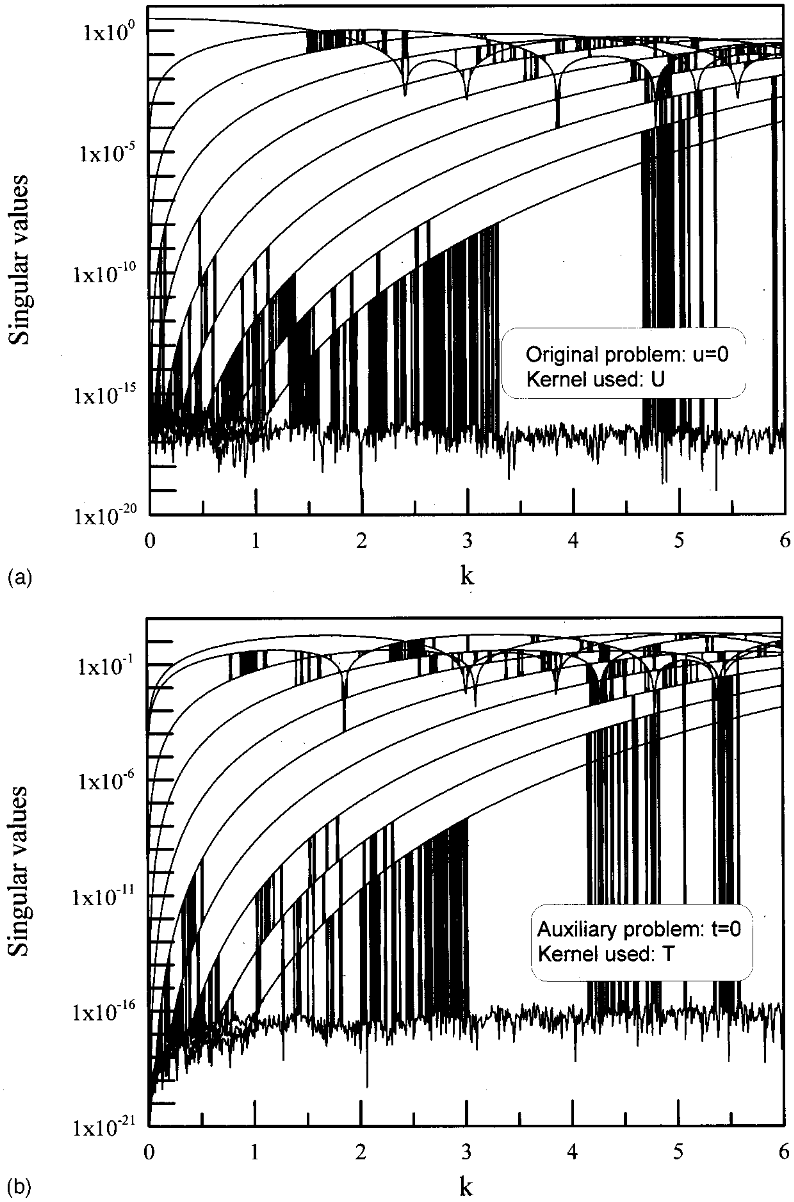


FIG. 2. (a) Numerical contamination exists by only performing the SVD technique for the original system. (b) Numerical contamination exists by only performing the SVD technique for the auxiliary system.

$$\operatorname{rank}\left(\begin{bmatrix} L_j W_i(\mathbf{x}_j) \\ L_j \dfrac{\partial W_i(\mathbf{x}_j)}{\partial n_j} \end{bmatrix}\right) < n, \tag{11}$$

where L_j is the element length of the jth element. The above equation is impossible to be achieved due to the linearly independent behavior of the base functions $W_i(\mathbf{x})$. Actually,

$$\operatorname{rank}\left(\begin{bmatrix} L_j W_i(\mathbf{x}_j) \\ L_j \dfrac{\partial W_i(\mathbf{x}_j)}{\partial n_j} \end{bmatrix}\right) = n.$$

This then leads to a contradiction and completes the proof.

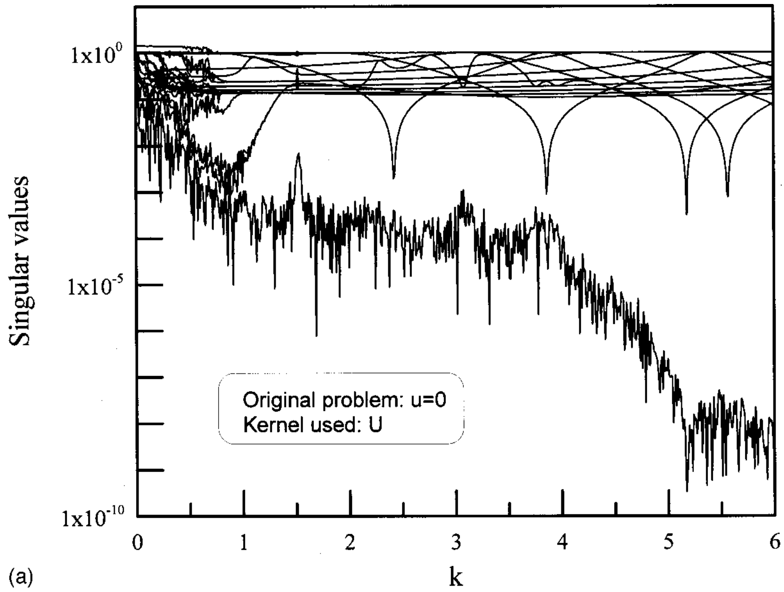
III. REGULARIZATION AND GENERALIZED SINGULAR-VALUE DECOMPOSITION METHODS TO DEAL WITH THE ILL-POSED PROBLEM

The Trefftz method adopts nonsingular base functions, and thus can be categorized into the regular BEM formulations.[14] However, the regular formulation leads to the ill-posed behaviors while the nodes (or elements) increase. Kuo *et al.*[14] explained the reason and proposed a method to fix it. Here, we simply introduce the method Kuo *et al.* suggested since we will use the same technique later on.

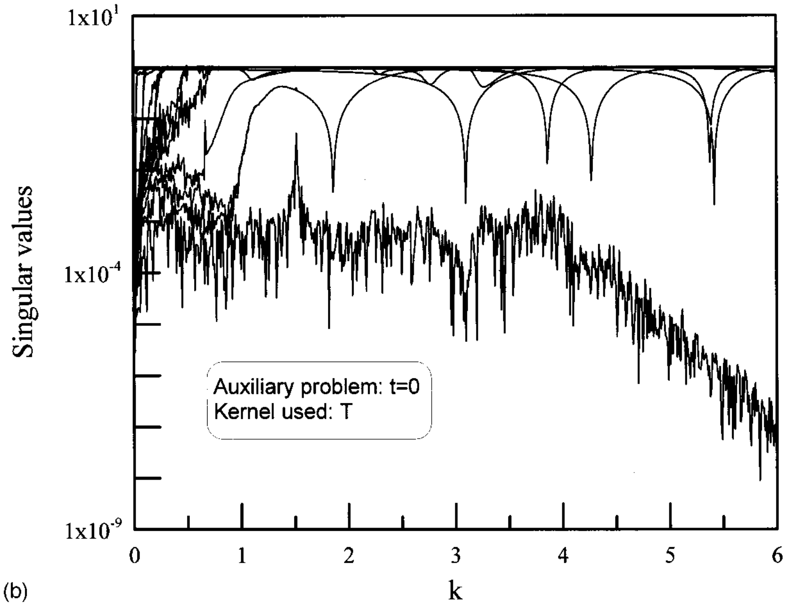
To treat the ill-posed behaviors, Kuo *et al.*[14] proposed using the Tikhonov's regularization method and generalized singular-value decomposition. Now, let us briefly introduce their idea. From Theorem 1, it can be seen that the spurious eigensolution will appear in two systems having homogeneous, linearly independent boundary conditions simultaneously. That is, we have a system as $[\mathbf{A}_1]_{n\times n}\psi_{n\times 1} = [\mathbf{A}_2]_{n\times n}\psi_{n\times 1} = \mathbf{0}$. Since both problems can have common spurious eigensolutions, we can intuitively decompose both matrices into the following form:

$$\mathbf{PW}_1\mathbf{x} = \mathbf{PW}_2\mathbf{x} = 0,$$

where $\mathbf{PW}_1 = \mathbf{A}_1$ and $\mathbf{PW}_2 = \mathbf{A}_2$. Then, spurious eigenvalues will result in the rank deficiency of matrix $\mathbf{P}$, and true eigenvalues will result in the rank deficiency of matrix $\mathbf{W}_1$ for the original problem. When the spurious eigenvalues are encoun-



(a)



(b)

FIG. 3. (a) Numerical contamination still exists after using Tikhonov's regularization method and the SVD technique for the original system. (b) Numerical contamination still exists after using the Tikhonov's regularization method and the SVD technique for the auxiliary system.

tered, basically we want to extract them by finding matrix **P**. That is, to perform a numerical operation of L'Hospital rule on an indefinite form of 0/0. The above-mentioned technique can be achieved using the QR factorization, which is the first step of the generalized singular-value decomposition.

Remember that the serious problem we encounter is not spurious eigensolution but numerical instability of this algorithm. To treat this, we will add some small quantities into the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ to make the numerically tiny singular values occurring in both matrices become "numerical spurious eigenvalues" such that the QR factorization can extract them. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ have the following singular value decompositions:

$$\mathbf{A}_1=\mathbf{P}\Sigma_1\mathbf{V}_1^*, \tag{12a}$$

$$\mathbf{A}_2=\mathbf{P}\Sigma_2\mathbf{V}_2^*, \tag{12b}$$

where $\mathbf{V}_i$ is the right unitary matrix of system i, the superscript "*" means take the transpose and complex conjugate of the matrix, and Σ_i is a singular value matrix of system i with singular values allocated in the diagonal line. When one of the singular values is numerically very small at a specific wave number, it can be said that the system has degenerated, i.e., that the wave number is an eigenvalue. However, when a nonsingular BEM is adopted, there exist many numerical tiny values in the singular values, which are not true zeros. This phenomenon becomes very severe when the number of elements increases and/or a direct eigenvalue search is used at a low wave number. Now, let us add two small quantities in the matrices to construct new influencing matrices as

$$\hat{\mathbf{A}}_1=\mathbf{P}(\Sigma_1+\varepsilon_1\mathbf{I})\mathbf{V}_1^*, \tag{13a}$$

$$\hat{\mathbf{A}}_2=\mathbf{P}(\Sigma_2+\varepsilon_2\mathbf{I})\mathbf{V}_2^*, \tag{13b}$$

where ε_i is the small value added to system i. The choice of ε_i is dependent on the problem itself; however, if they are larger than the unreasonable tiny values of singular values in the original two systems, but still small enough not to overcoat the true eigenvalue, one can then successfully extract

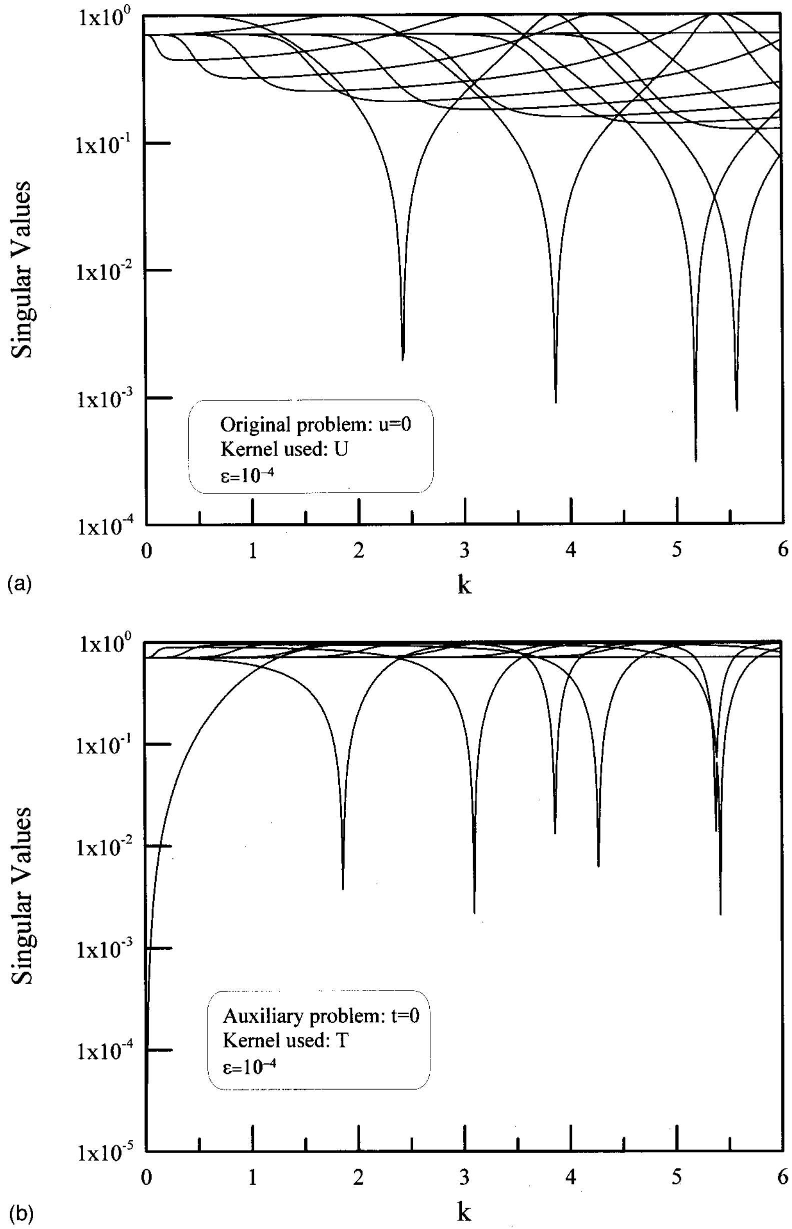


FIG. 4. (a) Numerical contamination is eliminated by using Tikhonov's regularization method, QR factorization, and the SVD technique for the original system. (b) Numerical contamination is eliminated by using Tikhonov's regularization method, QR factorization, and the SVD technique for the auxiliary system.

the contaminated tiny value. If one takes the QR factorization of $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$, the unreasonable ones can be extracted. The idea can be seen in Fig. 2–Fig. 4. Before treatments shown in Fig. 2, at low wave numbers some singular values are very small for both systems. However, when the minimum singular value of system one occurs for "mode shape p" (or pth singular vector), the corresponding singular value of system two for the same pth singular vector may not be the smallest one in system two. After QR factorization, this singular value remains the smallest in system one such that we still cannot distinguish if it is an eigenvalue, as shown in Fig. 3. After treatment as shown in Fig. 4, the contaminated singular values for both systems are elevated. The QR factorization method extracts such singular values out and changes the order of the singular values. Adding such a small value (for instance, $\varepsilon = 10^{-4}$ in Fig. 4) in the singular value cannot change the facts of true degenerated singular value. That is, at the true eigenvalue, the singular value of system one should approach zero but its corresponding part in system two will not be close to zero. Using this method, we can successfully treat the ill-posed behaviors; numerical examples will be given in the next section.

IV. NUMERICAL EXAMPLES

A. Example 1

A circular domain with radius $R_o = 1.0$ and the Dirichlet boundary condition, $u = 0$, is given.

Fifty-one constant elements are used, and the Neumann condition problem, $t = 0$, is chosen as the auxiliary problem. Using the Tikhonov's regularization method and generalized singular-value decomposition, eigenvalues are found successfully, as shown in Fig. 5. In this figure, the value in the bracket is the analytical solution.

B. Example 2

A circular domain with radius $R_o = 1.0$ and the Neumann boundary condition, $t = 0$, is given.

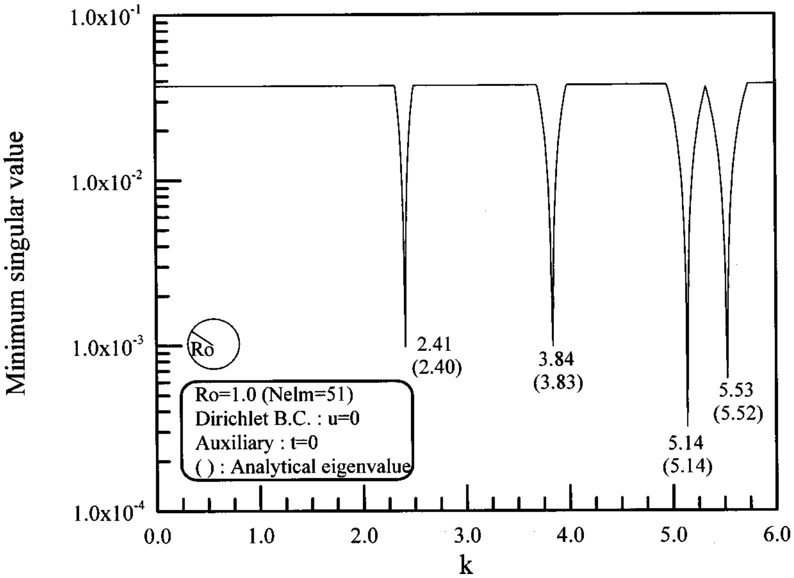


FIG. 5. Eigenvalues searching for the Dirichlet boundary condition of a unit circle by using the direct Trefftz method.

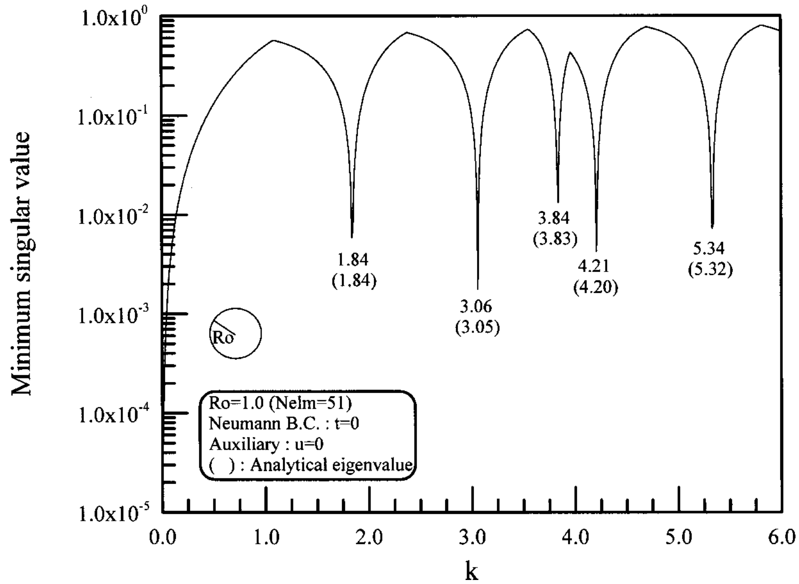


FIG. 6. Eigenvalues searching for the Neumann boundary condition of a unit circle by using the direct Trefftz method.

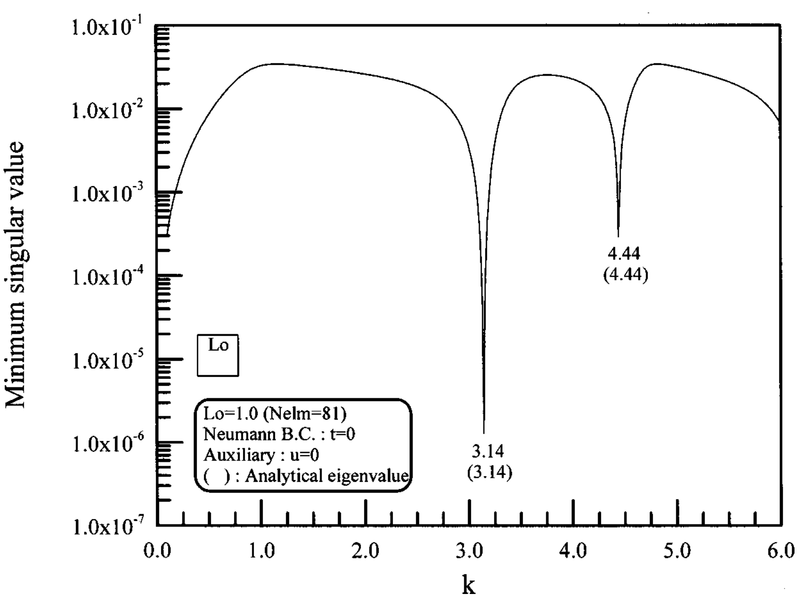


FIG. 7. Eigenvalues searching for the Neumann boundary condition of a square by using the direct Trefftz method with an auxiliary system, $u=0$.

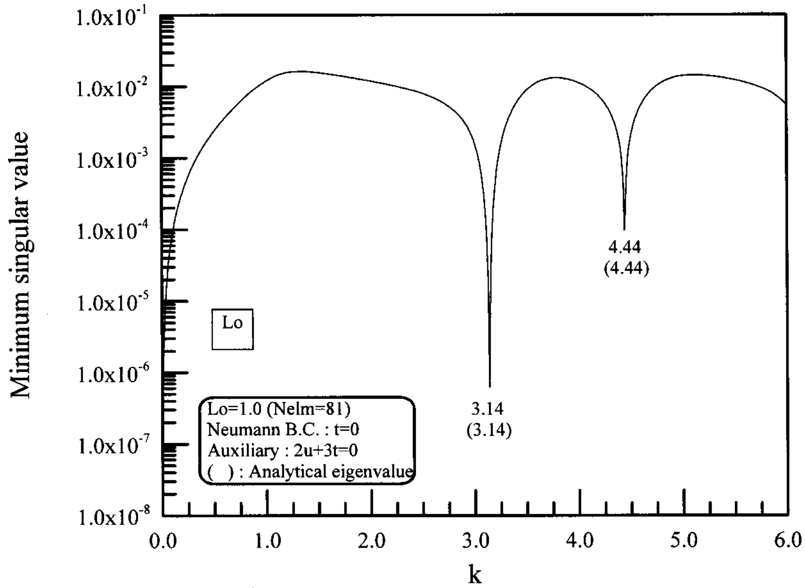


FIG. 8. Eigenvalues searching for the Neumann boundary condition of a square by using the direct Trefftz method with a different auxiliary system, $2u+3t=0$.

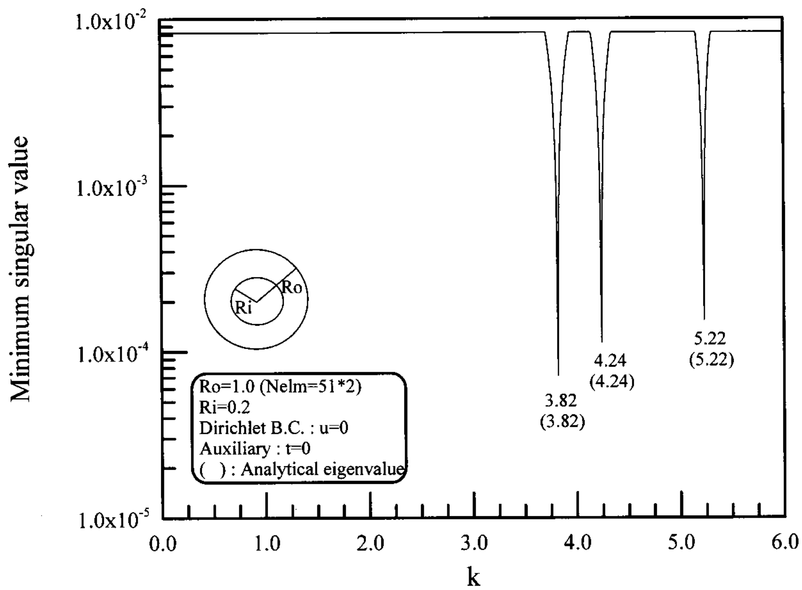


FIG. 9. Eigenvalues searching for the Dirichlet boundary condition of an annular region.

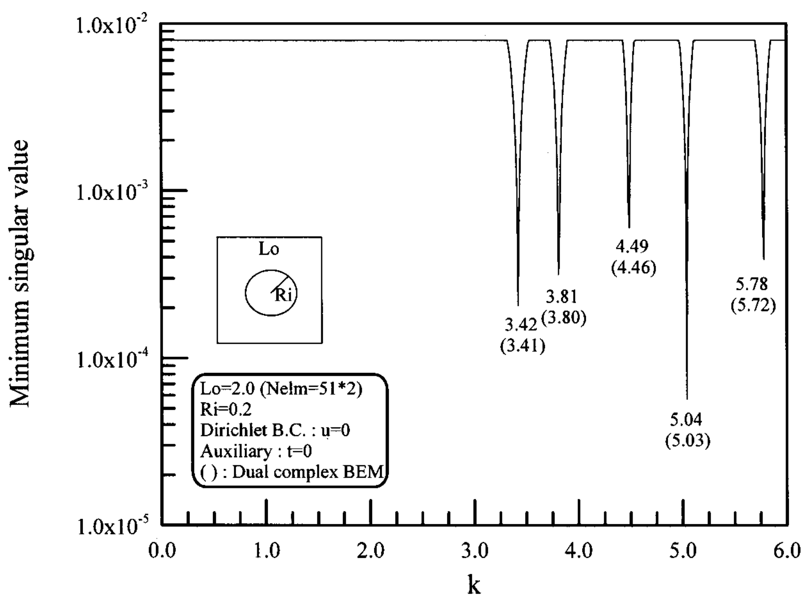


FIG. 10. Eigenvalues searching for the Dirichlet boundary condition of the multiply connected domain.

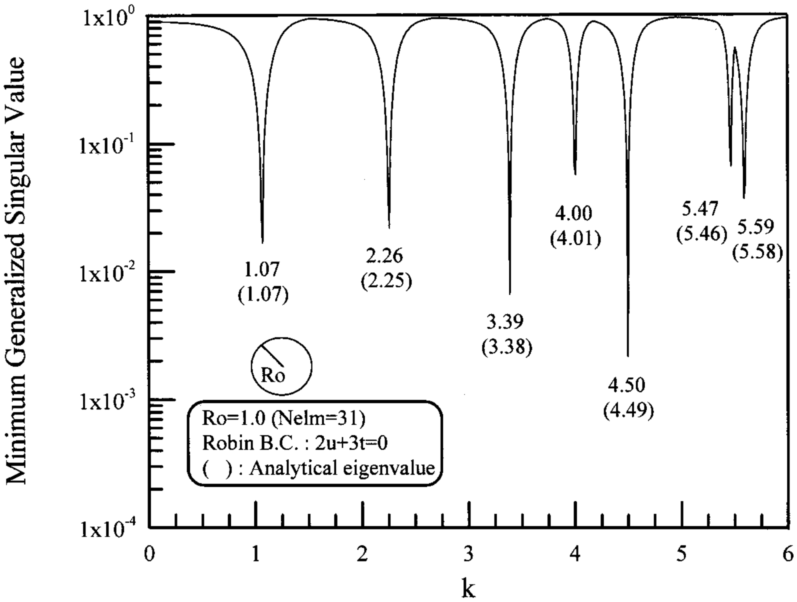


FIG. 11. Eigenvalues searching for the Robin-type boundary condition of a unit circle.

In this example, we can see that our method is valid for all kinds of boundary conditions. Again, 51 constant elements are used and the Dirichlet condition problem, $u=0$, is used as the auxiliary problem. Using the proposed method, eigenvalues are successfully found and are very close to the analytical values, as shown in Fig. 6.

C. Example 3

A square membrane with edge length $L_o=1.0$ and the Neumann boundary condition, $t=0$, is prescribed.

In this example, a domain without radial symmetry is illustrated. Eighty-one constant elements are used and the Dirichlet boundary problem, $u=0$, is chosen as the auxiliary problem. It can be found in Fig. 7 that the numerical results match the analytical solutions very well.

We have claimed that any problem having a linearly independent boundary condition to the original problem can be used as an auxiliary problem. In this example, we use another auxiliary problem $2u+3t=0$. The results are shown in Fig. 8, and our approach works as expected.

D. Example 4

An annular region with the outer radius $R_o=1.0$ and inner radius $R_i=0.2$, and a Dirichlet boundary condition, $u=0$, is prescribed on the boundary.

The domain is a multiply connected domain, which shows the superiority of the current approach over Kuo's method.[14] Their methods were proven to fail when a multiply connected domain is treated. However, the direct Trefftz method can easily overcome this problem by putting the origin inside the hole. In this example, 51 elements are used for the outer and inner boundaries. The auxiliary problem is the Neumann problem, $t=0$. As shown in Fig. 9, eigenvalues can be found successfully. The analytical values are obtained using the eigenequation[27]

$$[J_m(kR_o)Y_m(kR_\mathrm{i})-Y_m(kR_o)J_m(kR_\mathrm{i})]=0.$$

E. Example 5

A multiply connected domain with the outer boundary is a square with edge length $L_o=2.0$, and the inner boundary is a circle with a radius $R_i=0.2$. The origin of the circular hole is the geometric center of the whole domain. The boundary condition is the Dirichlet condition, $u=0$.

In this example, no analytical solution is available. We compared our results with those obtained from the complex-valued dual BEM. The auxiliary system is the Neumann problem, $t=0$. As shown in Fig. 10, numerical results obtained from the direct Trefftz method are close to those obtained from the complex-valued dual BEM. The reason why a complex-dual BEM is required is explained in Chang's dissertation.[27] He explained that solving an eigenvalue problem of a multiply connected domain by the complex-valued singular integral equation or the complex-valued hypersingular integral equation will result in an unreasonable numerical resonance. He named this kind of degeneracy of the direct BEM the pseudofictitious eigenvalue. To treat this unreasonable degeneracy, a combined use of singular and hypersingular integral equations was suggested. For more detail, readers can refer to Chang's dissertation.[27]

F. Example 6

A circular domain with radius $R_o=1.0$ and the Robin-type boundary condition, $2u+3t=0$, is given on the boundary.

The analytical values for this case can be obtained by using the true eigenequation as:[14] $2J_m(kR_o)+3J_m'(kR_o)=0$. In this case, 31 elements and 31 bases are used correspondingly. When the Tikhonov's regularization method and generalized singular-value decomposition are adopted, eigenvalues are found successfully and numerical results match analytical solutions very well, as shown in Fig. 11.

V. CONCLUSIONS

In this paper, the direct Trefftz method was used to solve the free-vibration problem of a membrane. It was found that the direct Trefftz method has no spurious eigenvalues, but leads to numerical instability when the number of elements increases and/or an eigenvalue search is conducted in the low wave number range. To treat this ill-posed behavior, we adopted the Tikhonov's regularization method and the generalized singular-value decomposition. The direct Trefftz method can easily treat a multiply connected domain of genus 1 by putting the origin inside the hole. Six numerical examples were provided to show the validity of the current approach, and good matches can be achieved in comparison with the analytical solutions or numerical results obtained from other methods.

ACKNOWLEDGMENTS

The first author would like to express his thank to the National Science Council, Taiwan, for their financial support under Contract Number: NSC-90-2611-E-019-018.

[1] R. W. Clough and J. Penzien, *Dynamics of Structures*, 2nd ed. (McGraw-Hill, New York, 1993).

[2] I. Stakgold, *Green's Functions and Boundary Value Problems*, 2nd ed. (Wiley, New York, 1998).

[3] N. Kamiya, E. Andoh, and K. Nogae, ''A new complex-valued formulation and eigenvalue analysis of the Helmholtz equation by boundary element method,'' Adv. Eng. Software **26**, 219–227 (1996).

[4] G. De Mey, ''A simplified integral equation method for the calculation of the eigenvalues of Helmholtz equation,'' Int. J. Numer. Methods Eng. **11**, 1340–1342 (1977).

[5] A. J. Nowak and C. A. Brebbia, ''The multiple reciprocity method—A new approach for transforming BEM domain integral to the boundary,'' Eng. Anal. Boundary Elem. **6**, 164–167 (1989).

[6] J. T. Chen and F. C. Wong, ''Analytical derivations for one-dimensional eigenproblems using dual BEM and MRM,'' Eng. Anal. Boundary Elem. **20**(1), 25–33 (1997).

[7] W. Yeih, J. T. Chen, and C. M. Chang, ''Applications of dual MRM for determining the natural frequencies and natural modes of an Euler–Bernoulli beam using the singular value decomposition method,'' Eng. Anal. Boundary Elem. **23**, 339–360 (1999).

[8] W. Yeih, J. R. Chang, C. M. Chang, and J. T. Chen, ''Application of dual BEM for determining the natural frequencies and natural modes of a rod using the singular value decomposition method,'' Adv. Eng. Software **30**(7), 459–468 (1999).

[9] W. Yeih, J. T. Chen, K. H. Chen, and F. C. Wong, ''A study on the multiple reciprocity method and complex-valued formulation for the Helmhotz equation,'' Adv. Eng. Software **29**(1), 1–6 (1998).

[10] Y. Y. Kim and J. H. Kang, ''Free vibration analysis of membranes using wave-type base functions,'' J. Acoust. Soc. Am. **99**, 2938–2946 (1996).

[11] S. W. Kang, J. M. Lee, and Y. J. Kang, ''Vibration analysis of arbitrarily shaped membranes using nondimensional dynamic influence function,'' J. Sound Vib. **221**(1), 117–132 (1999).

[12] S. K. Kang and J. M. Lee, ''Eigenmode analysis of arbitrarily shaped two-dimension cavities by the method of point-matching,'' J. Acoust. Soc. Am. **107**, 1153–1160 (2000).

[13] J. T. Chen, S. R. Kuo, and K. H. Chen, ''A nonsingular integral formulation for the Helmholtz eigenproblems of a circular domain,'' J. Chin. Inst. Eng. **22**(6), 729–739 (1999).

[14] S. R. Kuo, W. Yeih, and Y. C. Wu, ''Application of the generalized singular-value decomposition on the eigenproblem using the incomplete boundary element formulation,'' J. Sound Vib. **235**(5), 813–845 (2000).

[15] W. G. Jin, Y. K. Cheung, and O. C. Zienkiewicz, ''Applications of the Trefftz method in plane elasticity problems,'' Int. J. Numer. Methods Eng. **30**, 1147–1161 (1990).

[16] W. G. Jin, Y. K. Cheung, and O. C. Zienkiewicz, ''Trefftz method for Kirchhoff plate bending problems,'' Int. J. Numer. Methods Eng. **36**, 765–781 (1993).

[17] I. Harari, P. Barai, and P. E. Barai, ''Three-dimensional infinite elements based on a Trefftz formulation,'' J. Comput. Acoust. **9**(2), 381–394 (2001).

[18] N. Kamiya and E. Kita, ''Editorial—Trefftz method 70 years,'' Adv. Eng. Software **24**, 1 (1995).

[19] J. Jirousek and A. Wróblewski, ''T-elements: State of the art and future trends,'' Arch. Comput. Methods Eng. **3**(4), 323–434 (1996).

[20] E. Kita and N. Kamiya, ''Trefftz method: An overview,'' Adv. Eng. Software **24**, 3–12 (1995).

[21] Y. K. Cheung, W. G. Jin, and O. C. Zienkiewicz, ''Solution of Helmholtz equation by Trefftz method,'' Int. J. Numer. Methods Eng. **32**, 63–78 (1991).

[22] N. Kamiya and S. T. Wu, ''Generalized eigenvalue formulation of the Helmholtz equation by Trefftz method,'' Eng. Comput. **11**, 177–186 (1994).

[23] E. Kita, N. Kamiya, and T. Iio, ''Application of a direct Trefftz method with domain decomposition to 2D potential problems,'' Eng. Anal. Boundary Elem. **23**, 539–548 (1999).

[24] W. G. Jin and Y. K. Cheung, ''Trefftz direct method,'' Adv. Eng. Software **24**, 65–69 (1995).

[25] A. N. Tikhonov, ''Solution of incorrectly formulated problems and regularization method,'' Dokl. Akad. Nauk SSSR **151**(3), 501–504 (1963).

[26] G. H. Golub and C. F. Van Loan, *Matrix Computation*, 2nd ed. (Johns Hopkins University Press, Baltimore, 1989).

[27] J. R. Chang, ''Applications of domain partitioning on vibration problems using the dual BEM,'' Ph.D. dissertation of Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan (1999).