Regularization of the divergent integrals in boundary integral equations

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Abstract. This paper considers divergent integrals with different type of singularities, which arise when the boundary integral equation (BIE) method is used to solve boundary value problems in the theory of potentials. The main equations related to formulation of the boundary integral equation and boundary element methods in 2-D and 3-D cases are discussed in details. For their regularization an approach based on the theory of distribution and application of the Green theorem has been used. The expressions, which allow an easy calculation of the weakly singular, singular and hypersingular integrals in 2-D case, have been constructed.

1. Introduction. In recent years, more and more of publications is devoted to the boundary integral equation methods (BEM) and its application science and engineering. It is because the BIE is a very powerful tool for solution of the mathematical problems science and engineering [1]. When the BIE are solved numerically divergent integrals have to be calculated. Numerical methods developed for regular integrals calculation can not be used for their calculation. There are many methods for calculation of the divergent integrals, for references see review articles [2-4] and references there. We will consider here in more details method of the divergent integrals regularization developed in [5-14] and it application in the 2-D and 3-D BIE. The method is based on the theory of distributions and idea of finite part integrals according to Hadamard.

In our previous publications approach based on the theory of distributions has been developed for regularization of the divergent integrals with different singularities. We apply the approach based on the theory of distributions and finite part integrals for the problems of fracture mechanics firstly in [5]. Then it was further developed for regularization of the hypersingular integrals in static and dynamic problems of fracture mechanics in [13, 14] respectively. Further development of this approach and application of the Green's theorems in the sense of theory of distribution has bean done in [5, 6] for piecewise constant and in [7, 8] piecewise linear approximation respectively. The equations presented in [12] permit transforms divergent hypersingular integrals into the regular ones. The developed approach can be applied not only for hypersingular integrals regularization but also for a wide class of divergent integral regularizations and any polynomial approximation.

In this paper, the approach for the divergent integral regularization which is based on the theory of distributions and Green's theorems is further developed and applied for the potential theory problems. The divergent integral regularization have been done for 2-D and 3-D the weakly singular and hypersingular integrals and regular formulas for their calculation have been obtained. The weakly singular and hypersingular integrals piecewise constant approximation have been considered for arbitrary convex polygon. It is important to mention that in presented equations all calculations can be done analytically, no numerical integration is needed.

2. Statement of problem and BIE. Let consider a homogeneous region, which in 2-D or 3-D Euclidean space occupies volume V with smooth boundary ∂V. The region V is an open bounded subset of the Euclidean space with a $C^{0,1}$ Lipschitzian regular boundary ∂V. In the region V we consider scalar function $u(\mathbf{x})$ that subjected to Poisson equation

$$\Delta u(\mathbf{x}) + f(\mathbf{x}) = 0, \qquad \forall \mathbf{x} \in V \qquad (1)$$

Here $\Delta = \sum_{i=1}^{n} \partial_i \partial_i$ is the Laplace operator, $\partial_i = \partial/\partial x_i$ denotes the partial derivatives with respect to space, $f(\mathbf{x})$ is given in the region function. Throughout this paper we use the Einstein summation convention.

If the eq (1) is defined in an infinite region, then its solution must satisfy additional conditions at the infinity in the form

$$\text{in 2-D case} \quad u(\mathbf{x}) = O(\ln(r^{-1})) \ , \ \partial_n u(\mathbf{x}) = p(\mathbf{x}) = O(r^{-1}) \ \text{ for } \ r \to \infty$$

$$\text{in 3-D case} \quad u(\mathbf{x}) = O(r^{-1}) \ , \ \partial_n u(\mathbf{x}) = p(\mathbf{x}) = O(r^{-2}) \ \text{ for } \ r \to \infty \tag{2}$$

Here n_i are components of the outward normal vector, $\partial_n = n_i \partial_i$ is a derivative in direction of the vector $\mathbf{n}(\mathbf{x})$ normal to the surface ∂V, r is the distance in the Euclidian space.

If the body occupied a finite region V with the boundary ∂V, it is necessary to establish boundary conditions. We consider the mixed boundary conditions in the form

$$u(\mathbf{x}) = \varphi(\mathbf{x}) \ , \ \forall \mathbf{x} \in \partial V_u \ , \ p(\mathbf{x}) = \psi(\mathbf{x}), \ \forall \mathbf{x} \in \partial V_p \tag{3}$$

The boundary contain two parts ∂V_u and ∂V_p such that $\partial V_u \cap \partial V_p = \varnothing$ and $\partial V_u \cup \partial V_p = \partial V$. On the part ∂V_u is prescribed unknown function $u(\mathbf{x})$ and on the part ∂V_p is prescribed it normal derivative $p(\mathbf{x})$ respectively.

In order to establish integral representations for the function $u(\mathbf{x})$ and it normal derivative $p(\mathbf{x})$ we start from second Green theorem in the form

$$\int_V (u^* \Delta u - u \Delta u^*) dV = \int_{\partial V} (u \partial_n u^* - u^* \partial_n u \ ]) dS \tag{4}$$

which take place for any two functions $u(\mathbf{x})$ and $u^*(\mathbf{x})$ with continuous first and second derivatives within the region V.

Let us consider solution of the elliptic partial differential eq (1) in an infinite space for the function $f^*(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})$

$$\Delta U(\mathbf{x} - \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0, \qquad \forall \mathbf{x}, \mathbf{y} \in \mathfrak{R}^3 \tag{5}$$

Solution of this equation is called the fundamental solutions. In 2-D and 3-D cases it has the form

$$U(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{r} \ , \ U(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi r} \tag{6}$$

Here $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and $r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ for 2-D and 3-D case respectively.

Now considering that

$$u^*(\mathbf{x}) = U(\mathbf{x} - \mathbf{y}) \text{ and } p_i^*(\mathbf{x}) = \partial_n u^*(\mathbf{x}) = W(\mathbf{x}, \mathbf{y}) \tag{7}$$

from eq (4) we obtain the integral representation for the unknown function $u(\mathbf{x})$

$$u(\mathbf{y}) = \int_{\partial V} (p(\mathbf{x}) U(\mathbf{x} - \mathbf{y}) - u(\mathbf{x}) - W(\mathbf{x}, \mathbf{y})) dS + \int_V b(\mathbf{x}) U(\mathbf{x} - \mathbf{y}) dV \tag{8}$$

The kernels $U(\mathbf{x} - \mathbf{y})$ and $W(\mathbf{x}, \mathbf{y})$ are called fundamental solutions for Laplace equation. After some transformations and simplifications the expression for the kernel $W(\mathbf{x}, \mathbf{y})$ will has the following form

$$W(\mathbf{x}, \mathbf{y}) = -\frac{n_i(\mathbf{x})(x_i - y_i)}{\alpha \pi r^\beta} \tag{9}$$

Applying to eq (8) the differential operator of normal derivative $\partial_n = n_i(\mathbf{x}) \partial_i$ we will find integral representation for the unknown function $p(\mathbf{x})$ in the form

$$p(\mathbf{y}) = \int_{\partial V} (p(\mathbf{x}) K(\mathbf{x}, \mathbf{y}) - u(\mathbf{x}) F(\mathbf{x}, \mathbf{y})) dS + \int_V b(\mathbf{x}) K(\mathbf{x}, \mathbf{y}) dV \tag{10}$$

The kernels $K(\mathbf{x}, \mathbf{y})$ and $F(\mathbf{x}, \mathbf{y})$ may be obtained applying the differential operator $\partial_n = n_i(\mathbf{y}) \partial_i$ to the kernels $U(\mathbf{x} - \mathbf{y})$ and $W(\mathbf{x}, \mathbf{y})$ respectively. After some transformations and simplifications the expression for the kernels $K_{ij}(\mathbf{x}, \mathbf{y})$ and $F_{ij}(\mathbf{x}, \mathbf{y})$ have the form

$$K(\mathbf{x}, \mathbf{y}) = \frac{n_i(\mathbf{y})(x_i - y_i)}{\alpha \pi r^\beta}, \ F(\mathbf{x}, \mathbf{y}) = \frac{1}{\alpha \pi r^\beta} \left(\beta \frac{n_i(\mathbf{x}) n_j(\mathbf{y})(x_i - y_i)(x_j - y_j)}{r^2} - n_i(\mathbf{x}) n_i(\mathbf{y}) \right) \tag{11}$$

In the eq (9) and (11) $\alpha = 4, 2$ and $\beta = 3, 2$ in 3-D and 2-D cases respectively.

The kernels $U(\mathbf{x} - \mathbf{y}), W(\mathbf{x}, \mathbf{y})$, $K_{ij}(\mathbf{x}, \mathbf{y})$ and $F_{ij}(\mathbf{x}, \mathbf{y})$ contain different kind singularities, therefore corresponding integrals are divergent. Simple observation shows that the kernels in the integral representations (8) and (10) tend to infinity when $r \to 0$.

In the 3-D case with $\mathbf{x} \to \mathbf{y}$

$$U(\mathbf{x}-\mathbf{y}) \to r^{-1}, W(\mathbf{x},\mathbf{y}) \to r^{-2}, K(\mathbf{x},\mathbf{y}) \to r^{-2}, F(\mathbf{x},\mathbf{y}) \to r^{-3} \tag{12}$$

In the 2-D case with $\mathbf{x} \to \mathbf{y}$

$$U(\mathbf{x}-\mathbf{y}) \to ln(r^{-1}), W(\mathbf{x},\mathbf{y}) \to r^{-1}, K(\mathbf{x},\mathbf{y}) \to r^{-1}, F(\mathbf{x},\mathbf{y}) \to r^{-2} \tag{13}$$

Tending $\mathbf{y}$ in eq (8) and (10) to the boundary ∂V and taking into consideration boundary properties of the kernels (9), (11) we obtain representation of the functions $u(\mathbf{x})$ and $p(\mathbf{x})$ on the smooth parts of boundary surface ∂V in the following form

$$\pm \frac{1}{2}u(\mathbf{y}) = \int_{\partial V}(p(\mathbf{x})U(\mathbf{x}-\mathbf{y})u(\mathbf{x}) - W(\mathbf{x},\mathbf{y}))dS + \int_V p(\mathbf{x})U(\mathbf{x}-\mathbf{y})dV,$$

$$\mp \frac{1}{2}p(\mathbf{y}) = \int_{\partial V}(p(\mathbf{x})K(\mathbf{x},\mathbf{y})u(\mathbf{x}) - F(\mathbf{x},\mathbf{y}))dS + \int_V p(\mathbf{x})K(\mathbf{x},\mathbf{y})dV \tag{14}$$

The plus and minus signs in these equations are used for the interior and exterior problems, respectively.

To transform the BIE into the finite dimensional BEM equations we have to split the boundary ∂V into a collection of finite boundary elements (BE)

$$\partial V = \bigcup_{n=1}^{N} \partial V_n, \quad \partial V_n \cap \partial V_k = \varnothing, \text{ if } n \neq k. \tag{15}$$

On each BE we shall choose Q nodes of interpolation and the shape functions $\varphi_q(\mathbf{x})$. Then the displacement and traction on each BE ∂V_n will be approximately represented in the form

$$u(\mathbf{x}) \approx \sum_{q=1}^{Q} u^n(\mathbf{x}_q)\varphi_q(\mathbf{x}), \ \mathbf{x} \in \partial V_n, \ p(\mathbf{x}) \approx \sum_{q=1}^{Q} p^n(\mathbf{x}_q)\varphi_q(\mathbf{x}) \ \mathbf{x} \in \partial V_n \tag{16}$$

and on the whole crack surface ∂V in the form

$$u(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} u^n(\mathbf{x}_q)\varphi_q(\mathbf{x}), \ \mathbf{x} \in \bigcup_{n=1}^{N}\partial V_n, \ p(\mathbf{x}) \approx \sum_{n=1}^{N}\sum_{q=1}^{Q} p^n(\mathbf{x}_q)\varphi_q(\mathbf{x}), \quad \mathbf{x} \in \bigcup_{n=1}^{N}\partial V_n \tag{17}$$

Substitution of the expressions (17) in eq (14), gives us the finite-dimensional representations for the vectors of displacements and traction on the boundary in the form

$$\frac{1}{2}u^m(\mathbf{y}_r) = \sum_{n=1}^{N}\sum_{q=1}^{Q}\left[U^{nq}(\mathbf{y}_r,\mathbf{x}_m)p_j^n(\mathbf{x}_m) - W^{nq}(\mathbf{x}_r,\mathbf{x}_m)u^n(\mathbf{x}) + U(\mathbf{f},\mathbf{y},V_n)\right],$$

$$\frac{1}{2}p^m(\mathbf{y}_r) = \sum_{n=1}^{N}\sum_{q=1}^{Q}\left[K^{nq}(\mathbf{y}_r,\mathbf{x}_m)p^n(\mathbf{y}_m) - F^{nq}(\mathbf{y}_r,\mathbf{x}_m)u^n(\mathbf{x}_m) + K(\mathbf{f},\mathbf{y},V_n)\right], \tag{18}$$

where

$$U^{nq}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} U(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS, \quad W^{nq}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} W(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS,$$

$$K^{nq}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} K(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS, \quad F^{nq}(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} F(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS. \tag{19}$$

In the case of piecewise constant approximation the finite-dimensional representations for the vectors of displacements and traction on the boundary take the form

$$\frac{1}{2}u^m(\mathbf{y}_r) = \sum_{n=1}^{N}\left[U^n(\mathbf{y}_r,\mathbf{x}_m)p_j^n(\mathbf{x}_m) - W^n(\mathbf{x}_r,\mathbf{x}_m)u^n(\mathbf{x}) + U(\mathbf{f},\mathbf{y},V_n)\right],$$

$$\frac{1}{2}p^m(\mathbf{y}_r) = \sum_{n=1}^{N}\left[K^{nq}(\mathbf{y}_r,\mathbf{x}_m)p^n(\mathbf{y}_m) - F^{nq}(\mathbf{y}_r,\mathbf{x}_m)u^n(\mathbf{x}_m) + K(\mathbf{f},\mathbf{y},V_n)\right], \tag{20}$$

where

$$U^n(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} U(\mathbf{y}_r,\mathbf{x})dS, \quad W^n(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} W(\mathbf{y}_r,\mathbf{x})dS,$$

$$K^n(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} K(\mathbf{y}_r,\mathbf{x})dS, \quad F^n(\mathbf{y}_r,\mathbf{x}_m) = \int_{\partial V_n} F(\mathbf{y}_r,\mathbf{x})dS. \tag{21}$$

2. Divergent integrals and distributions. Let us consider function $f(\mathbf{x})$ that contain singular points in the region $\mathbf{x} \in V$ in n-D space and definite integral

$$I_0 = \int_V f(\mathbf{x})d\mathbf{x} \tag{22}$$

The classical approach can not provide the meaning of the integral I_0. The integrals with singularities can not be considered in usual (Riemann or Lebegue) sense. In order to such integrals have sense it is necessary special consideration of them. Following [5] we consider here the above divergent integrals in the sense of distribution. To do that we introduce function $g(\mathbf{x})$, such that the function $f(\mathbf{x})$ can be presented in the form

$$f(\mathbf{x}) = \Delta^k g(\mathbf{x}), \tag{23}$$

where $\Delta^k = \partial_1^{2k} + \partial_2^{2k}$, which is called the k – dimensional Laplace's operator.

This representation of the function $f(\mathbf{x})$ has to be considered in the sense of distribution in the region V. To do that we introduce test function $\varphi(\mathbf{x}) \in C^\infty(R^n)$. Then the eq (22) can be presented in the form of distributions

$$(f,\varphi) = \int_{V^\varepsilon} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_{V^\varepsilon} \varphi(\mathbf{x})\Delta^k g(\mathbf{x})d\mathbf{x}. \tag{24}$$

Application of the Green theorem gives the following identity, which also take place only in the sense of distributions

$$\int_V [\varphi(\mathbf{x})\Delta^k g(\mathbf{x}) - (-1)^k g(\mathbf{x})\Delta^k \varphi(\mathbf{x})]dV = \sum_{i=0}^{k-1}(-1)^{i+1}\int_{\partial V}[\varphi(\mathbf{x})\partial_n \Delta^{k-i-1}g(\mathbf{x}) - g(\mathbf{x})\partial_n \Delta^{k-i-1}\varphi(\mathbf{x})]dS. \tag{25}$$

Here, $\partial_n = n_i \partial_i$ is the normal derivative on the surface with respect to $\mathbf{x}$ and $n_i(\mathbf{x})$ is a unit normal to the surface.

Taking into account eq (24) we obtain equality

$$F.P.\int_V f(\mathbf{x})\varphi(\mathbf{x})dV = \sum_{i=0}^{k-1}(-1)^{i+1}\int_{\partial V}[\varphi(\mathbf{x})\partial_n \Delta^{k-i-1}g(\mathbf{x}) - g(\mathbf{x})\partial_n \Delta^{k-i-1}\varphi(\mathbf{x})]dS + (-1)^k \int_V g(\mathbf{x})\Delta^k \varphi(\mathbf{x})]dV \tag{26}$$

which can be consider as definition of the finite part ($F.P.$) of the divergent integral according to Hadamard in the sense of distribution in n-D case. This equation can be used for the divergent integrals calculation. For the singular function $f(\mathbf{x})$ of the form $f(\mathbf{x}) = \dfrac{1}{r^m}$ we have

$$F.P.\int_V \frac{\varphi(\mathbf{x})}{r^m}dV = \sum_{i=0}^{k-1}(-1)^{i+1}\int_{\partial V}[\Delta^{k-i-1}\varphi(\mathbf{x})\partial_n \frac{P_i}{r^{m-2i}} - \frac{P_i}{r^{m-2i}}\partial_n \Delta^{k-i-1}\varphi(\mathbf{x})]dS + (-1)^k \int_V \frac{1}{r^{m-2k}}\Delta^{k+1}\varphi(\mathbf{x})]dV, \tag{27}$$

where $P_k = (-1)^k \prod_{i=0}^{k-1}\dfrac{1}{(m+2i)^2}$ for $k, m > 1$.

In the case $\varphi(\mathbf{x}) = 1$ the above equations are significantly simplified. The eq (26) has the form

$$I_0 = F.P.\int_V f(\mathbf{x})dV = \int_{\partial V}\partial_n \Delta^{k-1}g(\mathbf{x})dS, \text{ for } k=1 \ \ I_0 = F.P.\int_V f(\mathbf{x})dV = \int_{\partial V}\partial_n g(\mathbf{x})dS, \tag{28}$$

From eqs (27) and (28) follows

$$J_k = F.P.\int_{S_n}\frac{dS}{r^k} = \frac{1}{(k-2)^2}\int_{\partial S_n}\partial_n \frac{1}{r^{k-2}}dl = -\frac{1}{(k-2)}\int_{\partial S_n}\frac{r_n}{r^k}dl \tag{29}$$

Here $r_n = x_\alpha n_\alpha$ and $\alpha = 1,2$.

For a circular area integral (29) can be easy calculated and the following result be obtained

$$J_k = -\int_{\partial S_n}\frac{r_n}{r^k}dl = -\frac{1}{(k-2)R^{k-2}}\int_0^{2\pi}d\varphi = -\frac{2\pi}{(k-2)R^{k-2}}, \ \ J_1 = 2\pi R, \ \ J_3 = -\frac{2\pi}{R} \tag{30}$$

Here polar coordinates are used, were R and φ are the circle radius and polar angle respectively.

In the 1-D case singular function of one variable $f(x)$ is defined in the region $x \in V = [a,b]$ and can be represented in the form

$$f(x) = \frac{d^k g(x)}{dx^k} \,, \tag{31}$$

which also has to be considered in the sense of distributions as it was shown in the eq (24).

In the same way as in 2-D case integrating by path we obtain

$$F.P. \int_a^b g(x) \frac{d^k \varphi(x)}{dx^k} dx = \frac{d^{k-1} g(x)}{dx^{k-1}} \Bigg|_{x=a}^{x=b} + (-1)^k \int_a^b \varphi(x) \frac{d^k g(x)}{dx^k} dx \,, \tag{32}$$

This equality can be consider as definition of the *F.P.* of the divergent integral according to Hadamard in the sense of distribution in 1-D case and can be used for the divergent integrals calculation in 1-D case. For the function $f(x)$ of the form $f(x) = \frac{1}{r^m}$ we have

$$F.P. \int_a^b \frac{\varphi(x)}{r^m} dx = \sum_{i=0}^{k-1} (-1)^{i+1} \frac{d^i}{dx^i} \frac{P_i}{r^{m-k}} \frac{d^{k-1-i} \varphi(x)}{dx^{k-1-i}} \Bigg|_{x=a}^{x=b} + (-1)^k \int_a^b \frac{P_k}{r^{m-k}} \frac{d^k \varphi(x)}{dx^k} \,, \tag{33}$$

where $P_k = (-1)^k \prod_{i=0}^{k-1} \frac{1}{(m+i)}$ for $k, m > 1$.

In the case $\varphi(\mathbf{x}) = 1$ the above equations are significantly simplified. The eq (32) has the form

$$I_0 = F.P. \int_a^b f(x) dx = \frac{d^{k-1} g(x)}{dx^{k-1}} \Bigg|_{x=a}^{x=b} , \text{ for } k = 1 \; I_0 = F.P. \int_a^b f(x) dx = g(x) \Big|_{x=a}^{x=b} \tag{34}$$

which is the well known Leibniz's formula for the definite integral.

Examples of the divergent integrals calculation in the 1-D case are presented below

$$F.P. \int_a^b \ln \left| \frac{1}{x-y} \right| dx = (b-y) \ln \left| \frac{1}{b-y} \right| - (a-y) \ln \left| \frac{1}{a-y} \right|$$

$$F.P. \int_a^b \frac{dx}{x-y} = \ln \left| \frac{b-y}{a-y} \right| , F.P. \int_a^b \frac{dx}{(x-y)^2} = -\frac{1}{b-y} + \frac{1}{a-y} \tag{35}$$

4.2. Piecewise constant approximation in the 1-D case.

Let us consider a straight BE of the length Δ_n. The piecewise constant approximation is the simplest one. We transform global coordinates in the way that they are related to the local coordinate $\xi \in [-1, 1]$ by the equations

$$x_1(\xi) = r = \Delta_n \xi \;, \; x_2(\xi) = 0 \;, \; n_1(\xi) = 0 \;, \; n_2(\xi) = 1. \tag{36}$$

Interpolation function has the form

$$\varphi_0(\xi) = \begin{cases} 1 & \forall \xi \in [-\Delta_n, \Delta_n] \\ 0 & \forall \xi \notin [-\Delta_n, \Delta_n]. \end{cases} \tag{37}$$

Fundamental solutions (6), (9), (11) have the following simple form

$$U(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{\Delta_n \xi} \,, \; W(\mathbf{x}, \mathbf{y}) = 0 \;, \; K(\mathbf{x}, \mathbf{y}) = 0 \;, \; F(\mathbf{x}, \mathbf{y}) = \frac{1}{\pi} \frac{1}{\Delta_n \xi^2} \,. \tag{38}$$

Applying corresponding formulas from [9] and considering divergent integrals as it was shown above we get

$$J_0 = \int_{-1}^1 \ln \frac{1}{\Delta_n \xi} \Delta_n d\xi = 2\Delta_n \left(1 + \ln \left(\frac{1}{\Delta_n} \right) \right) \quad , \quad J_2 = \int_{-1}^1 \frac{1}{\Delta_n \xi^2} d\xi = -\frac{2}{\Delta_n} \tag{39}$$

5. Piecewise constant approximation in the 2-D case. The piecewise constant approximation is the simplest one. Interpolation functions in this case do not depend on the FE form and dimension of the domain. In 2-D case they have the form

$$\varphi_q(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in S_n, \\ 0 & \forall \mathbf{x} \notin S_n. \end{cases} \tag{40}$$

In order to simplify situation we transform global system of coordinates such that the origin is located at the nodal point, where $\mathbf{y}^0 = 0$, the coordinate axes x_1 and x_2 are located in the plane of the element, while the axis x_3 is perpendicular to that plane. In this case $x_3 = 0$ and $n_1 = 0$, $n_2 = 0$, $n_3 = 1$ and fundamental solutions have the following simple form

$$U(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi r}, \ W(\mathbf{x}, \mathbf{y}) = 0 \ , \ K(\mathbf{x}, \mathbf{y}) = 0 \ , \ F(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi r^3} \ . \tag{41}$$

Regular representations for integrals with these kernels can be easy calculated using above approach. From the eq (29) follows

$$J_1 = {}'F.P. \int_{S_n} \frac{dS}{r} = \int_{\partial S_n} \frac{r_n}{r} dl \ , \qquad J_3 = F.P. \int_{S_n} \frac{dS}{r^3} = -\int_{\partial S_n} \frac{r_n}{r^3} dl \ . \tag{42}$$

Calculations of the integrals (42) will be done using the local rectangular coordinate system with its origin located in the point $\mathbf{y}^q$, the x_1 and x_2 axis located in the plane of the polygon and the x_3 axis perpendicular to this plane as it is shown on Fig. 1.
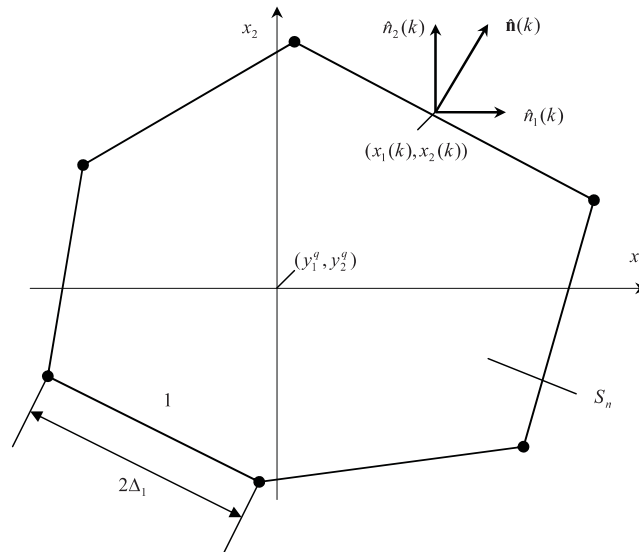

Fig. 1.

Global coordinates of the vertexes are (x_1^k, x_2^k). They can be calculate though the nodal points and unit vector normal to the contour

$$x_i(k) = \frac{x_i^{k+1} + x_i^k}{2} \ , \hat{n}_1(k) = \frac{x_2^{k+1} - x_2^k}{2\Delta_k} \ , \ \hat{n}_2(k) = -\frac{x_1^{k+1} - x_1^k}{2\Delta_k} \ . \tag{43}$$

The coordinates of an arbitrary point on the contour ∂V_n may be represented in the form

$$x_1(\xi) = x_1(k) - \xi\Delta_k\hat{n}_2(k) \text{ and } x_2(\xi) = x_2(k) + \xi\Delta_k\hat{n}_1(k) \tag{44}$$

where $x_1(k)$ and $x_2(k)$ are the coordinates of the k-th side of the contour, $\hat{\mathbf{n}}(\hat{n}_1, \hat{n}_2)$ is a unit vector normal to the contour and $\xi \in [-1,1]$ is a parameter of integration along the k-th side, $2\Delta_k$ is the length of a k-th side.

These are some useful notations

$$r(\xi) = \sqrt{\Delta_k^2\xi^2 + 2\xi\Delta_k r_-(k) + r^2(k)} \ , \ r(k) = \sqrt{x_1^2(k) + x_2^2(k)} \ , \ r_n(k) = x_\alpha(k)\hat{n}_\alpha(k) \ ,$$

$$r_+(k) = x_1(k)\hat{n}_2(k) + x_2(k)\hat{n}_1(k) \ , \ r_n(\xi) = r_n(k) \ , 2\Delta_k = \sqrt{(x_1^{k+1} - x_1^k)^2 + (x_2^{k+1} - x_2^k)^2} \ . \tag{45}$$

Using these notations the integrals under consideration may be represented in a convenient form for the calculation.

$$U^n(\mathbf{y}_r, \mathbf{x}_m) = \int_{S_n} U(\mathbf{y}_r, \mathbf{x})dS = \sum_{k=1}^K \int_{l_k} U(\mathbf{y}_r, \mathbf{x})dl,$$

$$F^n(\mathbf{y}_r, \mathbf{x}_m) = \int_{S_n} F(\mathbf{y}_r, \mathbf{x})dS = \sum_{k=1}^K \int_{l_k} F(\mathbf{y}_r, \mathbf{x})dl. \tag{46}$$

Here indexes r and m indicate number of nodes.

Substituting eqs (43)-(45) into eqs (42) we obtain formulas for calculation of the corresponding integrals over each side of polygon in the form

$$J_1(k) = \int_{-1}^1 \frac{r_n(k)}{r(\xi)}\Delta_k d\xi \ , \quad J_3(k) = -\Delta_k \int_{-1}^1 \frac{r_n(k)}{r^3(\xi)}d\xi \tag{47}$$

Now these integrals can be calculated over polygon using the formulas

$$J_1 = \sum_{k=1}^K r_n(k)I_{1,0} \ , \quad J_3 = -\sum_{k=1}^K r_n(k)I_{3,0} \tag{48}$$

Here we use the following notation for the integrals

$$I_{p,l} = (\Delta_k)^{l+1}\int_{-1}^1 \frac{\xi^l}{r^p(\xi)}d\xi \tag{49}$$

These integrals may be calculated analytically

$$I_{1,0} = \Delta_k\int_{-1}^1 \frac{1}{r(\xi)}d\xi = \ln\left|r_+(k) + \Delta_k\xi + r(\xi)\right|\Big|_{-1}^1, \quad I_{3,0} = \Delta_k\int_{-1}^1 \frac{1}{r(\xi)^3}d\xi = \frac{\Delta_k\xi + r_+(k)}{(r^2(k) - r_+^2(k))r(\xi)}\Big|_{-1}^1 \tag{50}$$

Integrals (47) were calculated for the cases of triangular and quadrangular domain of integration. For a regular triangle with unit side we obtain $J_1 = 2,281$, $J_3 = -18$, and for a square with a unit side we obtain $J_1 = 3,525$, $J_3 = -11.31$ respectively.

Then integrals in eqs (46), over any convex polygon may be represented in the form

$$U^n(\mathbf{y}_r, \mathbf{x}_q) = \frac{1}{4\pi}J_1 \ , W^n(\mathbf{y}_r, \mathbf{x}_q) = K^n(\mathbf{y}_r, \mathbf{x}_q) = 0 \ , \ F^n(\mathbf{y}_r, \mathbf{x}_q) = \frac{1}{4\pi}J_3 \tag{51}$$

It is important to mention that all calculations here can be done analytically, no numerical integration is needed.

Conclusions. Based on the theory of distribution and Green theorems the approach for the divergent hypersingular integrals regularization is developed here and applied for the BIE methods. We consider the 1-D and 2-D divergent integrals over arbitrary convex polygon for piecewise constant approximation.. The divergent integrals over the BE have been transformed to regular ones over contour of the BE. Convenient for their calculation regular formulae have been obtained. In the presented equations all calculations can be done analytically, no numerical integration is needed. It is important to mention that proposed methodology easy can be applied for regularization of the divergent integrals in elastostatics and elastodynamics and for calculation regular integrals when collocation point situated outside BE. Also developed here methodology can applied to regularization of the divergent integrals in the case of quadratic and higher BE.

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