Computer Facilitated Generalized Coordinate Transformations of Partial Differential Equations With Engineering Applications

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ABSTRACT: Partial differential equations (PDEs) play an important role in describing many physical, industrial, and biological processes. Their solutions could be considerably facilitated by using appropriate coordinate transformations. There are many coordinate systems besides the well-known Cartesian, polar, and spherical coordinates. In this article, we illustrate how to make such transformations using Maple. Such a use has the advantage of easing the manipulation and derivation of analytical expressions. We illustrate this by considering a number of engineering problems governed by PDEs in different coordinate systems such as the bipolar, elliptic cylindrical, and prolate spheroidal. In our opinion, the use of Maple or similar computer algebraic systems (e.g. Mathematica, Reduce, etc.) will help researchers and students use uncommon transformations more frequently at the very least for situations where the transformations provide smarter and easier solutions. Comput Appl Eng Educ; Published online in Wiley InterScience (www.interscience. wiley.com); DOI 10.1002/cae.20318

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INTRODUCTION

Partial differential equations (PDEs) are mathematical models describing physical laws such as chemical processes, electrostatic distributions, heat flow, and fluid motion. The solution to a number of PDEs is shaped by the boundaries of the geometry, and thus the coordinate system selected is influenced by these boundaries. By choosing a curvilinear coordinate system (ξ_1, ξ_2, ξ_3) such that the boundary surface is one of the coordinate surfaces, it is possible to express the solution of the PDE in terms of these new coordinates, that is, ξ_1, ξ_2, ξ_3. There are many coordinate systems besides the usual Cartesian, polar, and spherical coordinates. For example, Figure 1 shows two identical pipes imbedded in a concrete slab. To find the steady-state temperature, the most suitable coordinate system is the bipolar coordinate system as will be demonstrated in more details in Application of the Bipolar Coordinate System Section. In addition, the separation of variables is a common method for solving linear PDEs. The separation is different for different coordinate systems. In other words, we find out in what coordinate system an equation will be amenable to a separation of variables solution. The properties of the solution can be related to the characteristics of the equations and the geometry of the selected coordinate system. It is possible to prove using the theory of analytic functions of the complex variable that there are a number of two- and three-dimensional separable coordinate systems. The coordinate system is defined by relationships between the rectangular coordinates (x, y, z) and the coordinates (ξ_1, ξ_2, ξ_3). The new coordinate axes are given by the equations $\xi_1(x, y, z) = \text{constant}$, $\xi_2(x, y, z) = \text{constant}$, and $\xi_3(x, y, z) = \text{constant}$.

For example, polar coordinates are useful for circular boundaries or ones consisting of two lines meeting at an angle (see Fig. 2). The families $r = \text{constant}$ and $\varphi = \text{constant}$ are, respectively, the concentric circles and the radial lines as illustrated in Figure 2. Coordinate systems more general than the polar are the elliptic coordinates consisting of ellipses and hyperbolas. These coordinates are suitable for elliptic boundaries or ones consisting of hyperbolas as illustrated in Figure 3. Parabolic cylindrical

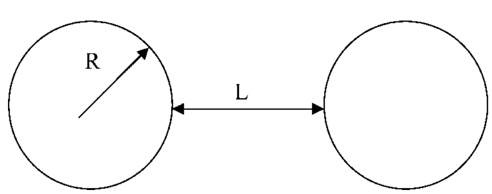

Figure 1 Two identical pipes imbedded in an infinite concrete slab.

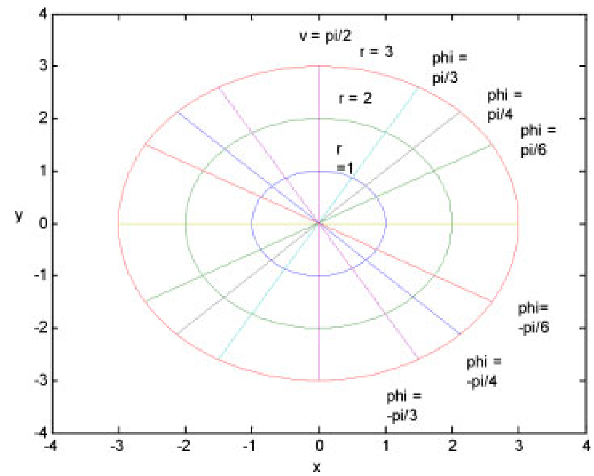

Figure 2 Circular cylindrical coordinates. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

coordinates, shown in Figure 4, are two orthogonal families of parabolas, with axes along the x-axis. These coordinates are suitable, for example, for a boundary consisting of the negative half of the x-axis. Generally speaking, the separable coordinate systems for two dimensions consisted of conic sections; that is, ellipses and hyperbolas or their degenerate forms (lines, parabolas, circles).

For three dimensions, the separable coordinate systems are quadratic surfaces or their degenerate forms. A common coordinate system is the spherical coordinates. The coordinate surfaces are spheres having centers at the origin, cones having vertices at the origin, and planes through the z-axis. Robertson's condition, which relates scale factors and the properties of Stäckel determinant, places a limit on

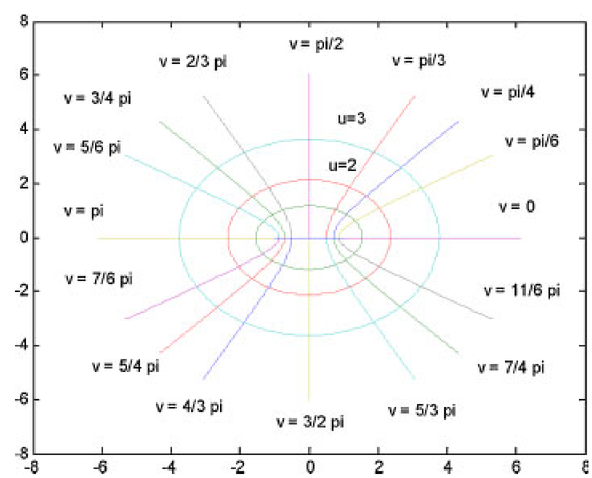

Figure 3 Elliptic cylindrical coordinates. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

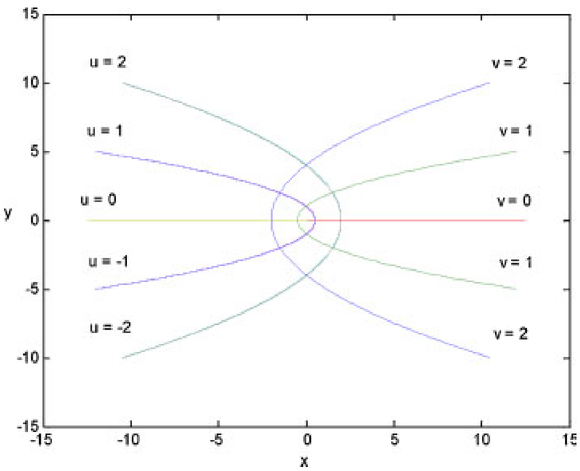

Figure 4 Parabolic coordinates. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

the number of possible coordinate systems. Table 1 lists a number of common coordinate systems. These coordinate systems are: rectangular coordinates, circular cylindrical coordinates, elliptic cylinder coordinates, parabolic cylinder coordinates, spherical coordinates, bipolar coordinates, conical coordinates, parabolic coordinates, prolate spheroidal coordinates, oblate spheroidal coordinates, ellipsoidal coordinates, bi-spherical coordinates, and toroidal coordinates. The names are generally descriptive of the coordinate systems. For example, the circular cylinder coordinates involve coordinate surfaces, which are cylinder coaxial. Maple implements, respectively, about 15 coordinate systems in two dimensions and 31 in three dimensions.

Let us illustrate one useful application of using these coordinate systems. For example, if the boundary conditions require the use of polar coordinates (shown in Fig. 1), the equation $\Delta^2\varphi + k^2\varphi = 0$ (k is a constant), for example, can be split (making use

Table 1 Definition of Common Coordinate Systems

Circular cylindrical (polar) coordinates (ρ, ϕ, z)	$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z$
Elliptic cylindrical coordinates (u, ϕ, z)	$x = d\cosh u\cos\phi,\ y = d\sinh u\sin\phi,\ z$
Parabolic cylindrical coordinates (u, v, z)	$x = (1/2)(u^2 - v^2),\ y = uv,\ z$
Bipolar coordinates (u, v, z)	$x = a\dfrac{\sinh v}{\cosh v - \cos u},\quad y = a\dfrac{\sin u}{\cosh v - \cos u},\quad z$
Spherical coordinates (r, ϕ, θ)	$x = r\cos\phi\sin\theta,\quad y = r\sin\phi\sin\theta,\quad z = r\cos\theta$
Conical coordinates (λ, μ, ν)	$x = \dfrac{\lambda\mu\nu}{ab},\quad y = \dfrac{\lambda}{a}\sqrt{\dfrac{(\mu^2 - a^2)(\nu^2 - a^2)}{a^2 - b^2}},\quad z = \dfrac{\lambda}{b}\sqrt{\dfrac{(\mu^2 - b^2)(\nu^2 - b^2)}{b^2 - a^2}}$
Parabolic coordinates (u, v, ϕ)	$x = uv\cos\phi, y = uv\sin\phi, z = \dfrac{1}{2}(u^2 - v^2)$
Prolate spheroidal coordinates (u, v, ϕ)	$x = d\sinh u\sin\nu\cos\phi,\quad y = d\sinh u\sin\nu\sin\phi,\quad z = d\cosh u\cos\nu$
Oblate spheroidal coordinates (u, v, ϕ)	$x = d\cosh u\sin\nu\cos\phi,\quad y = d\cosh u\sin\nu\sin\phi,\quad z = d\sinh u\cos\nu$
Ellipsoidal coordinates (ξ_1, ξ_2, ξ_3)	$x = \sqrt{\dfrac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{(a^2 - c^2)(a^2 - b^2)}},\quad y = \sqrt{\dfrac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{(b^2 - c^2)(b^2 - a^2)}},$ $z = \sqrt{\dfrac{(\xi_1^2 - c^2)(\xi_2^2 - c^2)(\xi_3^2 - c^2)}{(c^2 - a^2)(c^2 - b^2)}}$
Paraboloidal coordinates (ξ_1, ξ_2, ξ_3)	$x = \sqrt{\dfrac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{(a^2 - b^2)}},\quad y = \sqrt{\dfrac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{(b^2 - a^2)}},$ $z = \dfrac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2 - a^2 - b^2)$
Bispherical coordinates (η, μ, ϕ)	$x = a\cos\phi\dfrac{\sin\eta}{\cosh\mu - \cos\eta},\quad y = a\sin\phi\dfrac{\sin\eta}{\cosh\mu - \cos\eta},\quad z = a\dfrac{\sinh\mu}{\cosh\mu - \cos\eta}$
Toroidal coordinates (η, μ, ϕ)	$x = a\cos\phi\dfrac{\sinh\mu}{\cosh\mu - \cos\eta},\quad y = a\sin\phi\dfrac{\sinh\mu}{\cosh\mu - \cos\eta},\quad z = a\dfrac{\sin\eta}{\cosh\mu - \cos\eta}$

Table 2 Scale Factors for Common Coordinate Systems

Circular cylindrical coordinates (ρ, ϕ, z)	$h_1 = 1$, $h_2 = \rho$, $h_3 = 1$
Elliptic cylindrical coordinates (u, ϕ, z)	$h_1 = d\sqrt{\sinh^2 u + \sin^2 \phi}, \quad h_2 = h_1, \quad h_3 = 1$
Parabolic cylindrical coordinates (u, v, z)	$h_1 = d\sqrt{u^2 + v^2}, \quad h_2 = h_1, \quad h_3 = 1$
Bipolar coordinates (η, μ, z)	$h_1 = \dfrac{\alpha}{\cosh\mu - \cos\eta}, \quad h_2 = h_1, \quad h_3 = 1$
Spherical coordinate (r, θ, ϕ)	h_1=1, $h_2 = r\sin\theta$, $h_3 = r$
Conical coordinates (λ, μ, ν)	$h_1 = 1, \quad h_2 = \lambda\sqrt{\dfrac{(\mu^2-\nu^2)}{(\mu^2-a^2)(b^2-\mu^2)}}, \; h_3 = \lambda\sqrt{\dfrac{(\mu^2-\nu^2)}{(\nu^2-a^2)(\nu^2-b^2)}}$
Parabolic coordinate (u, v, ϕ)	$h_1 = \sqrt{\lambda^2 + \mu^2}, \quad h_2 = h_1, \quad h_3 = \lambda\mu$
Prolate spheroidal coordinates (u, v, ϕ)	$h_1 = d\sqrt{\sinh^2 u + \sin^2 \nu}, \quad h_2 = h_1, \quad h_3 = d\sinh u \sin\nu$
Oblate spheroidal coordinates (u, v, ϕ)	$h_1 = d\sqrt{\sinh^2 u + \sin^2 \nu}, \quad h_2 = h_1, \quad h_3 = d\cosh u \sin\nu$
Ellipsoidal coordinates (ξ_1, ξ_2, ξ_3)	$h_1 = \sqrt{\dfrac{(\xi_1^2-\xi_2^2)(\xi_1^2-\xi_3^2)}{(\xi_1^2-a^2)(\xi_1^2-b^2)(\xi_1^2-c^2)}}, \quad h_2 = \sqrt{\dfrac{(\xi_2^2-\xi_1^2)(\xi_2^2-\xi_3^2)}{(\xi_2^2-a^2)(\xi_2^2-b^2)(\xi_2^2-c^2)}},$ $h_3 = \sqrt{\dfrac{(\xi_3^2-\xi_1^2)(\xi_3^2-\xi_2^2)}{(\xi_3^2-a^2)(\xi_3^2-b^2)(\xi_3^2-c^2)}}$
Paraboloidal coordinates (ξ_1, ξ_2, ξ_3)	$h_1 = \sqrt{\dfrac{(\xi_1^2-\xi_2^2)(\xi_1^2-\xi_3^2)}{(\xi_1^2-a^2)(\xi_1^2-b^2)}}, \quad h_2 = \sqrt{\dfrac{(\xi_2^2-\xi_1^2)(\xi_2^2-\xi_3^2)}{(\xi_2^2-a^2)(\xi_2^2-b^2)}},$ $h_3 = \sqrt{\dfrac{(\xi_3^2-\xi_1^2)(\xi_3^2-\xi_2^2)}{(\xi_3^2-a^2)(\xi_3^2-b^2)}}$
Bispherical coordinates (η, μ, ϕ)	$h_1 = \dfrac{a}{(\cosh\mu - \cos\eta)\sinh\mu}, \quad h_2 = \dfrac{a}{(\cosh\mu - \cos\eta)\sin\eta},$ $h_3 = \dfrac{a\sin\eta}{(\cosh\mu - \cos\eta)\sin\phi}$
Toroidal coordinates (η, μ, ϕ)	$h_1 = \dfrac{a}{\cosh\mu - \cos\eta}, \quad h_2 = h_1, \quad h_3 = \dfrac{a\sinh\mu}{\cosh\mu - \cos\eta}$

of Tables 1–3) into two ordinary differential equations, each for a single independent variable

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \frac{1}{r^2}(r^2k^2 - \alpha^2)f = 0, \quad \frac{d^2 g}{d\phi^2} + \alpha^2 g = 0 \tag{1}$$

where $\varphi(r,\ \phi) = f(r)g(\phi)$ and α is the separation constant (which must be integer in the case of polar coordinates since φ is a periodic coordinate). We will apply the method of separation of variables to the PDEs in the examples.

So, the basic technique is to transform a given boundary value problem in the xy plane (or x, y, z space) into a simpler one in the plane $\xi_1\xi_2$ (or $\xi_1\xi_2\xi_3$ space) and then write the solution of the original problem in terms of the solution obtained for the simpler equation. This is explained in the next three sections. The transformations will make the solution more tractable and convenient to find.

Nowadays, high-performance computers coupled with highly efficient user-friendly symbolic computation software tools such as Maple, Mathematica, Matlab, Reduce [http:www.maplesoft.com, http://www.wolfram.com, http://www.mathworks.com, http://www.reduce-algebra.com] are very useful in teaching mathematical methods involving tedious algebra and manipulations. In this paper, the powerful software tool Maple is used. Maple facilitates the manipulation and derivation of analytical expressions, and can

Table 3 Line and Volume Elements Along With Differential Operators in Curvilinear Orthogonal Coordinate System

Scale factors, h_n	$h_n = \sqrt{\left(\frac{\partial x}{\partial \xi_n}\right)^2 + \left(\frac{\partial y}{\partial \xi_n}\right)^2 + \left(\frac{\partial z}{\partial \xi_n}\right)^2} = \left[\sqrt{\left(\frac{\partial \xi_n}{\partial x}\right)^2 + \left(\frac{\partial \xi_n}{\partial y}\right)^2 + \left(\frac{\partial \xi_n}{\partial z}\right)^2}\right]^{-1}$
Line element, ds	$ds = \sqrt{h_n^2 (d\xi_n)^2}$
Volume element, dV	$dV = h_1 h_2 h_3 d\xi_1 d\xi_2 d\xi_3$
Gradient, $\nabla\psi$	$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial\xi_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial\xi_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial\xi_3}\vec{e}_3$
Divergence, $\nabla\cdot\vec{A}$	$\nabla\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\sum_{n=1}^{3}\frac{\partial}{\partial\xi_1}\left(h_1 h_2 h_3 \frac{A_n}{h_n}\right)$
Curl, $\nabla\times\vec{A}$	$\nabla\times\vec{A} = \frac{1}{h_1 h_2 h_3}\sum_{m,n,p} h_m \vec{e}_m \left[\frac{\partial}{\partial\xi_n}(h_p A_p) - \frac{\partial}{\partial\xi_p}(h_n A_n)\right], \quad m,n,p = 1,2,3, \text{ or } 2,3,1 \text{ or } 3,1,2$
Laplacian, $\nabla^2\psi$	$\nabla^2\psi = \frac{1}{h_1 h_2 h_3}\sum_{n=1}^{3}\frac{\partial}{\partial\xi_n}\left[\frac{h_1 h_2 h_3}{h_n^2}\frac{\partial\psi}{\partial\xi_n}\right]$

be used to perform tedious algebra, complicated integrals, and differential equations [1–3]. A secondary objective of this paper is to expose the student to different skills in using Maple to perform the algebra and work with differential equations calculations and solutions in different coordinate systems.

For the sake of readers not familiar with Maple, a brief introduction will follow. Maple is a powerful symbolic computational tool used to perform analytical derivations and numerical calculations. It is easy to use, and its commands are often straightforward to know even for a first-time user. In this paper, the student version of Maple is used. We recommend that the student uses ";" and not ":" at the end of a command statement so that Maple prints the results. This helps in fixing mistakes in the program since the results are printed after every command statement. In addition, the user might have to manipulate the resulting expressions from a Maple command to obtain the equation in the simplest or desired form. All the mathematical manipulations involved can be performed in the same program, and Maple can be used to perform all the required steps from setting up the equations to interpreting plots in the same sheet. Please note that equations containing ":=" are results printed by Maple.

APPLICATION OF THE BIPOLAR COORDINATE SYSTEM

There are a number of real applications for bipolar coordinates such as pairs of ducts, pipes, transmission lines, and bubbles [4–9]. We will illustrate the use of the bipolar orthogonal coordinate system in this section by the following example. Two identical circular pipes of identical radius R are imbedded in an infinite concrete slab as shown in Figure 1. The uniform temperature of both pipes is T_0. We want to solve the temperature distribution in the concrete slab by solving the following differential Laplace equation:

```
> restart: with(student):
> eq:=diff(T(x,y),x$2)+
diff(T(x,y),y$2)=0;
```

$$eq := \left(\frac{\partial^2}{\partial x^2}T(x,y)\right) + \left(\frac{\partial^2}{\partial y^2}T(x,y)\right) = 0$$

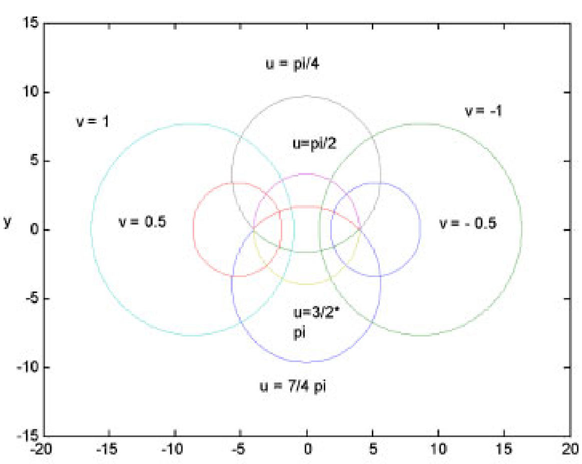

Figure 5 Bipolar coordinates. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

The boundary conditions of this problem dictate the use of bipolar coordinates (see Fig. 5). According to Table 1, the following equations define bipolar coordinates:

$$x = \frac{c \sinh v}{\cosh v - \cos u}, \quad y = \frac{c \sin u}{\cosh v - \cos u} \tag{2}$$

```
> eq1:=x-c*sinh(v(x,y))/(
  cosh(v(x,y))-cos(u(x,y)));
```

$$\mathrm{eq1} := x - \frac{c \sinh(v(x,y))}{\cosh(v(x,y)) - \cos(u(x,y))}$$

```
> eq2:=y=c*sin(u(x,y))/
  (cosh(v(x,y))-cos(u(x,y)));
```

$$\mathrm{eq2} := y = \frac{c \sin(u(x,y))}{\cosh(v(x,y)) - \cos(u(x,y))}$$

We will show how tractable and convenient it is to solve the Laplace equation in the bipolar coordinates, whereas in terms of x, y, and z the temperature field expression is complex. So, the bipolar coordinates are the "natural" coordinates for this type of problem.

First of all, we show that the bipolar coordinate system is an orthogonal coordinate system. This means that the two families of the coordinate surfaces $u(x, y)$ and $v(x, y)$ are mutually orthogonal. The lines of intersection of these surfaces constitute two families of lines. At the point (u, v), we have unit vectors $\vec{e}^1$ and $\vec{e}^2$ each, respectively, tangent to the coordinate line of the bipolar coordinate system which goes through the point. Since the coordinate system is orthogonal, $\vec{e}^1$ and $\vec{e}^2$ are mutually perpendicular everywhere

$$\vec{e}^1 \cdot \vec{e}^2 = 0 \quad \text{or} \quad \frac{1}{h^1}\frac{\partial \vec{r}}{\partial u}\frac{1}{h^2}\frac{\partial \vec{r}}{\partial v} = 0 \tag{3}$$

where $\vec{r}$ is a position vector and is given by

$$\vec{r} = x\vec{e}^1 + y\vec{e}^2 \tag{4}$$

h_1 and h_2 are scale factors for the bipolar coordinates u and v. We will write about them later on Equation (2) becomes

$$\frac{1}{h^1h^2}\left[\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right)\right] = 0 \tag{5}$$

Equation (1) is a conformal transformation to x, y from u, v coordinates. u, u_y, v_x, and v_y are computed using Maple as follows:

```
> eq3:=diff(eq1,x):
  eq4:=diff(eq2,x):
vx:=solve(eq4,diff(v(x,y),x)):
eq31:=simplify(subs(diff(v(x,y),x)
=vx,eq3)):ux:=solve(eq31,diff
(u(x,y),x));
```

$$ux := -\frac{\sin(u(x,y))\sinh(v(x,y))}{c}$$

```
> vx:=simplify(subs(diff
(u(x,y),x)=ux,vx));
```

$$-\frac{\cos(u(x,y))\cosh(v(x,y)) - 1}{c}$$

```
> eq5:=diff(eq1,y):eq6:=diff
  (eq2,y):uy:=solve(eq5,diff
  (u(x,y),y)):eq61:=simplify
  (subs(diff(u(x,y),y)=uy,eq6)):
  vy:=solve(eq61,diff(v(x,y),y));
```

$$vy := -\frac{\sin(u(x,y))\sinh(v(x,y))}{c}$$

```
uy:=simplify(subs(diff
(v(x,y),y)=vy,uy));
```

$$uy := \frac{\cos(u(x,y))\cosh(v(x,y)) - 1}{c}$$

The Cauchy−Riemann conditions are satisfied since:

```
simplify(ux-vy);simplify(uy+vx);
```

$$0$$
$$0$$

and thus we conclude that the bipolar coordinate system is orthogonal.

Next, we obtain the scale factors h_i ($I = 1, 2$) given by

$$h_i = \left|\frac{\partial \vec{r}}{\partial \xi_i}\right|, \quad \xi_1 = u \quad \text{and} \quad \xi_2 = v \tag{6}$$

The scale factor h_i can be interpreted as follows: a change du in the bipolar coordinate system produces a displacement $h_1\,\mathrm{d}u$ along the coordinate line. Now, we notice that the rate of displacement along u due to a displacement along the x-axis is $h_1(\partial u/\partial x)$ which is the same as the rate of change of x due to a displacement $h_1\,\mathrm{d}u$. The same argument goes for the scale factor h_2. The scales of the new coordinates and the change of scale from point to point determine the important properties of the coordinate system. The scale factors play a role in expressing the differential/

integral operators, line, surface, and volume elements, and for the sake of completeness, they are shown in Table 2 for common coordinate systems. After some algebraic manipulations, we find that

```
> h1square:=factor(simplify
(1/(ux^(2)+uy^(2)))):h[1]:
=simplify(sqrt(h1square),
sqrt,symbolic);
```

$$h1 := -\frac{c}{\cos(u(x,y)) - \cosh(v(x,y))}$$

So,

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2} = \frac{c}{\cosh v - \cos u} \tag{7}$$

```
> h2square:=factor(simplify
(1/(vx^(2)+vy^(2)))):
h[2]:=simplify(sqrt
(h2square),sqrt,symbolic);
```

$$h_2 := -\frac{c}{-\cosh(v(x,y)) + \cos(u(x,y))}$$

and so,

$$h_2 = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2} = \frac{c}{\cosh v - \cos u} \tag{8}$$

Next, we need to obtain expressions for the parameter c in terms of R and L. The y-axis lies in the middle between the two cylinders, while the x-axis crosses the centers of the cylinders. After algebraic manipulations, the relationship between the Cartesian and bipolar coordinates can be expressed as follows:

$$x^2 + (y - c \cot u)^2 = c^2 \csc^2 u \tag{9}$$

$$(x - c \coth v)^2 + y^2 = c^2 \csc h^2 v \tag{10}$$

This form of writing the relationship between the "old" set of coordinates, that is, x, y and the "new" set of coordinates u, v provides us a further insight since we can notice that Equations (9) and (10) are circles. For an arbitrary $v = \eta_0$, from Equation (10) we have a circle of radius $c \csc h\eta_0$ and center ($c \coth \eta_0$, 0). Also, when $v = -\eta_0$, we have a circle of radius $c \csc h\eta_0$ and center ($-c \coth \eta_0$, 0). Now, when

$$c \csc h\eta_0 = R \tag{11}$$

and,

$$c \coth \eta_0 = \frac{L}{2} + R \tag{12}$$

Then

$$c = R \sinh\left[\cosh^{-1}\left(1 + \frac{L}{2R}\right)\right] \tag{13}$$

The obvious next step is to transform the differential equation from x, y to u, v coordinates. Using Maple, we can simply use the Laplacian command. However, Laplace form does not govern many differential equations, and so we will follow the approach needed to map a differential equation from a set of coordinates to another. So, first, we need to get the second derivatives of u, v with respect to x, y

```
> uxx:=diff(ux,x):uxx:=simplify
(subs(diff(u(x,y),x)=ux,
diff(v(x,y),x)=vx,uxx)):
> vxx:=diff(vx,x):vxx:=simplify
(subs(diff(u(x,y),x)=ux,
diff(v(x,y),x)=vx,vxx)):
> uyy:=diff(uy,y):uyy:=simplify
(subs(diff(u(x,y),y)=uy,
diff(v(x,y),y)=vy,uyy)):
> vyy:=diff(vy,y):vyy:=simplify
(subs(diff(u(x,y),y)=uy,
diff(v(x,y),y)=vy,vyy)):
```

Then, we map the differential equation into the bipolar coordinates:

```
> eq1:=eval(subs(T(x,y)=TT
(u(x,y),v(x,y)),eq)):
eq2:=subs(diff(v(x,y),y$2)=
-diff(v(x,y),x$2),diff
(u(x,y),y$2)=-diff(u(x,y),x$2),
diff(v(x,y),y)=diff (u(x,y),x),
diff(v(x,y),x)=-diff
(u(x,y),y),eq1):
eq3:=factor(expand(eq2)):
eq4:=D[2,2](TT)(u(x,y),v(x,y))
+D[1,1](TT)(u(x,y),v(x,y)):
eq5:=subs(u(x,y)=u,v(x,y)
=v,eq4):Eq:=convert(eq5,diff);
```

$$\mathrm{Eq} := \left(\frac{\partial^2}{\partial u^2} TT(u,v)\right) + \left(\frac{\partial^2}{\partial v^2} TT(u,v)\right)$$

which can be rewritten as

$$\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0 \tag{14}$$

which shows that Laplace's equation is invariant under a mapping to bipolar coordinates. Now, if T changes with respect to z as well, then the transformed PDE can be shown to be

$$\frac{1}{h_1 h_2}\left(\frac{\partial^2 T}{\partial u^2}+\frac{\partial^2 T}{\partial v^2}\right)+\frac{\partial^2 T}{\partial z^2}=0 \qquad (15)$$

but since the temperature does not change with respect to z, that is, $\partial T/\partial z=0$, then Equation (14) is reduced to (13). The transformed boundary conditions in the u, v coordinates are

$$T=T_1 \text{ at } v=\eta_0, \quad T=T_2 \text{ at } v=-\eta_0 \qquad (16)$$

At this point, we can use the method of separation of variables covered in many undergraduate and graduate textbooks [10,11] (so $T(u, v)=X(u)Y(v)$). The separation of variables splits the PDE into two ODEs. We assume that the temperature $T(u, v)$ is only dependent on v. So, in our particular case, the separation constant is $\lambda_n=0$. So, we end up solving the following differential equation with its Dirichlet boundary conditions:

```
> Eq:=diff(Y(v),v$2);
```

$$\frac{d^2}{dv^2}Y(v)=0$$

```
> bc1:=subs(v=eta[0],Y(v))=T[1];
```

$$\text{bc1} := Y(h_0)=T_1$$

```
> bc2:=subs(v=-eta[0],
Y(v))=T[2];
```

$$\text{bc2} := Y(-h_0)=T_2$$

```
> sol:=dsolve(Eq,Y(v)):
eq1:=subs(v=eta[0],
Y(eta[0])=T[1],sol):
eq2:=subs(v=-eta[0],
```

$$eq3 := \left\{\_C1=\frac{T_1-T_2}{2\eta_0}, \_C_2=\frac{1}{2}T_2+\frac{1}{2}T_1\right\}$$

```
Y(-eta[0])=T[2],sol):
eq3:=solve({eq1,eq2},
{_C1,_C2});
```

and thus:

$$Y(v)=\frac{(T_1-T_2)v}{2h_0}+\frac{1}{2}T_2+\frac{1}{2}T_1$$

which is equal to $T(u, v)$.

Generally, when we conduct the separation of variables, one gets

```
> eqs:=subs(TT(u,v)=X(u)*Y
(v),Eq):eqs:=expand(eqs):
eqs:=eqs/(X(u)*Y (v)):
eqs:=expand(eqs);
```

$$\text{eqs} := \frac{\frac{d^2}{du^2}X(u)}{X(u)}+\frac{\frac{d^2}{dv^2}Y(v)}{Y(v)}$$

we obtain two ODEs in $X(u)$ and $Y(v)$ and the first one is

```
> eqs1:=diff(X(u),u$2)
+lambda^2*X(u):bc1:=subs
(u=0,D(X)(u))=0:bc2:
=subs(u=Pi,D(X)(u))=0;
```

$$eqs1 := \left(\frac{d^2}{du^2}\times(u)\right)+\lambda^2\times(u)$$

$$\text{bc1} := D(X)(0)=0$$

$$\text{bc2} := D(X)(p)=0$$

The ODE in $X(u)$ is a regular Sturm−Liouville boundary value problem [10], with separation constant $\lambda_n=n$ (where $n=1, 2, \ldots$) with corresponding eigenfunctions $X_n(u)=\sin(nu)$. The ODE in $Y(v)$ is

```
> eqs2:=diff(Y(v),v$2)
-n^2*Y(v);
```

$$eqs2 := \left(\frac{d^2}{dv^2}Y(v)\right)-n^2Y(v)$$

```
> dsolve(eqs2);
```

$$Y(v)=\_C1\,e^{(-nv)}+\_C2\,e^{(n,v)}$$

In order to satisfy the boundary conditions (Eq. 16), we consider the following solution which is a linear combination of products, and for which the separation constant is not zero:

$$T(u,v)=\frac{T_1-T_2}{2\eta_0}v+\frac{T_1+T_2}{2}+\sum_{n=1}^{\infty}(c_n e^{nv}+d_n e^{-nv})\sin(nu)$$

where c_n and d_n are computed by Maple as follows:

```
> an:=(2/Pi)*int(T[1]*sin(n*u),
u=0..Pi):bn:=(2/Pi)*int(T[2]
*sin(n*u),u=0..Pi) : assume(n:
:integer) : assume(n,odd):
eval(an):eval(bn): eqc1:=cn*exp
(n*eta[0])+dn*exp(-n*eta[0])=an:
eqc2:= cn*exp(-n*eta[0])+dn
```

$$solc := \left\{ cn = \frac{4(-e^{(-n+\eta_0)}T_2 - T_1 e^{(n+\eta_0)}}{\pi n\left(-\left(e^{(-n+\eta_0)}\right)^2 + \left(e^{(n-\eta_0)^2}\right)\right)}, dn = \frac{4(-T_1 e^{(-n+\eta_0)} - T_2 e^{(n+\eta_0)}}{\pi n\left(-\left(e^{(-n+\eta_0)}\right)^2 + \left(e^{(n-\eta_0)}\right)^2\right)} \right\}$$

```
*exp(n*eta[0])=bn:solc:=solve
({eqc1,eqc2},{cn,dn});
```

where n denotes an odd number (i.e. $n=2m-1$, $m=1, 2\ldots$).

APPLICATION OF THE ELLIPTIC CYLINDRICAL COORDINATE SYSTEM

Another type of an orthogonal curvilinear coordinates system is the elliptic cylinder. The coordinate surfaces are elliptic cylinders ($u=$ constant) and hyperbolic cylinders ($v=$ constant) in the two coordinate systems. Many problems are amenable to this type of coordinate systems such as coils, solar, and heat concentrators, metallurgical junctions, material flaw shapes, shells, fluid flow past an obstacle [12–17]. The transformations to the Cartesian coordinates from the elliptic cylindrical coordinates is listed in Table 1, and are

$$x = a\cosh(u)\cos(v), \quad y = a\sinh(u)\sin(v)$$

where a is the length of the semi-major axis of the ellipse. For example, if an elliptical hole (Fig. 6) is cut in a region as we will see in this section as an example, then it is more tractable to solve the Laplace equation in the elliptic cylindrical coordinates. We take the center of the coordinate system to be that of the hole. The scale factors are given by

```
> restart:with(student):
```

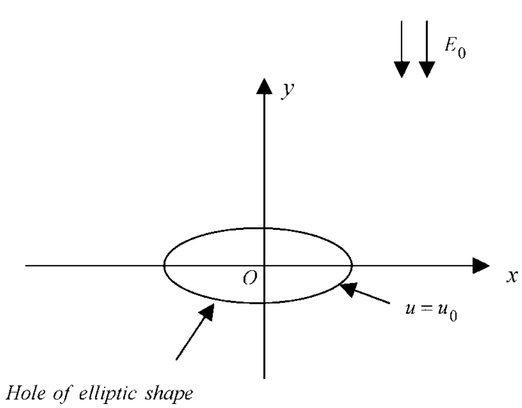

Figure 6 The shape of the elliptic hole which is illuminated with the field E_0.

```
> x:=a*cosh(u)*cos(v):
y:=a*sinh(u)*sin(v):h[u]:=sqrt
(diff(x,u)^2+diff(y,u)^2):h[u]
:=simplify(h[u]):h[u]:=subs
(cosh(u)^2=1+sinh(u)^2,
cos(v)^2=1-sin(v)^2,h[u]):
h[u]:=simplify(h[u],sqrt,
symbolic);
```

$$h_u := a\sqrt{\sin(v)^2 + \sinh(u)^2}$$

Likewise, we obtain h_v

$$h_v := a\sqrt{\sin(v)^2 + \sinh(u)^2}$$

The Laplace equation in elliptic cylindrical coordinates, assuming $a=1$, is

```
> with(linalg):
> eq1:=laplacian(phi(u,v),
[u,v],coords=elliptic):
eq1:=numer(eq1);
```

$$\text{eq1} := \left(\frac{\partial^2}{\partial u^2}\phi(u,v)\right) + \left(\frac{\partial^2}{\partial v^2}\phi(u,v)\right)$$

Using separation of variables, we have $\phi(u, v) = \phi_1(u, v)\phi_2(u, v)$:

```
> eq2:=subs(phi(u,v)=phi[1]
(u)*phi[2](v),eq1) : eq3:=
expand(eq2):eq4:=eq3*
(1/(phi[1](u)*phi[2](v))):
eq4:=expand(eq4);
```

$$\text{eq4} := \frac{\frac{\partial^2}{\partial u^2}\phi_1(u)}{\phi_1(u)} + \frac{\frac{\partial^2}{\partial v^2}\phi_2(v)}{\phi_2(v)}$$

We end up with two differential equations for $\phi_1(u, v)$ and $\phi_2(u, v)$.

```
eq41:=diff(phi[1](u),u$2)
-p^2*phi[1](u);
```

$$\text{eq41} := \left(\frac{\partial^2}{\partial u^2}\phi_1(u)\right) - p^2\phi_1(u)$$

```
eq42:=diff(phi[2](v),v$2)
+p^2*phi[2](v);
```

$$\text{eq42} := \left(\frac{\partial^2}{\partial v^2}\phi_2(v)\right) - p^2\phi_2(v)$$

The two boundary conditions labeled bc1 and bc2 are such that:

```
bc1:=subs(u=infinity,
phi(u,v))=phi[0]+E
[0]*sinh(u)*sin(v);
```

$$\text{bc1} := \phi(\infty, v) = \phi_0 + E_0 \sinh(u)\sin(v)$$

```
bc2:=subs(u=u[0],
D[ 1](phi)(u,v))=0;
```

$$\text{bc2} := D_1(\phi)(u_0, v) = 0$$

Assuming that $\phi(u,\ v) = A_0 + A_1 \sinh u \sin v + A_1 \cosh \sin v$, then enforcing the first boundary condition and since $\sinh u = \cosh u$ for large u, we have

```
eq5:=phi(u,v)=_A0+_A1*
sinh(u)*sin(v)+_A2*cosh
(u)*sin(v) : eq6:=subs
(u=infinity,_A0=phi[0],
eq5):eq7:=phi[0]+E[0]*
sinh(u)*sin(v)=phi[0]+
(_A1+_A2)*sinh(u)*sin(v);
```

$$\begin{aligned}\text{eq7} : &= \phi_0 + E_0 \sinh(u)\sin(v) = \phi_0 \\ &+ (\_A1 + \_A2)\sinh(u)\sin(v)\end{aligned}$$

$$E(u,v) : = -\left[\frac{-\dfrac{E_0 \sinh(u0)\cosh(u)\sin(v)}{\cosh(u0)-\sinh(u0)} + \dfrac{\cosh(u0)\,E_0 \sinh(u)\sin(v)}{\cosh(u0)-\sinh(u0)}}{\sqrt{\sinh(u)^2 + \sin(v)^2}},\right.$$

$$\left.\frac{-\dfrac{E_0 \sinh(u0)\sinh(u)\cos(v)}{\cosh(u0)-\sinh(u0)} + \dfrac{\cosh(u0)\,E_0 \cosh(u)\cos(v)}{\cosh(u0)-\sinh(u0)}}{\sqrt{\sinh(u)^2 + \sin(v)^2}}\right]$$

We assume that $A_0 = E_0$ which is equal to the potential at $y = 0$ as shown in Figure 3. The second boundary condition gives

```
phi(u,v):=_A0+_A1*sinh
(u)*sin(v)+_A2*cosh(u)*sin(v):
eq8:=diff(phi(u,v),u):
eq8:=subs(u=u[0],eq8)=0;
```

$$\begin{aligned}\text{eq8} : &= \_A1 \cosh(u_0)\sin(v) \\ &+ \_A2 \sinh(u_0)\sin(v) = 0\end{aligned}$$

So, now we can solve for the constants A_1 and A_2:

```
eq9:=solve({eq7, eq8},
{_A1,_A2});
```

$$\text{eq9} : = \left\{\_A1 = -\frac{E_0 \sinh(u_0)}{\cosh(u_0) - \sinh(u_0)}\right.$$

$$\left.\_A2 = \frac{\cosh(u_0)E_0}{\cosh(u_0) - \sinh(u_0)}\right\}$$

Substituting A_1 and A_2 in the potential $\phi(u, v)$, one obtains

```
eq10:=subs(_A0=phi[0],eq5):
eq11:=subs(eq9,eq10);
```

$$\begin{aligned}\text{eq11} : &= \phi_0 - \frac{E_0 \sinh(u0)\sinh(u)\sin(v)}{\cosh(u0) - \sinh(u0)} \\ &+ \frac{\cosh(u0)E_0 \cosh(u)\sin(v)}{\cosh(u0) - \sinh(u0)}\end{aligned}$$

The electrostatic field E is the gradient of the potential, and thus is given by

```
E(u,v):=-grad(rhs(eq11),
[u,v],coords=elliptic);
```

A field plot is shown Figure 7 reflecting the elliptic cylindrical coordinate system.

APPLICATION OF THE PROLATE SPHEROIDAL COORDINATES

In three dimensions, as shown in Table 1, there are a number of three-dimensional coordinates. In here, we will give as an example the case of the prolate

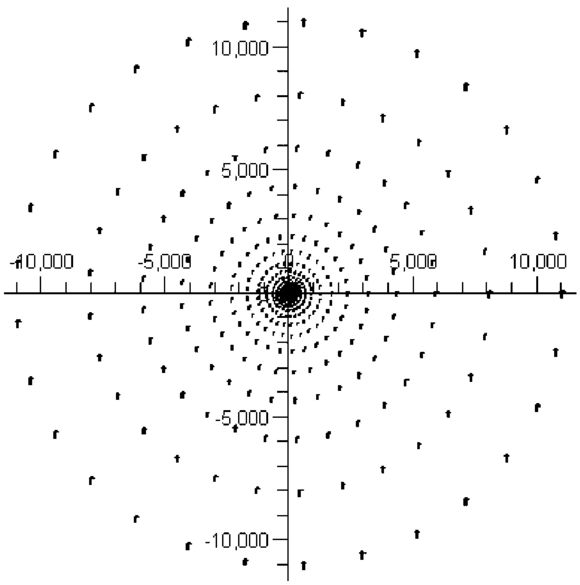

Figure 7 Field plot of E(u, v/E_0) which reflects the elliptic coordinate system.

spheroidal coordinates system illustrated in Figure 8. This type of coordinates system has been used extensively in many fields. For example, raindrops, dust grains in plasma, molten regions in laser welding, ground rod connection, conducting electrodes, ventricle, diatomic molecules, hydrogen molecular ions, and biological cells [11,18–24] are modeled as prolate spheroidal objects. As an example, the shape of a football ball is a prolate spheroid. The transformation to the Cartesian coordinates from the prolate spheroidal coordinates is listed in Table 1. We will repeat here for the sake of clarity:

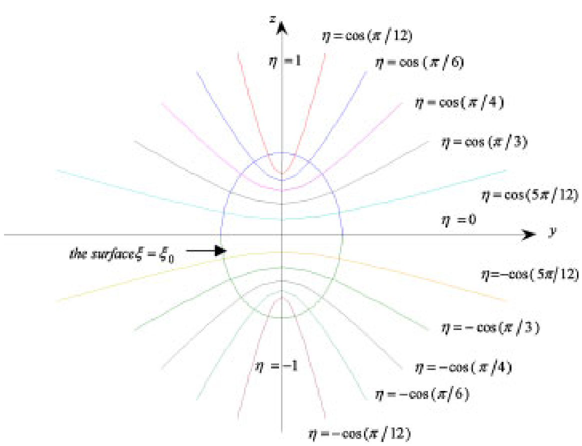

Figure 8 Prolate spherical coordinates. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

$$x = d\sinh(u)\sin(v)\cos(\phi),$$
$$y = d\sinh(u)\sin(v)\sin(\phi), \quad z = d\cosh(u)\cos(w)$$

where d is the focal length for the prolate spheroidal system. However, we need to be aware that there are other equivalent transformations. When $\xi = \cosh(u)$, $\eta = \cos v$, then one gets

$$x = d\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos(\phi),$$
$$y = d\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin(\phi), \quad z = d\xi\eta$$

where $1 \le \xi \le \infty$, $-1 \le \eta \le 1$, $0 \le \phi \le 2\pi$.

To compute the volume and surface area in the prolate spheroidal coordinate system, the Jacobian matrix associated with the coordinate transformation must be calculated. The Jacobian matrix elements are the partial derivatives of the transformation from prolate spherical to Cartesian coordinates. So, the infinitesimal change in volume is the determinant of the Jacobian matrix:

```
restart:with(student):
with(linalg):with (PDEtools):
T:=[d*sinh(u)*sin(v)
*cos(phi),d*sinh(u)*sin(v)
*sin(phi),d*cosh(u)
*cos(v)]:J:=jacobian
(T,[u,v,phi]):dV:=simplify
(det(J))*d(u)*d(v)*d(phi);
```

$$dV := d^3\sin(v)\sinh(u)(\cosh(u)^2 - \cos(v)^2)\,\mathrm{d}(u)\,\mathrm{d}(v)\,\mathrm{d}(\phi)$$

Similarly, for the computation of surfaces, we need to compute the two-dimensional Jacobian matrix in the desired direction.

In this section, we want to obtain an expression for the equilibrium temperature distribution of a metal spheroid. Let us suppose that because of an internal heating system within the spheroid ($\xi = \xi_i$), its surface temperature is

$$\Psi = 10 + 30\cos^2\eta$$

The prolate spheroid made of metal is immersed in a large container filled with insulating powder. The ambient temperature is 20°C. So, basically, we need to solve for the Laplace equation in prolate spheroidal coordinates:

```
eq1:=laplacian(psi(xi,
eta,phi),[xi,eta,phi],
coords=prolatespheroidal(a)):
eq2:=expand(eq1):
```

$$\mathrm{eq2} := \frac{\coth(\xi)\left(\frac{\partial}{\partial\xi}\psi(\xi,\eta,\phi)\right)}{a^2(\sinh(\xi)^2+\sin(\eta)^2)} + \frac{\frac{\partial^2}{\partial\xi^2}\psi(\xi,\eta,\phi)}{a^2(\sinh(\xi)^2+\sin(\eta)^2)} + \frac{\cot(\eta)\left(\frac{\partial}{\partial\eta}\psi(\xi,\eta,\phi)\right)}{a^2(\sinh(\xi)^2+\sin(\eta)^2)} + \frac{\frac{\partial^2}{\partial\eta^2}\psi(\xi,\eta,\phi)}{a^2(\sinh(\xi)^2+\sin(\eta)^2)} + \frac{\frac{\partial^2}{\partial\phi^2}\psi(\xi,\eta,\phi)}{a^2(\sinh(\xi)^2+\sin(\eta)^2)\sin(\eta)^2\sinh(\xi)^2}$$

The system of prolate spheroidal coordinate is separable. So writing the dependent variable $\Psi(\xi, \eta, \phi)$ as the product of three functions $\Xi(\xi)H(\eta)\Phi(\phi)$ will split the Laplacian equation in three ordinary differential equations:

```
eq3:=subs(psi(xi,eta,
phi)=Xi(xi)*Eta(eta)
*Phi(phi),eq2):
eq4:=eq3*(1/(Xi(xi)
*Eta(eta)*Phi(phi))):
eq4:=expand(eq4):eq4:
=expand(a^2*eq4);
```

The three differential equations are:

$$\mathrm{ode1} := \left(\frac{\partial^2}{\partial\phi^2}\Phi(\phi)\right) + q^2\Phi(\phi) = 0$$

$$\mathrm{eq42} := \left(\frac{\partial^2}{\partial\eta^2}\mathrm{H}(\eta)\right) + \cot(\eta)\left(\frac{\partial}{\partial\eta}\mathrm{H}(\eta)\right) + \left(p(p+1) + \frac{q^2}{\sin(\eta)^2}\right)\mathrm{H}(\eta) = 0$$

$$\mathrm{eq43} := \left(\frac{\partial^2}{\partial\xi^2}\Xi(\xi)\right) + \coth(\xi)\left(\frac{\partial}{\partial\xi}\Xi(\xi)\right) - \left(p(p+1) + \frac{q^2}{\sinh(\xi)^2}\right)\Xi(\xi) = 0$$

q and p are the introduced separation variables. The differential equations in $H(\eta)$ and $\Xi(\xi)$ are transformed using the following change of variables:

$$\lambda(\xi) := \cosh(\xi)$$

$$\mu(\eta) := \cos(\eta)$$

Then, we will get the following differential equations:

```
eq430:=changevar(Xi(xi)=
f(lambda(xi)),eq43):eq431:
=numer(simplify(subs(lambda
(xi)=cosh(xi),eq430))):eq432:
=subs(cosh(xi)=lambda,eq431):
eq432:=convert(eq432,diff):
eq432a:=collect(eq432,diff(f
(lambda),lambda$2)):eq432b:
=collect(eq432,diff(f(lambda),
lambda)):eq432c:=collect
(eq432,f(lambda)):a:=factor
(1+lambda^4-2*lambda^2):b:
=factor(2*lambda^3-2*lambda):
fac:=1-lambda^2:a1:=simplify
(a/fac):b1:=simplify(b/fac):c:
=simplify((p^2*lambda^2-p^2+p
*lambda^2-p-q^2)/fac):c1:
=collect(c,p^2):ode3:=a1*diff
(f(lambda),lambda$2)+b1*diff
(f(lambda),lambda)+c1*f(lambda);
```

$$\mathrm{ode3} := (1-\lambda^2)\left(\frac{\partial^2}{\partial\lambda^2}f(\lambda)\right) - 2\lambda\left(\frac{\partial}{\partial\lambda}f(\lambda)\right) + \left(p(p+1) + \frac{q^2}{\lambda^2-1}\right)f(\lambda)$$

and the same Maple technique is used for the equation in $H(\eta)$,

$$\mathrm{ode2} := (1-\mu^2)\left(\frac{\partial^2}{\partial\mu^2}g(\mu)\right) - 2\mu\left(\frac{\partial}{\partial\mu}g(\mu)\right) + \left(p(p+1) + \frac{q^2}{\mu^2-1}\right)g(\mu)$$

So, we need to solve the differential equations labeled above by ode1, ode2, and ode3. We rename, respectively, f and g to $\Xi(\xi)$ and $H(\eta)$.

```
Phi:=rhs(dsolve(ode1,
Phi(phi)));alias(P=LegendreP,
Q=LegendreQ):f:=rhs(dsolve
(ode3,f(lambda)));g:=rhs
(dsolve(ode2,g(mu))):
Xi:=subs(lambda=cosh(xi),f)
:Eta:=subs(mu=cos(eta),g);
```

$$\Phi := \_C1\sin(q\phi) + \_C2\cos(q\phi)$$
$$\Xi := \_C1\,\mathrm{Q}(p,q,\cosh(\xi)) + \_C2\,\mathrm{P}(p,q,\cosh(\xi))$$
$$\mathrm{H} := \_C1\,\mathrm{P}(p,q,\cos(\eta)) + \_C2\,\mathrm{Q}(p,q,\cos(\eta))$$

where $\_C1$ and $\_C2$ are constants. We notice that both $\Xi(\xi)$ and $H(\eta)$ solutions are functions of associated Legendre functions of the first and second kind.

The solution $\Psi(\xi, \eta, \phi)$ has axial symmetry about the z-axis and so it is independent of ϕ and is given by $\Psi(\xi, \eta, \phi) = \Xi(\xi)H(\eta)$. Let the boundary conditions be $\Psi(\xi, \xi_0, \eta) = M(\eta) = 10 + 30\cos^2\eta$, and hence $\mu = \cos\eta$ ranges only between -1 and 1. The associated Legendre functions of the second kind $Q_n(\cos\eta)$ are unbounded for $\mu = 1$, and so the solution $\Psi(\xi, \eta)$ depends only on the associated Legendre functions of the first kind illustrated in Figure 9 and has the form $P_n(\cos\eta)P_n(\cosh\xi)$, in other words:

```
N:=infinity;Psi(xi):=sum('A[i]
*P(i,mu)*(P(i,cosh(mu))/
P(i,cosh(xi0)))','i'=0...N);
```

$$\Psi(\xi) := \sum_{i=0}^{\infty} \frac{A_i\, P(i, \mu)\, P(i, \cosh(\xi))}{P(i, \cosh(\xi 0))}$$

Enforcing the boundary condition that $\Psi(\xi, \xi_0, \eta) = M(\eta)$, one obtains

```
Psi(xi):=sum('A[i]*P
(i,mu)*(P(i,cosh(xi))/P
(i,cosh(xi0)))','i'=0...N):
M(mu):=subs(xi=xi0,Psi(xi));
```

$$M(\mu) := \sum_{i=0}^{\infty} A_i P(i, \mu)$$

But, when we truncate N to 5,

$$M(\mu) : = A_0 + A_1\mu + A_2P(2, \mu) + A_3P(3, \mu) + A_4P(4, \mu) + A_5P(5, \mu)$$

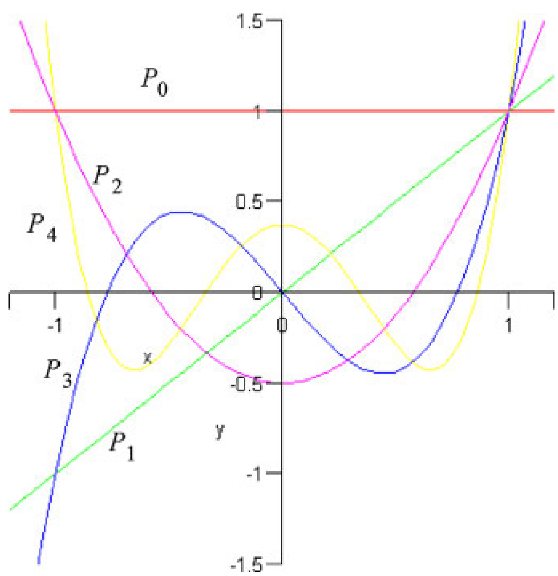

Figure 9 Legendre polynomials of the first kind $P_0(x) \ldots P_4(x)$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

and so to compute A_n, we will have to perform the following computation:

$$A_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} M(\mu) P_n(\mu)\, d\mu$$

```
> for i from 0 to 10 do A[i]=evalf
((i+0.5)*int((10+30*mu^2)
*P(i,mu),mu=-1...1)); od;
```

$$A_0 = 20.0, A_1 = 0, A_2 = 20.00000000$$
$$+0.1, A_3 = 0 + 0.1, \quad A_4 = -0.450 \times 10^{-11}$$
$$+0.1, A_5 = 0 + 0.1$$

Then, substituting the coefficients A_n into the solution $\Psi(\xi, \eta)$, one gets

```
sol1:=subs(A[0]=20,A[2]
=20,A[1]=0,A[3]=0,A[5]
=0,A[4]=0,mu=cos(eta),
Psi(xi));
```

$$\text{sol1} := 20 + \frac{20P(2, \cos(\eta))P(2, \cosh(\xi))}{P(2, \cosh(\xi 0))}$$

whose solution $\Psi(\xi, \eta)$ is illustrated in Figure 10 for $\xi = 2$.

CONCLUSION

In this paper, examples of solving problems in coordinate systems other than the most famous Cartesian, polar, and spherical coordinates are given. In this

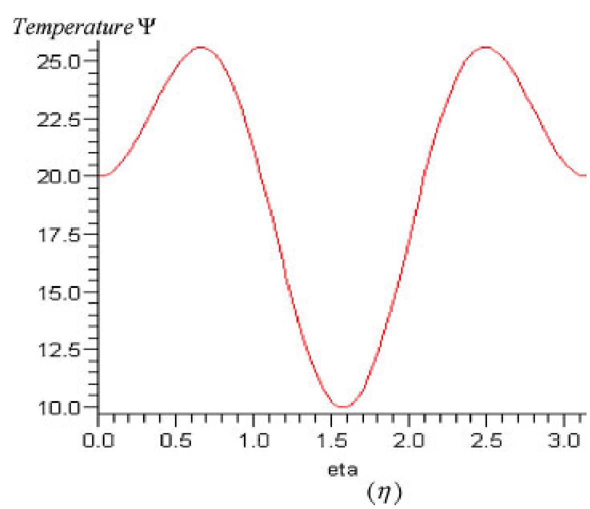

Figure 10 Temperature $\Psi(\eta)$ for $\xi = 2$.

article, we gave examples in the two-dimensional bipolar and elliptic cylindrical coordinates, and the three-dimensional prolate spheroidal coordinates. Maple proved to be very useful tool to perform the required transformations from one coordinate system to another, to simplify expressions, to manipulate the mathematical expressions, and to plot the solutions. The techniques used in this paper apply to other coordinate systems such as the parabolic cylindrical coordinates, conical coordinates, parabolic coordinates, oblate spheroidal coordinates, ellipsoidal coordinates, bispherical coordinates, and toroidal coordinates. The boundaries involved in solving the PDE provide us with the clue of the coordinate system to be used. Transforming the PDE to one of these coordinates makes the solution more tractable. For example, we saw that for the case of boundaries consisting of two circles, the most appropriate coordinate system to use is the bipolar coordinates. The aforementioned coordinates are the most commonly used because they are separable coordinates. In other words, the method of separation of variables can be used to solve the PDE. However, one needs to be aware that there are other coordinate systems such as the hyperbolodoidal, the exponential, and the three-dimensional bipolar coordinates, which are not considered as separable coordinates, and they are currently the focus of our endeavors.

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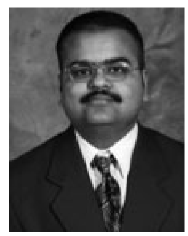

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