反平面力場孔洞與剛性夾雜應力集中因子(SCF)互易性研究

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摘要

利用邊界積分方程(BIE)，搭配分離核和傅立葉級數解析探討圓形(孔洞與剛性夾雜)與橢圓形(孔洞與剛性夾雜)在遠端反平面剪力負載下之位移、應力與SCF。將封閉型式的基本解以分離核型式在極座標及橢圓座標展開。並且發現由分離核導得之結果，也可以從複變中的柯西-黎曼關係式來解釋孔洞與剛性夾雜在不同方向負載下之互換關係。

問題描述

反平面剪力位移場

$(u_x, u_y, u_z) = (0, 0, w(x, y))$

控制方程式

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = 0$$

邊界積分方程

$$2\pi u(\mathbf{x}) = \int_B T^e(\mathbf{s},\mathbf{x}) u(\mathbf{s}) dB(\mathbf{s}) - \int_B U^e(\mathbf{s},\mathbf{x}) t(\mathbf{s}) dB(\mathbf{s}),\ \mathbf{x} \in D \cup B$$

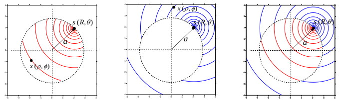
零場邊界積分方程

$$0 = \int_B T^i(\mathbf{s},\mathbf{x}) u(\mathbf{s}) dB(\mathbf{s}) - \int_B U^i(\mathbf{s},\mathbf{x}) t(\mathbf{s}) dB(\mathbf{s}),\ \mathbf{x} \in D^c \cup B$$

分離核極座標展開

$$U(\mathbf{s},\mathbf{x}) = \begin{cases} U^i(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos m(\theta-\phi), & R \ge \rho \\ U^e(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos m(\theta-\phi), & \rho > R \end{cases}$$

$$T(\mathbf{s},\mathbf{x}) = \begin{cases} T^i(R,\theta;\rho,\phi) = -\left(\frac{1}{R} + \sum_{m=1}^{\infty} \left(\frac{\rho^m}{R^{m+1}}\right) \cos m(\theta-\phi)\right), & R > \rho \\ T^e(R,\theta;\rho,\phi) = \sum_{m=1}^{\infty} \left(\frac{R^{m-1}}{\rho^m}\right) \cos m(\theta-\phi), & \rho > R \end{cases}$$

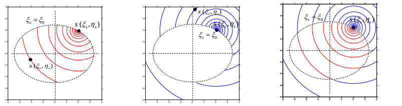


內域 $(\rho \le R)$ 外域 $(\rho > R)$ 全域 $(0 < \rho < \infty)$

分離核橢圓座標展開

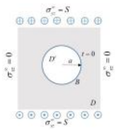
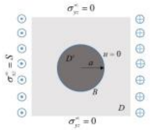
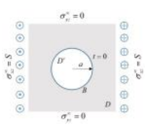
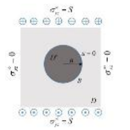
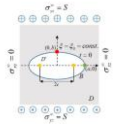
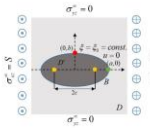
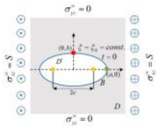
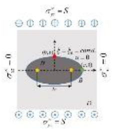
$$U(s,x) = \begin{cases} U^i(\xi_s,\eta_s;\xi_x,\eta_x) = \xi_s + \ln\frac{c}{2} - \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_s} \cosh m\xi_x \cos m\eta_x \cos m\eta_s - \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_s} \sinh m\xi_x \sin m\eta_x \sin m\eta_s, & \xi_s \ge \xi_x \\ U^e(\xi_s,\eta_s;\xi_x,\eta_x) = \xi_x + \ln\frac{c}{2} - \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_x} \cosh m\xi_s \cos m\eta_x \cos m\eta_s - \sum_{m=1}^{\infty} \frac{2}{m} e^{-m\xi_x} \sinh m\xi_s \sin m\eta_x \sin m\eta_s, & \xi_s < \xi_x \end{cases}$$

$$T(s,x) = \begin{cases} T^i(\xi_s,\eta_s;\xi_x,\eta_x) = \frac{-1}{J(\xi_s,\eta_s)} \left(1 + 2\sum_{m=1}^{\infty} e^{-m\xi_s} \cosh m\xi_x \cos m\eta_x \cos m\eta_s + 2\sum_{m=1}^{\infty} e^{-m\xi_s} \sinh m\xi_x \sin m\eta_x \sin m\eta_s\right), & \xi_s > \xi_x \\ T^e(\xi_s,\eta_s;\xi_x,\eta_x) = \frac{1}{J(\xi_s,\eta_s)} \left(2\sum_{m=1}^{\infty} e^{-m\xi_x} \sinh m\xi_s \cos m\eta_x \cos m\xi_s + 2\sum_{m=1}^{\infty} e^{-m\xi_x} \cosh m\xi_s \sin m\eta_x \sin m\eta_s\right), & \xi_s < \xi_x \end{cases}$$

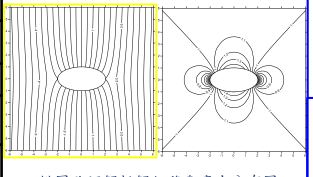


內域 $(\xi_x \le \xi_s)$ 外域 $(\xi_x > \xi_s)$ 全域 $(0 < \xi_x < \infty)$

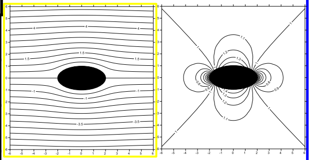
結果與討論

關係	互易關係		互易關係		互易關係		互易關係	
圖示 / 函數								
邊界條件	Neumann $t(x) = \frac{\partial w}{\partial n} = 0, x \in B$	Dirichlet $u(x) = 0, x \in B$	Neumann $t(x) = \frac{\partial w}{\partial n} = 0, x \in B$	Dirichlet $u(x) = 0, x \in B$	Neumann $t(x) = \frac{\partial w}{\partial n} = 0, x \in B$	Dirichlet $u(x) = 0, x \in B$	Neumann $t(x) = \frac{\partial w}{\partial n} = 0, x \in B$	Dirichlet $u(x) = 0, x \in B$
遠端反平面剪力位移	$\sigma_{yz}^\infty = S, u^\infty = \frac{Sy}{\mu}, \lvert y\rvert \to \infty$	$\sigma_{xz}^\infty = S, u^\infty = \frac{Sx}{\mu}, \lvert x\rvert \to \infty$	$\sigma_{xz}^\infty = S, u^\infty = \frac{Sx}{\mu}, \lvert x\rvert \to \infty$	$\sigma_{yz}^\infty = S, u^\infty = \frac{Sy}{\mu}, \lvert y\rvert \to \infty$	$\sigma_{yz}^\infty = S, u^\infty = \frac{Sy}{\mu}, \lvert y\rvert \to \infty$	$\sigma_{xz}^\infty = S, u^\infty = \frac{Sx}{\mu}, \lvert x\rvert \to \infty$	$\sigma_{xz}^\infty = S, u^\infty = \frac{Sx}{\mu}, \lvert x\rvert \to \infty$	$\sigma_{yz}^\infty = S, u^\infty = \frac{Sy}{\mu}, \lvert y\rvert \to \infty$
τ 切向導微 $\tau^m = \frac{\partial u_z}{\partial m} = \frac{\mu}{\rho}\frac{\partial u_z}{\partial \phi} = \frac{\mu}{h}\frac{\partial u_z}{\partial \eta_x}$	$2S\cos\phi$	0	$-2S\sin\phi$	0	$\frac{Se^{\xi_0}\cos\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}$	0	$\frac{-Se^{\xi_0}\sin\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}$	0
τ 法向導微 $\tau^n = \frac{\partial u_z}{\partial n} = \mu\frac{\partial u_z}{\partial \rho} = \frac{\mu}{h}\frac{\partial u_z}{\partial \xi_x}$	0	$2S\cos\phi$	0	$2S\sin\phi$	0	$\frac{Se^{\xi_0}\cos\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}$	0	$\frac{Se^{\xi_0}\sin\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}$
$SCF = \frac{\lvert\tau^m\rvert}{\sigma^\infty}$ or $\frac{\lvert\tau^n\rvert}{\sigma^\infty}$	$SCF(\phi) = \lvert 2\cos\phi\rvert$	$SCF(\phi) = \lvert 2\cos\phi\rvert$	$SCF(\phi) = \lvert -2\sin\phi\rvert$	$SCF(\phi) = \lvert 2\sin\phi\rvert$	$SCF(\eta_x) = \left\lvert\frac{e^{\xi_0}\cos\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}\right\rvert$	$SCF(\eta_x) = \left\lvert\frac{e^{\xi_0}\cos\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}\right\rvert$	$SCF(\eta_x) = \left\lvert\frac{-e^{\xi_0}\sin\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}\right\rvert$	$SCF(\eta_x) = \left\lvert\frac{e^{\xi_0}\sin\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}\right\rvert$
$\mathrm{Re}(f(z))$		全位移場 u_z $\mathrm{Re}(f(z)) = \frac{S}{\mu}\rho(1 - \frac{a^2}{\rho^2})\cos\phi$	全位移場 u_z $\mathrm{Re}(f(z)) = \frac{S}{\mu}\rho(1 + \frac{a^2}{\rho^2})\cos\phi$			全位移場 u_z $\mathrm{Re}(f(z)) = \frac{S}{\mu}c\cos\eta_x(\cosh\xi_x - e^{\xi_0-\xi_x}\cosh\xi_0)$	全位移場u_z $\mathrm{Re}(f(z)) = \frac{S}{\mu}c\cos\eta_x(\cosh\xi_x + e^{\xi_0-\xi_x}\sinh\xi_0)$	
$\mathrm{Im}(f(z))$	全位移場 u_z $\mathrm{Im}(f(z)) = \frac{S}{\mu}\rho(1 + \frac{a^2}{\rho^2})\sin\phi$			全位移場 u_z $\mathrm{Im}(f(z)) = \frac{S}{\mu}\rho(1 - \frac{a^2}{\rho^2})\sin\phi$	全位移場 u_z $\mathrm{Im}(f(z)) = \frac{S}{\mu}c\sin\eta_x(\sinh\xi_x + e^{\xi_0-\xi_x}\cosh\xi_0)$			全位移場 u_z $\mathrm{Im}(f(z)) = \frac{S}{\mu}c\sin\eta_x(\sinh\xi_x - e^{\xi_0-\xi_x}\sinh\xi_0)$
解析函數 $f(z) = u(x,y) + iv(x,y)$ $z = x + iy$	$f(z) = \frac{S}{\mu}(z - \frac{a^2}{z})$		$f(z) = \frac{S}{\mu}(z + \frac{a^2}{z})$		$f(z) = \frac{S}{\mu}\frac{1}{a-b}(a\sqrt{z^2 - c^2} - bz)$		$f(z) = \frac{S}{\mu}\frac{1}{a-b}(az - b\sqrt{z^2 - c^2})$	
柯西黎曼關係式 $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial m}, \frac{\partial u}{\partial m} = -\frac{\partial v}{\partial n}$	滿足		滿足		滿足		滿足	

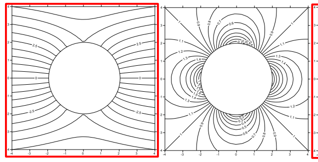
相同顏色之位移場圖互為正交關係
$(S = 1, \mu = 1)$



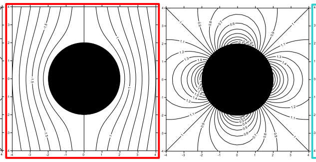
橢圓孔洞解析解位移與應力分布圖
$(a = 2, b = 1, \sigma_{yz}^\infty = 0\ \&\ \sigma_{xz}^\infty = S)$



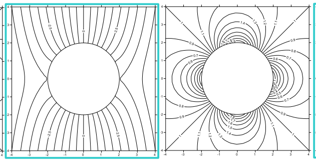
橢圓剛性夾雜解析解位移與應力分布圖
$(a = 2.b = 1, \sigma_{yz}^\infty = S\ \&\ \sigma_{xz}^\infty = 0)$



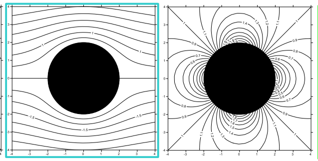
圓形孔洞解析解位移與應力分布圖
$(a = 2, \sigma_{yz}^\infty = S\ \&\ \sigma_{xz}^\infty = 0)$



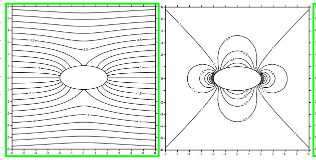
圓形剛性夾雜解析解位移與應力分布圖
$(a = 2, \sigma_{yz}^\infty = 0\ \&\ \sigma_{xz}^\infty = S)$



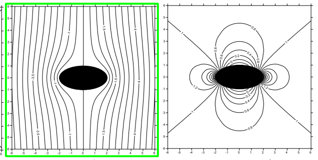
圓形孔洞解析解位移與應力分布圖
$(a = 2, \sigma_{yz}^\infty = 0\ \&\ \sigma_{xz}^\infty = S)$



圓形剛性夾雜解析解位移與應力分布圖
$(a = 2, \sigma_{yz}^\infty = S\ \&\ \sigma_{xz}^\infty = 0)$



橢圓孔洞解析解位移與應力分布圖
$(a = 2, b = 1, \sigma_{yz}^\infty = S\ \&\ \sigma_{xz}^\infty = 0)$



橢圓剛性夾雜解析解位移與應力分布圖
$(a = 2.b = 1, \sigma_{yz}^\infty = 0\ \&\ \sigma_{xz}^\infty = S)$

結論

我們成功使用邊界積分方程中的分離核，求出圓形(孔洞與剛性夾雜)與橢圓形(孔洞與剛性夾雜)在遠端反平面剪力負載下之位移、應力與SCF。並且從中發現，於相同形狀之孔洞與剛性夾雜在不同方向之負載下，SCF具有互易性，此現象亦可由複變中柯西-黎曼關係式驗證。能判斷兩個解位移分別是屬於解析解中的實部與虛部，並且從位移圖中看出實虛之位移互為正交關係。由其結果推導出與複變相同之解析函數，驗證了我們的發現。

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