A novel boundary meshless method for radiation and scattering problems

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Keywords: Singular boundary method, meshless, singular fundamental solution, unbounded domain, radiation, scattering

Abstract. This paper proposes a novel meshless singular boundary method (SBM) to solve time-harmonic exterior acoustic problems. Compared with the other boundary-type meshless methods, the innovative point of the SBM is to employ a novel inverse interpolation technique to circumvent the singularity of the fundamental solution at origin. The method is mathematically simple, easy-to-program, meshless and integration-free. This study tests the method to three benchmark radiation and scattering problems under unbounded domains. Our numerical experiments reveal that the SBM is a competitive numerical technique to the exterior acoustic problems.

1. Introduction

The finite element method (FEM) [3-5] is one of the most popular methods in numerical acoustics but requires the effective treatment of unbounded domains, among which are the local and nonlocal absorbing boundary conditions [6-8], infinite elements [9], and absorbing layers [10,11]. These boundary treatments could be very tricky and arbitrary and are largely based on trial-error experiences.

On the other hand, the boundary element method (BEM) [12-17] appears very attractive to handle the unbounded domain problems because its basis function is the fundamental solution which satisfies the governing equation and the Sommerfeld radiation condition at infinity [9]. And no special treatment for unbounded domains is required. However, the treatment of singularity and hyper-singularity [17] is mathematically complex and computationally very expensive.

To avoid the singularities of fundamental solutions, the method of fundamental solutions (MFS) [18-20] distributes the boundary knots on a fictitious boundary outside the physical domain, and the location of fictitious boundary is vital for the accuracy and reliability. However, despite great effort of decades, the optimal placement of fictitious boundary is still arbitrary and tricky and is largely based on experiences. Recently, Young et al. [21] proposed an alternative meshless method, called regularized meshless method (RMM) [22], to remedy this drawback. By employing the desingularization of subtracting and adding-back technique, the RMM places the source points on the real physical boundary. In addition, the ill-conditioned interpolation matrix of BEM and MFS is also remedied. However, the original RMM requires the uniform distribution of nodes and severely reduces its applicability to complex-shaped boundary problems. Similar to the RMM, Sarler [23] proposes the modified method of fundamental solution (MMFS) to solve potential flow problems. However, the MMFS demands a complex calculation of the diagonal elements of interpolation matrix. It is worthy of noting that the MFS, RMM and MMFS do not require any mesh and are all truly meshless.

This paper proposes a novel numerical method, called singular boundary method (SBM), to calculate the exterior acoustic problems. The SBM is developed to overcome the above-mentioned major shortcomings in the MFS, RMM, and MMFS while retaining their merits. The key point of the SBM is to use a simple numerical approach to calculate diagonal elements when the collocation and source nodes are coincident and are all placed on the physical boundary. This study also examines the efficiency, stability, and accuracy of the proposed technique in testing three benchmark exterior radiation and scattering problems. Based on the results reported here, some remarks will be concluded in section 4.

2. Singular boundary method for exterior Helmholtz problems

The problem under consideration is the Helmholtz equation in the domain D exterior to a closed bounded curve

S. To be precise, we consider propagation of time-harmonic acoustic waves in a homogeneous isotropic acoustic medium which is described by the Helmholtz equation

$$\nabla^2 u(x) + k^2 u(x) = 0, \qquad x \in D, \tag{1}$$

subjected to the boundary conditions:

$$u(x) = \overline{u} \qquad x \in \Gamma_D \tag{2a}$$

$$t(x) = \frac{\partial u(x)}{\partial n} = \overline{t} \qquad x \in \Gamma_N \tag{2b}$$

where u is the total acoustic wave (velocity potential or acoustic pressure), $k = \omega / c$ the wave number, ω the angular frequency, c the wave speed in the exterior acoustic medium D, and n denotes the unit inward normal on physical boundary. Γ_D, Γ_N denote the essential boundary (Dirichlet) and the natural boundary (Neumann) conditions, respectively, which construct the whole closed bounded curve S.

For the exterior acoustic problems, it requires ensuring the physical requirement that all scattered and radiated waves are outgoing. This is accomplished by imposing an appropriate radiation condition at infinity, which is termed as the Sommerfeld radiation condition [9]:

$$\lim_{r\to\infty} r^{\frac{1}{2}(\dim-1)} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \tag{2c}$$

where *dim* is the dimension of the acoustic problems (*dim*=2 in this study), and $i = \sqrt{-1}$.

The solution $u(x)$ of the acoustics problem (Eqs. (1) and (2)) can be approximated by a linear combination of the two-dimensional fundamental solution G

$$u(x_m) = \sum_{j=1}^{N} \alpha_j \mathrm{G}(x_m, s_j), \qquad x \in D \tag{3}$$

where N denotes the number of source points, α_j is the jth unknown coefficient, and the fundamental solution $\mathrm{G}(x, s_j) = -\frac{i\pi}{2} H_0^{(1)}\left(k \left\| x - s_j \right\|_2\right)$, $H_n^{(1)}$ is the nth order Hankel function of the first kind. We can find that the fundamental solution G satisfies both the governing equation (1) and the Sommerfeld radiation condition (2c). Thus, the formulation (3) only requires satisfying the boundary conditions (2a) and (2b).

If the collocation points x_m and source points s_j coincide, i.e., $x_m = s_j$, we will encounter well-known singularity at origin, i.e., $\mathrm{G}(x_m, s_j) = -\frac{i\pi}{2} H_0^{(1)}(0)$. In order to remedy this troublesome problem, the MFS places the source nodes on an artificial boundary outside the physical domain. However, despite of great effort, the placement of this artificial boundary remains a perplexing issue when dealing with complex-shaped boundary or multiply-connected domain problems.

The SBM places all source and boundary collocation nodes on the same physical boundary. Moreover, the source points and the boundary collocation points are the same set of boundary nodes. The SBM formulation is given by

$$u(x_m) = \sum_{j=1}^{N} \alpha_j \mathrm{G}(x_m, s_j), \qquad x_m \in \Omega^e, x_m \notin \Gamma_D \tag{4a}$$

$$u(x_m) = \sum_{j=1, j\neq m}^{N} \alpha_j \mathrm{G}(x_m, s_j) + \alpha_m \mathrm{G_S}(m), \qquad x_m \in \Gamma_D \tag{4b}$$

$$t(x_m) = \sum_{j=1}^{N} \alpha_j \frac{\partial \mathrm{G}(x_m, s_j)}{\partial n}, \qquad x_m \in \Omega^e, x_m \notin \Gamma_N \tag{4c}$$

$$t(x_m) = \sum_{j=1, j\neq m}^{N} \alpha_j \frac{\partial \mathrm{G}(x_m, s_j)}{\partial n} + \alpha_m \overline{\mathrm{G}}_\mathrm{S}(m), \qquad x_m \in \Gamma_N \tag{4d}$$

where G_S and $\overline{G}_S$ are defined as the source intensity factors, namely, the diagonal elements of the SBM interpolation matrix. This study employs a simple numerical technique, called the inverse interpolation technique (IIT), to determine the source intensity factors. In the first step, the IIT requires choosing a known sample solution

u_S of the Helmholtz acoustic problem and locating some sample points y_k inside the physical domain. It is noted that the sample points y_k do not coincide with the source points s_j, and the sample points number NK should not be fewer than the source node number N on physical boundary. By using the interpolation formula (3), we can then determine the influence coefficients β_j and $\bar{\beta}_j$ by the following linear equations

$$\left\{G\left(y_k,s_j\right)\right\}\left\{\beta_j\right\}=\left\{u_S\left(y_k\right)\right\} \tag{5a}$$

$$\left\{\frac{\partial G\left(y_k,s_j\right)}{\partial n}\right\}\left\{\bar{\beta}_j\right\}=\left\{\frac{\partial u_S\left(y_k\right)}{\partial n}\right\} \tag{5b}$$

Replacing the sample points y_k with the boundary collocation points x_m, the SBM interpolation matrix of the Helmholtz problem (Eqs. (1) and (2)) can be written as

$$\begin{bmatrix} \mathrm{G_S}(1) & \mathrm{G}\left(x_1,s_2\right) & \cdots & \mathrm{G}\left(x_1,s_N\right) \\ \mathrm{G}\left(x_2,s_1\right) & \mathrm{G_S}(2) & \cdots & \mathrm{G}\left(x_2,s_N\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{G}\left(x_N,s_1\right) & \mathrm{G}\left(x_N,s_2\right) & \cdots & \mathrm{G_S}(N) \end{bmatrix}\left\{\beta_j\right\}=\left\{u_S\left(x_m\right)\right\} \tag{6a}$$

$$\begin{bmatrix} \bar{\mathrm{G}}_\mathrm{S}(1) & \frac{\partial \mathrm{G}\left(x_1,s_2\right)}{\partial n} & \cdots & \frac{\partial \mathrm{G}\left(x_1,s_N\right)}{\partial n} \\ \frac{\partial \mathrm{G}\left(x_2,s_1\right)}{\partial n} & \bar{\mathrm{G}}_\mathrm{S}(2) & \cdots & \frac{\partial \mathrm{G}\left(x_2,s_N\right)}{\partial n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{G}\left(x_N,s_1\right)}{\partial n} & \frac{\partial \mathrm{G}\left(x_N,s_2\right)}{\partial n} & \cdots & \bar{\mathrm{G}}_\mathrm{S}(N) \end{bmatrix}\left\{\bar{\beta}_j\right\}=\left\{\frac{\partial u_S\left(x_m\right)}{\partial n}\right\} \tag{6b}$$

The source intensity factors can be calculated by the following formulations:

$$G_S(m)=\frac{u_S\left(x_m\right)-\sum\limits_{j=1,s_j\neq x_m}^{N}\beta_j \mathrm{G}\left(x_m,s_j\right)}{\beta_j} \qquad x_m=s_j, x_m\in\Gamma_D \tag{7a}$$

$$\bar{G}_S(m)=\frac{\frac{\partial u_S\left(x_m\right)}{\partial n}-\sum\limits_{j=1,s_j\neq x_m}^{N}\beta_j\frac{\partial \mathrm{G}\left(x_m,s_j\right)}{\partial n}}{\bar{\beta}_j} \qquad x_m=s_j, x_m\in\Gamma_N \tag{7b}$$

It is stressed that the source intensity factors only depends on the distribution of the source points, the fundamental solution of the governing equation and the boundary conditions. Theoretically speaking, the source intensity factors remain unchanged with different sample solutions in the IIT. Therefore, by employing this novel inverse interpolation technique, we circumvent the singularity of the fundamental solution upon the coincidence of the source and collocation points. It is noted that like the MFS, the SBM does not require considering the Sommerfeld radiation condition (2c) and is a truly meshless numerical technique; unlike the MFS, the SBM avoids the perplexing issue of the fictitious boundary.

3. Numerical results and discussions

In this section, the efficiency, accuracy and convergence of the present SBM are tested to the exterior acoustics problems. It is stressed that the boundary conditions are discontinuous in Cases 1 and 2. The present SBM is compared with the exact solution, the RMM and the MFS. *Lerr(u)* represents the L_2 norm error, which are defined as below

$$Lerr(u)=\sqrt{\frac{1}{NT}\sum_{k=1}^{NT}\left|u\left(k\right)-\bar{u}\left(k\right)\right|^2}, \tag{8}$$

where $\bar{u}\left(k\right)$ and $u\left(k\right)$ are the analytical and numerical solutions at x_i, respectively, and NT is the total number of points in the interest domain which are used to test the solution accuracy, the sample solution $u_s(r,\theta)=\dfrac{H_2^{(1)}(kr)}{H_2^{(1)}(ka)}\cos 2\theta$ for Dirichlet boundary problem, $u_s(r,\theta)=\dfrac{H_1^{(1)}(kr)}{H_1^{(1)\prime}(ka)}e^{i\theta}$ for Neumann boundary problem. The number of inner sample points is equal to the boundary knots, and the distribution of sample points depends on the shape of the physical domain. In the MFS, according to the boundary shape of the physical domain, we typically place the source points outside physical domain with a parameter d defined as

$$d=\frac{x_i-s_i}{x_i-op} \tag{9}$$

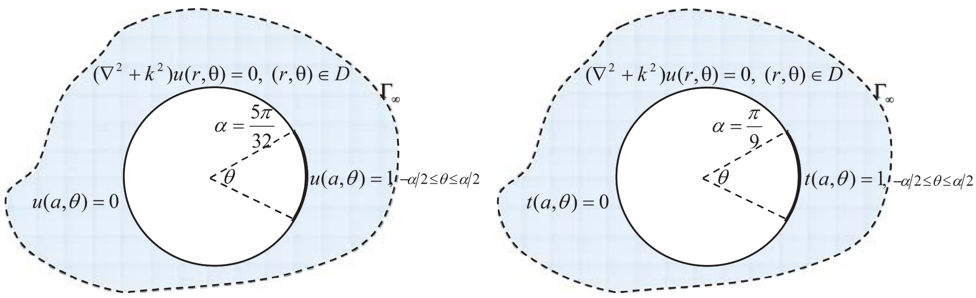
in which op is the geometric center, namely the origin point in this paper.

3.1 Radiation problems

Case 1: Nonuniform radiation problem (Dirichlet boundary condition) for a circular cylinder.
We first consider a nonuniform radiation problem (Dirichlet) from a sector of a cylinder as shown in Fig. 1(a). The boundary condition has a constant inhomogeneous value on the arc $-\alpha/2\le\theta\le\alpha/2$ and vanishing elsewhere. Two discontinuous boundary points can be found on the physical boundary. The analytical solution [8] is

$$u(r,\theta)=\frac{\alpha}{2\pi}\frac{H_0^{(1)}(kr)}{H_0^{(1)}(ka)}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n}\frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\cos n\theta \tag{10}$$

where $H_n^{(1)}(kr)$ is the first kind Hankel function of the n order. We choose the parameters $\alpha=\dfrac{5\pi}{32}$, ka=1. The analytical solution is obtained by using 20 terms in the series representations.
Fig. 2(a) shows the comparison of the L_2 norm errors between the MFS with different fictitious boundary parameters (d=0.01,0.2,0.5) and the present SBM. It can be observed that the arbitrary placing of the off-set boundary points may cause numerical stability. The present SBM avoids such trial-error placement of the fictitious boundary and is more efficient than the MFS with the boundary nodes of the same number.



(a) Dirichlet (Case 1) (b) Neumann (Case 2)

Fig. 1 Nonuniform radiation (a) Dirichlet (Case 1) and (b) Neumann (Case 2) problem of a circular cylinder

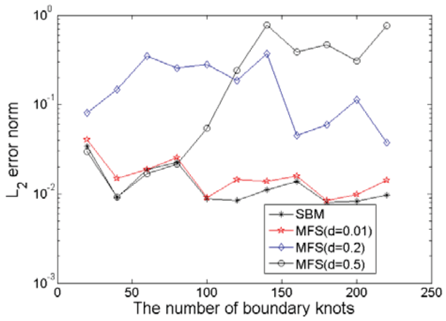
Case 2: Nonuniform radiation problem (Neumann boundary condition) of a circular cylinder.
A nonuniform radiation problem (Neumann) from a sector of a cylinder is considered as shown in Fig. 1(b) [24]. The discontinuous boundary condition is

$$t(a,\theta)=\begin{cases}1, & -\alpha/2\le\theta\le\alpha/2\\ 0, & \text{otherwise}\end{cases} \tag{11}$$

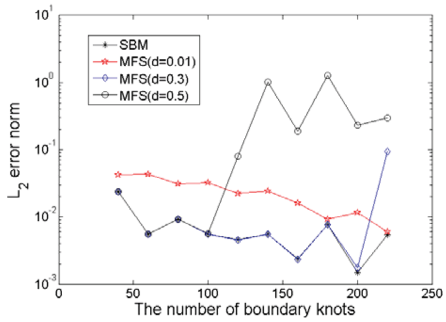
The analytical solution [24] is

$$u(r,\theta)=-\frac{\alpha}{2\pi k}\frac{H_0^{(1)}(kr)}{H_0^{(1)\prime}(ka)}-\frac{1}{\pi k}\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n}\frac{H_n^{(1)}(kr)}{H_n^{(1)\prime}(ka)}\cos n\theta \tag{12}$$

Here we choose the parameters $\alpha = \frac{\pi}{9}$, *ka*=1. The analytical solution is obtained by using 20 terms in the series representations. Fig. 2(a) shows the convergence curves of the MFS with different fictitious boundary parameters (*d*=0.01,0.3,0.5) and the present SBM. It can be observed that the fictitious boundary has a big influence on the MFS solution and its optimal placement is problem-dependent. The present SBM can obtain the acceptable results by using only 40 boundary nodes and outperforms the MFS in computational accuracy.



(a) Dirichlet (Case 1)



(b) Neumann (Case 2)

Fig.2 The accuracy variation of Case 1 and 2 against the number of interpolation knots by the MFS with *d*=0.01,0.2,0.5 for Case 1 and *d*=0.01,0.3,0.5 for Case 2 and the present SBM.

3.2 Scattering problems

The scattering problem with the incident wave can be divided into two parts, (a) incident wave field and (b) radiation field. And the radiation boundary condition in part (b) can be obtained as the minus value of the incident wave function, i.e. $t_R= -t_I$ for hard scatter or $u_R=-u_I$ for soft scatter, where the superscripts *R* and *I* denote radiation and incidence, respectively.

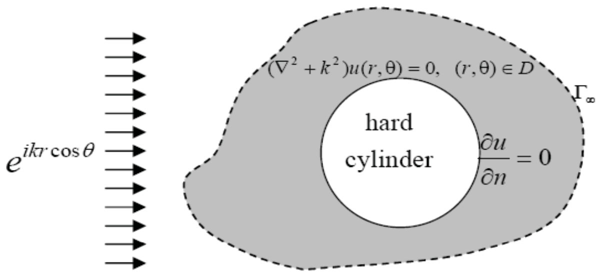
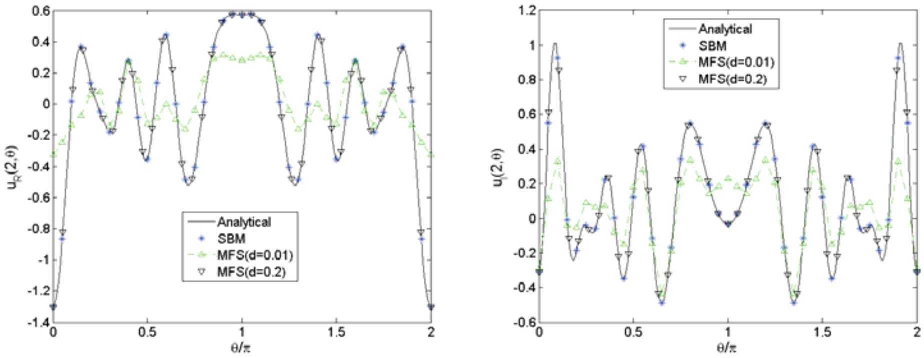


Fig. 3 The problem of a plane wave scattered by a rigid infinite circular cylinder (Neumann) in Case 3

Case 3: Scattering problem (Neumann boundary condition) of a rigid infinite circular cylinder
We consider a plane wave scattered by a rigid infinite circular cylinder as shown in Fig. 3 [20]. The analytical solution of this scattering field [20] is

$$u(r,\theta) = -\frac{J_0'(ka)}{H_0^{(1)\prime}(ka)} H_0^{(1)}(kr) - 2\sum_{n=1}^{\infty} i^n \frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} H_n^{(1)}(kr)\cos n\theta \tag{13}$$

The analytical solution in the following figures is calculated by using the first 20 terms in the above series representation (13). Figs. 4(a) and 4(b) plot both the real and imaginary parts of *u* on *r*=*2a* for $ka = 4\pi$ by using the SBM and the MFS (*d*=0.01,0.2) with 100 boundary nodes. It can be found that both the SBM and the MFS with fictitious boundary parameter d=0.2 agree the analytical solution very well. However, the MFS with d=0.01 can not obtain the right result. Thus, the determination of such a parameter d is very tricky and delicate in applications. It is noted that the present SBM avoids the headachy choice of the optimal fictitious boundary and is superior to the MFS. Figs. 5, 6(a) and 6(b) display the contour plot of the real-part potential by using the analytical solution, the present SBM and the MFS with 100 boundary nodes. It can be seen from Fig. 10 that the SBM solution matches the analytical solution very well.



(a) Real part (b) Imaginary part

Fig. 4 Plane wave scattered by a rigid infinite circular cylinder (Neumann) in case 3 for $ka = 4\pi$,r=2a: (a) Real part, (b) Imaginary part

4. Conclusions

This study proposes a novel singular boundary method formulation to calculate the exterior radiation and scattering problems. Numerical results demonstrate that the SBM performs more stably than the MFS and more accurate than the RMM, while retaining their merits. The present SBM appears a promising numerical technique to the exterior acoustic problems.

In addition, the present SBM is mathematically simple, easy-to-program, accurate, meshless and integration-free and avoids the controversy of the fictitious boundary in the MFS, the uniform boundary node requirement of the RMM, and the expensive calculation of diagonal elements in the MMFS.

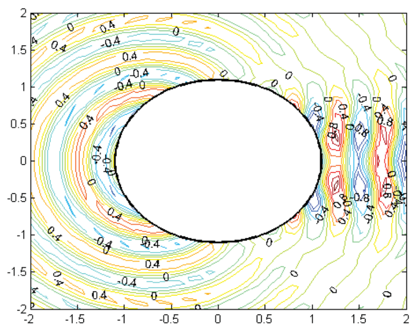
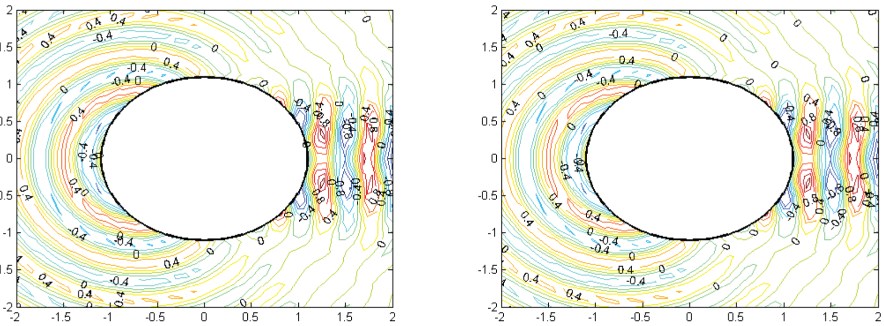


Fig. 5 The contour plot of the real-part analytical solution of a plane wave scattered by a rigid infinite circular cylinder in Case 3



(a) SBM solution (b) MFS (d=0.2) solution

Fig. 6 The contour plot of the real-part (a) SBM and (b) MFS (d=0.2) solution of a plane wave scattered by a rigid infinite circular cylinder in Case 3

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