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國立臺灣海洋大學 National Taiwan Ocean University

反平面線裂縫與線剛性夾雜之*J*積分解析推導與BEM驗證

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摘要：

在破壞力學中，應力強度因子是十分重要的。本研究以解析與數值探討反平面力場中線裂縫與線剛性夾雜問題。解析方面，引入橢圓座標之分離核與傅立葉展開，導得孔洞的位移與應力。再利用ξ_0=0，使其橢圓退化成線。在橢圓的座標中，透過橢圓曲線解析探討與路徑無關之J積分。此外，BEM之數值結果將與解析解進行驗證。本文提供三種觀點求得應力強度因子，位移或應力兩種觀點，由邊界到端點、域內逼近等取極限的兩種方法之外，還有一種方法是J積分以能量觀點來探討，三種方法之結果吻合一致。

問題描述

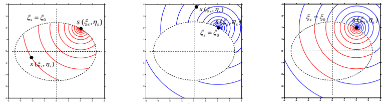
控制方程：$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\nabla^2 w=0$, 邊界條件

- 孔洞：$t(\mathbf{x})=\frac{\partial w}{\partial n}=0, \mathbf{x}\in B$
- 剛性夾雜：$u(\mathbf{x})=0, \mathbf{x}\in B$

分離核橢圓座標展開

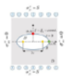
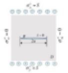
$$U(s,x)=\begin{cases}U^i(\xi_s,\eta_s;\xi_x,\eta_x)=\xi_s+\ln\frac{c}{2}-\sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_s}\cosh m\xi_x\cos m\eta_x\cos m\eta_s-\sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_s}\sinh m\xi_x\sin m\eta_x\sin m\eta_s, & \xi_s\ge\xi_x\\ U^e(\xi_s,\eta_s;\xi_x,\eta_x)=\xi_x+\ln\frac{c}{2}-\sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_x}\cosh m\xi_s\cos m\eta_x\cos m\eta_s-\sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_x}\sinh m\xi_s\sin m\eta_x\sin m\eta_s, & \xi_s<\xi_x\end{cases}$$

$$T(s,x)=\begin{cases}T^i(\xi_s,\eta_s;\xi_x,\eta_x)=\frac{-1}{J(\xi_s,\eta_s)}\left(1+2\sum_{m=1}^{\infty}e^{-m\xi_s}\cosh m\xi_x\cos m\eta_x\cos m\eta_s+2\sum_{m=1}^{\infty}e^{-m\xi_s}\sinh m\xi_x\sin m\eta_x\sin m\eta_s\right), & \xi_s>\xi_x\\ T^e(\xi_s,\eta_s;\xi_x,\eta_x)=\frac{1}{J(\xi_s,\eta_s)}\left(2\sum_{m=1}^{\infty}e^{-m\xi_x}\sinh m\xi_s\cos m\eta_x\cos m\xi_s+2\sum_{m=1}^{\infty}e^{-m\xi_x}\cosh m\xi_s\sin m\eta_x\sin m\eta_s\right), & \xi_s<\xi_x\end{cases}$$

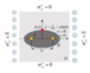
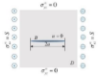


內域($\xi_x\le\xi_s$)　外域($\xi_x>\xi_s$)　全域($0<\xi_x<\infty$)

橢圓孔洞退化成線裂縫

反平面力場問題	橢圓孔洞	線裂縫
圖 \ 函數		$\xi_0=0$ 
邊界條件	Neumann $t(x)=\frac{\partial w}{\partial n}=0, x\in B$	Neumann $t(x)=\frac{\partial w}{\partial n}=0, x\in B$
位移 $u_z(\xi_x,\eta_x)$	$u_z(\xi_x,\eta_x)=\frac{S}{\mu}c\sin\eta_x(\sinh\xi_x+e^{\xi_0-\xi_x}\cosh\xi_0)$.	$\frac{S}{\mu}a\sin\eta_x(\sinh\xi_x+e^{-\xi_x})$
應力 $\sigma_{xz}(\xi_x,\eta_x)$	$\frac{-Se^{\xi_0}\cosh\xi_0\cos\eta_x\sin\eta_x}{(\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2}$	$\frac{-S\sin\eta_x\cos\eta_x}{(\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2}$
應力 $\sigma_{yz}(\xi_x,\eta_x)$	$\frac{S(\cosh\xi_x\sin^2\eta_x(\cosh\xi_x-e^{\xi_0-\xi_x}\cosh\xi_0)+\sinh\xi_x\cos^2\eta_x(\sinh\xi_x+e^{\xi_0-\xi_x}\cosh\xi_0))}{((\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2)}$	$\frac{S\sinh\xi_x\cosh\xi_x}{(\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2}$

橢圓剛性夾雜退化成線剛性夾雜

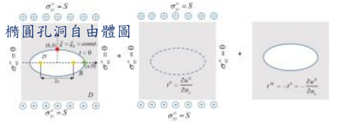
反平面力場問題	橢圓剛性夾雜	線剛性夾雜
圖 \ 函數		$\xi_0=0$ 
邊界條件	Dirichlet $u(x)=0, x\in B$	Dirichlet $u(x)=0, x\in B$
位移 $u_z(\xi_x,\eta_x)$	$u_z(\xi_x,\eta_x)=\frac{S}{\mu}c\cos\eta_x(\cosh\xi_x-e^{\xi_0-\xi_x}\cosh\xi_0)$.	$\frac{S}{\mu}a\cos\eta_x(\cosh\xi_x-e^{-\xi_x})$
應力 $\sigma_{xz}(\xi_x,\eta_x)$	$=\frac{S(\cosh\xi_x\sin^2\eta_x(\cosh\xi_x-e^{\xi_0-\xi_x}\cosh\xi_0)+\sinh\xi_x\cos^2\eta_x(\sinh\xi_x+e^{\xi_0-\xi_x}\cosh\xi_0))}{((\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2)}$	$\frac{S\sinh\xi_x\cosh\xi_x}{(\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2}$
應力 $\sigma_{yz}(\xi_x,\eta_x)$	$\frac{Se^{\xi_0}\cosh\xi_0\sin\eta_x\cos\eta_x}{((\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2)}$	$\frac{S\sin\eta_x\cos\eta_x}{(\sinh\xi_x\cos\eta_x)^2+(\cosh\xi_x\sin\eta_x)^2}$

邊界積分方程

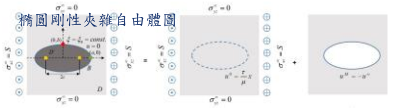
$$2\pi u(\mathbf{x})=\int_B T^e(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s})-\int_B U^e(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}),\ \mathbf{x}\in D\cup B$$

零場邊界積分方程

$$0=\int_B T^i(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s})-\int_B U^i(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}),\ \mathbf{x}\in D^c\cup B$$

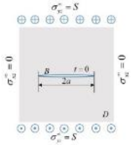
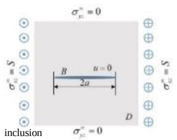
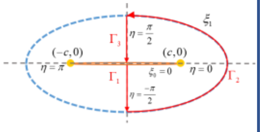
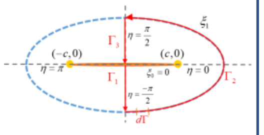
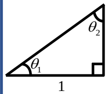
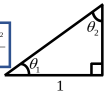


橢圓孔洞自由體圖

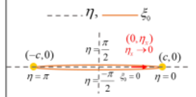
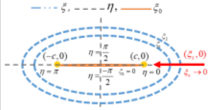
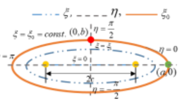


橢圓剛性夾雜自由體圖

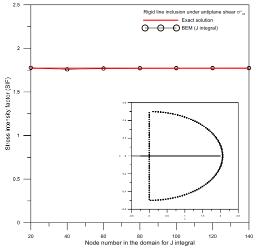
結果與討論：

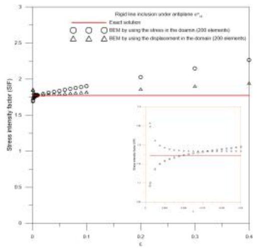
圖說 \ 函數	Crack	Rigid-line inclusion
		
積分路程		
$J_{\Gamma_1}=\int_{\Gamma_1}Wdy-\int_{\Gamma_1}T_i\frac{\partial u_i}{\partial x}d\Gamma=\int_{\Gamma_1}(Wn_x-T_i\frac{\partial u_i}{\partial x})d\Gamma$	$\frac{-S^2c}{2\mu}(\sinh\xi_x-\tan^{-1}(\sinh\xi_x))$ ⟷ 差一負號	$\frac{S^2c}{2\mu}(\sinh\xi_x-\tan^{-1}(\sinh\xi_x))$
$J_{\Gamma_2}=\int_{\Gamma_2}Wdy-\int_{\Gamma_2}T_i\frac{\partial u_i}{\partial x}d\Gamma=\int_{\Gamma_2}(Wn_x-T_i\frac{\partial u_i}{\partial x})d\Gamma$	$\frac{S^2c}{\mu}\tan^{-1}(1/\sinh\xi_1)+\frac{S^2b}{\mu}$ ⟷ 差一負號	$\frac{-S^2c}{\mu}\tan^{-1}(1/\sinh\xi_1)-\frac{S^2b}{\mu}$
$J_{\Gamma_3}=\int_{\Gamma_3}Wdy-\int_{\Gamma_3}T_i\frac{\partial u_i}{\partial x}d\Gamma=\int_{\Gamma_3}(Wn_x-T_i\frac{\partial u_i}{\partial x})d\Gamma$	$\frac{-S^2c}{2\mu}(\sinh\xi_x-\tan^{-1}(\sinh\xi_x))$ ⟷ 差一負號	$\frac{S^2c}{2\mu}(\sinh\xi_x-\tan^{-1}(\sinh\xi_x))$
J-Integral: $J_{\Gamma}=J_{\Gamma_1}+J_{\Gamma_2}+J_{\Gamma_3}$	$J=\frac{S^2c}{2\mu}(\tan^{-1}(1/\sinh\xi_1)+\tan^{-1}(\sinh\xi_1))=\frac{S^2c\pi}{\mu}$  $\sinh\xi_x$ $\boxed{\frac{S^2a\pi}{2\mu}=\frac{K_{III}^2}{2\mu}}$ ⟷ 差一負號	$J=\frac{-S^2c}{2\mu}(\tan^{-1}(1/\sinh\xi_1)+\tan^{-1}(\sinh\xi_1))=\frac{-S^2c\pi}{\mu}$ $\boxed{\frac{-S^2a\pi}{2\mu}=\frac{-K_{III}^2}{2\mu}}$  $\sinh\xi_x$

求應力強度因子之方法

Observation point \ Approach	Boundary density	Interior field	Field and boundary
			
Displacement	$K_{III}=\lim_{\varepsilon\to0}\frac{\mu\sqrt{\pi}u_z(0,\eta_x)}{\sqrt{2\varepsilon}}=S\sqrt{\pi a}$	$K_{III}=\lim_{\varepsilon\to0}\frac{\mu\sqrt{\pi}u_z(\xi_x,0)}{\sqrt{2\varepsilon}}=S\sqrt{\pi a}$	
Stress	$K_{III}=\lim_{\varepsilon\to0}\sqrt{2\pi\varepsilon}\,\sigma(0,\eta_x)=S\sqrt{\pi a}$	$K_{III}=\lim_{\varepsilon\to0}\sqrt{2\pi\varepsilon}\,\sigma(\xi_x,0)=S\sqrt{\pi a}$	
J-integral			$J=\frac{K_{III}^2}{2\mu}=\frac{S^2\pi c}{2\mu}$

ε is the infinitesimal distance near the crack tip either from the boundary or from the domain





BEM之數值結果

結論：

有別於傳統利用複變求解反平面剪力場問題，本研究以BIEM/BEM進行解析與數值驗證。透過橢圓座標之分離核與傅立葉展開，利用基底函數的正交性可導得相關物理量且無須主值計算。過去在求解線裂縫或線剛性夾雜受反平面力場作用之應力強度因子，大多透過複變方法將一般外型映射成圓形，以極座標來描述路徑導得應力強度因子。而本文的解析方法以分離核做推導，透過橢圓座標來描述路徑，使橢圓路徑在求解應力集中因子時可導得更簡潔的解析解，並且在線裂縫及線剛性夾雜近尖端透過漸進行為，解析探討應力強度因子。而應力強度因子可由位移、應力及J積分三種方法去做推導，亦將三種方法做解析推導以及BEM驗證。然而在使用J積分時，發現J積分出來的結果為常數，證明了J積分與路徑無關。且發現線裂縫與線剛性夾雜分別獲得一正一負的J積分。本文利用三角函數以及幾何關係解析探討，並以BEM數值驗證結果吻合一致。

文獻回顧

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