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A boundary element method for the Dirichlet eigenvalue problem of the Laplace operator

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Abstract The solution of eigenvalue problems for partial differential operators by using boundary integral equation methods usually involves some Newton potentials which may be resolved by using a multiple reciprocity approach. Here we propose an alternative approach which is in some sense equivalent to the above. Instead of a linear eigenvalue problem for the partial differential operator we consider a nonlinear eigenvalue problem for an associated boundary integral operator. This nonlinear eigenvalue problem can be solved by using some appropriate iterative scheme, here we will consider a Newton scheme. We will discuss the convergence and the boundary element discretization of this algorithm, and give some numerical results.

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1 Introduction

As a model problem we consider the interior Dirichlet eigenvalue problem of the Laplace operator,

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x) \quad \text{for } x \in \Omega, \quad u_\lambda(x) = 0 \quad \text{for } x \in \Gamma = \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. The variational formulation of (1.1) reads as follows: find $u_\lambda \in H_0^1(\Omega)$ such that

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$$\int_\Omega \nabla u_\lambda(x) \cdot \nabla v(x) dx = \lambda \int_\Omega u_\lambda(x) v(x) dx \tag{1.2}$$

is satisfied for all $v \in H_0^1(\Omega)$. It is well known that the eigenvalues are countable, and that there is a set of eigenfunctions $\{u_{\lambda_k}\}$ which forms a complete orthonormal system in $L_2(\Omega)$ and in $H_0^1(\Omega)$, respectively. Moreover, all eigenvalues λ_k have a finite multiplicity, and we have $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ as well as $\lambda_k \to \infty$ as $k \to \infty$.

A finite element approximation of the variational formulation (1.2) results in the linear algebraic eigenvalue problem

$$K_h \underline{u}_{\lambda_k} = \lambda_k M_h \underline{u}_{\lambda_k} \tag{1.3}$$

where K_h is the finite element stiffness matrix, and M_h is the related mass matrix. For a numerical analysis of this approach, and for appropriate eigenvalue solvers for (1.3), see, for example, [2,6,12,19].

Instead of a finite element approach, which always requires a discretization of the computational domain Ω, we will use boundary integral formulations and related boundary element methods [20,23] to solve the eigenvalue problem (1.1). Then, only a discretization of the boundary $\Gamma = \partial\Omega$ is required, and when using fast boundary element methods [17] the computational complexity is lower than in a finite element approach.

Considering the eigenvalue problem (1.1) as a Poisson equation with a certain right hand side λu_λ, we obtain a boundary–domain integral formulation [8,11] to be solved. By using the so-called Multiple Reciprocity Method (MRM) [8] it is possible to approximate the volume integrals by some boundary integrals. Then, a polynomial eigenvalue problem has to be solved. Since our approach is in some sense equivalent with the latter, we first describe the multiple reciprocity method.

By using the fundamental solution of the Laplace operator,

$$U^*(x, y) = \frac{1}{4\pi} \frac{1}{|x - y|} \quad \text{for } x, y \in \mathbb{R}^3,$$

the solution of the eigenvalue problem (1.1) is given by the representation formula

$$u_\lambda(x) = \int_\Gamma U^*(x, y) t_\lambda(y) ds_y + \lambda \int_\Omega U^*(x, y) u_\lambda(y) dy \quad \text{for } x \in \Omega \tag{1.4}$$

where $t_\lambda(x) = n_x \cdot \nabla u_\lambda(x)$, $x \in \Gamma$, is the associated normal derivative of the eigensolution u_λ. The basic idea of the multiple reciprocity method is to rewrite the volume integral in the representation formula (1.4) by using integration by parts recursively. The fundamental solution of the Laplace operator can be written as

$$U^*(x, y) = \frac{1}{4\pi} \frac{1}{|x-y|} = \Delta_y \left(\frac{1}{8\pi} |x - y| \right) = \Delta_y U_1^*(x, y), \quad U_1^*(x, y) = \frac{1}{8\pi} |x - y|.$$

Hence, by using Green's second formula, we obtain

$$\begin{aligned}\int_\Omega U^*(x,y)u_\lambda(y)dy &= \int_\Omega \Delta_y U_1^*(x,y)u_\lambda(y)dy \\ &= \int_\Omega U_1^*(x,y)\Delta_y u_\lambda(y)dy - \int_\Gamma U_1^*(x,y)\frac{\partial}{\partial n_y}u_\lambda(y)ds_y \\ &\quad + \int_\Gamma u_\lambda(y)\frac{\partial}{\partial n_y}U_1^*(x,y)ds_y \\ &= -\lambda\int_\Omega U_1^*(x,y)u_\lambda(y)dy - \int_\Gamma U_1^*(x,y)t_\lambda(y)ds_y\end{aligned}$$

when u_λ is a solution of the eigenvalue problem (1.1). The representation formula (1.4) is therefore equivalent to

$$u_\lambda(x) = \int_\Gamma \left[U^*(x,y) - \lambda U_1^*(x,y)\right]t_\lambda(y)ds_y - \lambda^2\int_\Omega U_1^*(x,y)u_\lambda(y)dy, \quad x \in \Omega.$$

By rewriting the volume integral in the same way as above recursively, we get for $x \in \Omega$ the representation formula

$$u_\lambda(x) = \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\left[\sum_{k=0}^{n}(-1)^k\lambda^k\frac{1}{(2k)!}|x-y|^{2k}\right]t_\lambda(y)ds_y + R_n(x,u_\lambda,\lambda) \quad (1.5)$$

where the remainder $R_n(x,u_\lambda,\lambda)$ is given by

$$R_n(x,u_\lambda,\lambda) = (-1)^n\lambda^{n+1}\frac{1}{(2n)!}\frac{1}{4\pi}\int_\Omega |x-y|^{2n-1}u_\lambda(y)dy.$$

In the approach of the multiple reciprocity method (MRM) the remainder R_n is neglected [7,8]. Then, due to the homogeneous Dirichlet boundary condition in (1.1), the polynomial eigenvalue problem

$$\frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\left[\sum_{k=0}^{n}(-1)^k\widetilde{\lambda}^k\frac{1}{(2k)!}|x-y|^{2k}\right]\widetilde{t}_\lambda(y)ds_y = 0 \quad \text{for } x \in \Gamma \quad (1.6)$$

is to be solved. In particular when taking the limit for $n \to \infty$ this results in the nonlinear eigenvalue problem

$$\frac{1}{4\pi}\int_\Gamma \frac{\cos(\sqrt{\lambda}|x-y|)}{|x-y|}t_\lambda(y)ds_y = 0 \quad \text{for } x \in \Gamma. \quad (1.7)$$

Let us now describe an alternative approach to derive the nonlinear eigenvalue problem (1.7). For $\lambda = \kappa^2 > 0$ we can write the interior Dirichlet eigenvalue problem (1.1) as a Helmholtz equation with homogeneous Dirichlet boundary conditions,

$$-\Delta u_\kappa(x) - \kappa^2 u_\kappa(x) = 0 \quad \text{for } x \in \Omega, \quad u_\kappa(x) = 0 \quad \text{for } x \in \Gamma. \tag{1.8}$$

Solutions of the boundary value problem (1.8) are given by the representation formula

$$u_\kappa(x) = \int_\Gamma U_\kappa^*(x, y) t(y) ds_y \quad \text{for } x \in \Omega \tag{1.9}$$

where the fundamental solution of the Helmholtz operator is given by

$$U_\kappa^*(x, y) = \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|} \quad \text{for } x, y \in \mathbb{R}^3. \tag{1.10}$$

Applying the interior trace operator to the representation formula (1.9), we obtain a boundary integral equation to find wave numbers $\kappa \in \mathbb{R}_+$ and related nontrivial eigensolutions $t \in H^{-1/2}(\Gamma)$ such that

$$\int_\Gamma U_\kappa^*(x, y) t(y) ds_y = 0 \quad \text{for } x \in \Gamma. \tag{1.11}$$

Since the eigenfunctions u_λ of the eigenvalue problem (1.1) and therefore the solutions u_κ of the boundary value problem (1.8) are real valued, instead of (1.11) we have to find nontrivial solutions $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ satisfying

$$\frac{1}{4\pi} \int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y) ds_y = 0 \quad \text{for } x \in \Gamma \tag{1.12}$$

and

$$\frac{1}{4\pi} \int_\Gamma \frac{\sin(\kappa|x-y|)}{|x-y|} t(y) ds_y = 0 \quad \text{for } x \in \Gamma. \tag{1.13}$$

Note that (1.12) corresponds to (1.7).

Summarizing the above we conclude that for any solution $(u_\lambda, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ of the eigenvalue problem (1.1) the nonlinear eigenvalue problem (1.12) is satisfied where the wave number is $\kappa = \sqrt{\lambda}$ and $t(x) = n_x \cdot \nabla u_\lambda(x)$, $x \in \Gamma$, is the corresponding normal derivative. On the other hand, if $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ is a solution of the nonlinear eigenvalue problem (1.12), the function

$$u_\kappa(x) = \frac{1}{4\pi} \int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y) ds_y \quad \text{for } x \in \Omega$$

solves the eigenvalue problem (1.1). Hence, the linear eigenvalue problem (1.1) is equivalent to the nonlinear eigenvalue problem (1.12). A boundary element approximation of the eigenvalue problem (1.11) would lead to a polynomial eigenvalue problem, see [10] for a collocation approach. Here we will first consider a Newton scheme to solve (1.12) and then we will apply a Galerkin boundary element discretization.

In Sect. 2, we consider an iterative solution approach for the nonlinear eigenvalue problem (1.12), which is an analogon of the inverse iteration for linear and for nonlinear matrix eigenvalue problems, see, e.g., [16,18,24]. In fact, we will apply a Newton scheme to solve the nonlinear equation where we introduce an appropriate scaling condition in $H^{-1/2}(\Gamma)$. In particular we will prove the invertibility of the related Fréchet derivative. However, our theoretical approach is restricted to simple eigenvalues only. A Galerkin boundary element method to solve the nonlinear eigenvalue problem is formulated and analyzed in Sect. 3, where we also prove optimal convergence for the approximate solutions. Numerical results given in Sect. 4 confirm not only the theoretical results, the experiments indicate that our approach also works for multiple eigenvalues.

2 Application of Newton's method

The nonlinear eigenvalue problem (1.12) can be written as

$$(V_\kappa t)(x) = \frac{1}{4\pi} \int_\Gamma \frac{\cos(\kappa|x-y|)}{|x-y|} t(y) ds_y = 0 \quad \text{for } x \in \Gamma \tag{2.1}$$

where for fixed κ the operator $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is linear and bounded, see, e.g., [15,20,23]. To normalize the eigensolutions $t \in H^{-1/2}(\Gamma)$ of (2.1) we introduce a scaling condition by using an equivalent norm in $H^{-1/2}(\Gamma)$,

$$\|t\|_V^2 = \langle Vt, t\rangle_\Gamma = \frac{1}{4\pi} \int_\Gamma t(x) \int_\Gamma \frac{1}{|x-y|} t(y) ds_y ds_x = 1, \tag{2.2}$$

where $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the single layer potential of the Laplace operator. Note that we have, see for example [23],

$$\langle Vt, t\rangle_\Gamma \geq c_1^V \|t\|^2_{H^{-1/2}(\Gamma)}, \quad \|Vt\|_{H^{1/2}(\Gamma)} \leq c_2^V \|t\|_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma).$$

Now we have to find solutions $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the nonlinear eigenvalue problem

$$F_1(t,\kappa) = (V_\kappa t)(x) = 0 \quad \text{for } x \in \Gamma, \quad F_2(t,\kappa) = \langle Vt, t\rangle_\Gamma - 1 = 0. \tag{2.3}$$

Let us define the operator $\boldsymbol{F} : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R}$ as

$$\boldsymbol{F}(t,\kappa) = \begin{pmatrix} F_1(t,\kappa) \\ F_2(t,\kappa) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{4\pi} \displaystyle\int_\Gamma \dfrac{\cos(\kappa|x-y|)}{|x-y|} t(y) ds_y \\ \langle Vt, t\rangle_\Gamma - 1 \end{pmatrix}. \tag{2.4}$$

Then, to obtain eigensolutions of the scaled eigenvalue problem (2.3) we have to find solutions $(t,\kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the nonlinear equation

$$\boldsymbol{F}(t,\kappa) = \boldsymbol{0} \tag{2.5}$$

which is to be solved by applying Newton's method. For the Fréchet derivative of $\boldsymbol{F}(t,\kappa)$ we obtain

$$\boldsymbol{F}'(t,\kappa) = \begin{pmatrix} V_\kappa & -A_\kappa t \\ 2\langle Vt, \cdot\rangle_\Gamma & 0 \end{pmatrix} : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R} \tag{2.6}$$

where

$$(A_\kappa t)(x) = \frac{1}{4\pi} \int_\Gamma \sin(\kappa|x-y|) t(y) ds_y \quad \text{for } x \in \Gamma.$$

As for the Laplace single layer potential, see for example [15,23], we conclude that the boundary integral operator $A_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded, i.e. for all $\kappa \in \mathbb{R}$ we have

$$\|A_\kappa t\|_{H^{1/2}(\Gamma)} \le c_{A_\kappa} \|t\|_{H^{-1/2}(\Gamma)} \quad \text{for all } t \in H^{-1/2}(\Gamma). \tag{2.7}$$

When applying a Newton scheme to find solutions $(t_*, \kappa_*) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ of the nonlinear equation (2.5), the new iterates $(t_{n+1}, \kappa_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ are the unique solutions of the linear operator equation

$$\boldsymbol{F}'(t_n,\kappa_n) \begin{pmatrix} t_{n+1} - t_n \\ \kappa_{n+1} - \kappa_n \end{pmatrix} + \boldsymbol{F}(t_n,\kappa_n) = \boldsymbol{0} \tag{2.8}$$

where for the previous iterates we assume $(t_n, \kappa_n) \in U_\varrho(t_*, \kappa_*)$ where $\varrho > 0$ is sufficiently small, i.e.

$$\|t_* - t_n\|^2_{H^{-1/2}(\Gamma)} + |\kappa_* - \kappa_n|^2 \le \varrho^2. \tag{2.9}$$

Note that the linearized equation (2.8) is equivalent to a saddle point problem to find $(t_{n+1}, \kappa_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n} t_{n+1}, w\rangle_\Gamma - \kappa_{n+1}\langle A_{\kappa_n} t_n, w\rangle_\Gamma &= -\kappa_n \langle A_{\kappa_n} t_n, w\rangle_\Gamma, \\ 2\langle V t_n, t_{n+1}\rangle_\Gamma &= \langle V t_n, t_n\rangle_\Gamma + 1 \end{aligned} \tag{2.10}$$

is satisfied for all $w \in H^{-1/2}(\Gamma)$.

The local convergence of the Newton method (2.8) to solve the nonlinear eigenvalue problem (2.3) is guaranteed if the Fréchet derivative $\boldsymbol{F}'(t_*, \kappa_*)$ is invertible for the solution (t_*, κ_*) of $\boldsymbol{F}(t, \kappa) = \boldsymbol{0}$. For this we first show that the Fréchet derivative $\boldsymbol{F}'(t, \kappa)$ satisfies a Gårdings inequality.

Lemma 2.1 *The Fréchet derivative $\boldsymbol{F}'(t, \kappa) : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R}$ is coercive for all $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$, i.e. there exists a compact operator*

$$\boldsymbol{C}(t, \kappa) : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma) \times \mathbb{R}$$

such that the following Gårdings inequality is satisfied,

$$\langle (\boldsymbol{F}'(t, \kappa) + \boldsymbol{C}(t, \kappa))(w, \alpha), (w, \alpha)\rangle \geq c \left[\|w\|^2_{H^{-1/2}(\Gamma)} + |\alpha|^2 \right] \tag{2.11}$$

for all $(w, \alpha) \in H^{-1/2}(\Gamma) \times \mathbb{R}$.

Proof For

$$\boldsymbol{C}(t, \kappa) \begin{pmatrix} w \\ \alpha \end{pmatrix} := \begin{pmatrix} C_1(w, \alpha) \\ C_2(w, \alpha) \end{pmatrix} = \begin{pmatrix} (V - V_\kappa) w + \alpha A_\kappa t \\ -2\langle w, t\rangle_\Gamma + \alpha \end{pmatrix}$$

we obtain

$$\boldsymbol{F}'(t, \kappa) + \boldsymbol{C}(t, \kappa) = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore Gårdings inequality (2.11) is fulfilled since the Laplace single layer potential V is $H^{-1/2}(\Gamma)$–elliptic. It remains to show that $\boldsymbol{C}(t, \kappa)$ is compact.

We first note that the operator $V - V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact, see, e.g., [20]. For a fixed $t \in H^{-1/2}(\Gamma)$ also the operator $\alpha A_\kappa t : \mathbb{R} \to H^{1/2}(\Gamma)$ is compact. Hence,

$$C_1(w, \alpha) = (V - V_\kappa) w + \alpha A_\kappa t, \quad C_1 : H^{-1/2}(\Gamma) \times \mathbb{R} \to H^{1/2}(\Gamma)$$

is compact. Finally, the operator $C_2 : H^{-1/2}(\Gamma) \times \mathbb{R} \to \mathbb{R}$ is compact since every linear and bounded operator which maps into a finite dimensional space is compact. □

Since for all $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ the Fréchet derivative $\boldsymbol{F}'(t, \kappa)$ satisfies a Gårdings inequality, it is sufficient to investigate the injectivity of $\boldsymbol{F}'(t_*, \kappa_*)$.

Lemma 2.2 *Let* $(t_*, \kappa_*) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ *be a solution of* $\boldsymbol{F}(t, \kappa) = \boldsymbol{0}$. *Assume*

(A1) κ_* *is a geometric simple eigenvalue of* $V_\kappa t = 0$,
(A2) $A_{\kappa_*} t_* \notin \mathcal{R}(V_{\kappa_*})$.

Then $\boldsymbol{F}'(t_*, \kappa_*)$ *is injective.*

Proof Consider

$$\boldsymbol{F}'(t_*, \kappa_*)(w, \alpha) = \begin{pmatrix} V_{\kappa_*} w - \alpha A_{\kappa_*} t_* \\ 2\langle V t_*, w\rangle_\Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.12}$$

for some $(w, \alpha) \in H^{-1/2}(\Gamma) \times \mathbb{R}$. In particular we have

$$V_{\kappa_*} w = \alpha A_{\kappa_*} t_*.$$

From the assumption (A2), $A_{\kappa_*} t_* \notin \mathcal{R}(V_{\kappa_*})$, it follows that $\alpha = 0$, and therefore $V_{\kappa_*} w = 0$. Hence we obtain that $w = \omega t_*$ for some $\omega \in \mathbb{R}$, since we assume in (A1) that κ_* is a simple eigenvalue of $V_\kappa t = 0$. Then we conclude, by using (2.12),

$$0 = 2\langle V t_*, w\rangle_\Gamma = 2\langle V t_*, \omega t_*\rangle_\Gamma = 2\omega\langle V t_*, t_*\rangle_\Gamma,$$

which implies that $\omega = 0$, since V is $H^{-1/2}(\Gamma)$-elliptic. Hence, $w \equiv 0$. Thus we have $(w, \alpha) = (0, 0)$, which proves the assertion. □

Remark 2.3 In the theory of nonlinear eigenvalue problems with holomorphic parameter dependence, see, e.g., [9,14], the assumption (A2) is equivalent to the property that the algebraic multiplicity of the eigenvalue is the same as the geometric multiplicity. For the Dirichlet Laplacian eigenvalue problem this assumption is fulfilled [1]. So far it is not proven if this property holds also for the equivalent eigenvalue problem of the associated boundary integral operator. However, numerical experiments indicate, by using results of Karma [9], that the assumption (A2) is fulfilled also for the boundary integral eigenvalue problem.

Corollary 2.4 *Let* (t_*, κ_*) *be a solution of* $\boldsymbol{F}(t, \kappa) = \boldsymbol{0}$ *and* (A1) *and* (A2) *be satisfied. Then* $\boldsymbol{F}'(t_*, \kappa_*)$ *is invertible. Since* $\boldsymbol{F}'(t, \kappa)$ *is continuous in* $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ *it follows that also* $\boldsymbol{F}'(t, \kappa)$ *is invertible for all* $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$ *for some* $\varrho > 0$.

Summarizing the above we now can formulate the main result of this section.

Theorem 2.5 *Let* (t_*, κ_*) *be a solution of* $\boldsymbol{F}(t, \kappa) = \boldsymbol{0}$ *and* (A1) *and* (A2) *be satisfied. Then* $\boldsymbol{F}'(t_*, \kappa_*)$ *is invertible and Newton's method converges for all initial values in a sufficiently small neighborhood* $U_\varrho(t_*, \kappa_*)$ *to* (t_*, κ_*).

Remark 2.6 For multiple eigenvalues κ_*, the Fréchet derivative $\boldsymbol{F}'(t_*, \kappa_*)$ is not invertible, because $\boldsymbol{F}'(t_*, \kappa_*)$ is not injective. Nevertheless, Newton's method may still converge [4,5]. The convergence rate may then be smaller and the convergence domain is not a small neighborhood of the solution but rather a restricted region which avoids the set on which $\boldsymbol{F}'$ is singular.

3 A boundary element method

Let us recall the variational formulation (2.10) to find $(t_{n+1}, \kappa_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n} t_{n+1}, w\rangle_\Gamma - \kappa_{n+1}\langle A_{\kappa_n} t_n, w\rangle_\Gamma &= -\kappa_n \langle A_{\kappa_n} t_n, w\rangle_\Gamma, \\ 2\langle V t_n, t_{n+1}\rangle_\Gamma &= \langle V t_n, t_n\rangle_\Gamma + 1 \end{aligned} \tag{3.1}$$

is satisfied for all $w \in H^{-1/2}(\Gamma)$. For a Galerkin discretization of (3.1) we first define trial spaces $S_h^0(\Gamma)$ of piecewise constant basis functions ψ_k which are defined with respect to a globally quasi–uniform boundary element mesh of mesh size h. Then the Galerkin discretization of (3.1) reads as follows: find $(t_{n+1,h}, \kappa_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\kappa_n} t_{n+1,h}, w_h\rangle_\Gamma - \kappa_{n+1,h}\langle A_{\kappa_n} t_n, w_h\rangle_\Gamma &= -\kappa_n \langle A_{\kappa_n} t_n, w_h\rangle_\Gamma, \\ 2\langle V t_n, t_{n+1,h}\rangle_\Gamma &= \langle V t_n, t_n\rangle_\Gamma + 1 \end{aligned} \tag{3.2}$$

is satisfied for all $w_h \in S_h^0(\Gamma)$.

Theorem 3.1 *Let (t_*, κ_*) be a solution of $\boldsymbol{F}(t, \kappa) = 0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(t_n, \kappa_n) \in U_\varrho(t_*, \kappa_*)$ be satisfied where ϱ is appropriately chosen as discussed in Corollary 2.4. Then, for a sufficiently small mesh size $h < h_0$, the Galerkin variational problem* (3.2) *has a unique solution $(t_{n+1,h}, \kappa_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ satisfying the error estimate*

$$\|t_{n+1} - t_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \kappa_{n+1,h}|^2 \le c \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)}. \tag{3.3}$$

Proof Since the Fréchet derivative $\boldsymbol{F}'(t_n, \kappa_n)$ is injective and satisfies a Gårding's inequality, the proof follows by applying standard arguments, see, e.g., [23]. □

In practical computations we have to replace in (3.2) $(t_n, \kappa_n) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ by previously computed approximations $(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) \in S_h^0(\Gamma) \times \mathbb{R}$. In particular we have to find $(\hat{t}_{n+1,h}, \hat{\kappa}_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\hat{\kappa}_{n,h}} \hat{t}_{n+1,h}, w_h\rangle_\Gamma - \hat{\kappa}_{n+1,h}\langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w_h\rangle_\Gamma &= -\hat{\kappa}_{n,h} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w_h\rangle_\Gamma, \\ 2\langle V \hat{t}_{n,h}, \hat{t}_{n+1,h}\rangle_\Gamma &= \langle V \hat{t}_{n,h}, \hat{t}_{n,h}\rangle_\Gamma + 1 \end{aligned} \tag{3.4}$$

is satisfied for all $w_h \in S_h^0(\Gamma)$. To analyze the perturbed variational problem (3.4) we also need to consider the continuous variational problem to find $(\hat{t}_{n+1}, \hat{\kappa}_{n+1}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_{\hat{\kappa}_{n,h}} \hat{t}_{n+1}, w\rangle_\Gamma - \hat{\kappa}_{n+1} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w\rangle_\Gamma &= -\hat{\kappa}_{n,h} \langle A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h}, w\rangle_\Gamma, \\ 2\langle V \hat{t}_{n,h}, \hat{t}_{n+1}\rangle_\Gamma &= \langle V \hat{t}_{n,h}, \hat{t}_{n,h}\rangle_\Gamma + 1 \end{aligned} \tag{3.5}$$

is satisfied for all $w \in H^{-1/2}(\Gamma)$. Note that (3.4) is the Galerkin discretization of (3.5).

Theorem 3.2 *Let (t_*, κ_*) be a solution of $\boldsymbol{F}(t, \kappa) = 0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) \in S_h^0(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$ be satisfied where ϱ is appropriately chosen as discussed in Corollary 2.4. Then, for a sufficiently small mesh size $h < h_0$, the Galerkin variational problem* (3.4) *has a unique solution $(\hat{t}_{n+1,h}, \hat{\kappa}_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ satisfying the error estimate*

$$\begin{aligned} &\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \\ &\quad \le c \left[\|t_n - \hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2 + \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)} \right] \end{aligned} \tag{3.6}$$

where the constant c depends on (t_*, κ_*), and on ϱ.

Proof Since (3.4) is the Galerkin formulation of the variational problem (3.5), the application of Theorem 3.1 gives the error estimate

$$\|\hat{t}_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\hat{\kappa}_{n+1} - \hat{\kappa}_{n+1,h}|^2 \le c \inf_{w_h \in S_h^0(\Gamma)} \|\hat{t}_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)}$$

and therefore, by using the triangle inequality twice,

$$\begin{aligned} &\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \\ &\quad \le 2(1+c) \Bigg[\|t_{n+1} - \hat{t}_{n+1}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1}|^2 \\ &\qquad + \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)} \Bigg] \end{aligned}$$

where the constant c is given as in (3.3).

Recall that the variational formulation (3.1) is equivalent to the Newton equation (2.8),

$$\boldsymbol{F}'(t_n, \kappa_n) \begin{pmatrix} t_{n+1} - t_n \\ \kappa_{n+1} - \kappa_n \end{pmatrix} + \boldsymbol{F}(t_n, \kappa_n) = \boldsymbol{0}.$$

In the same way we can write the variational problem (3.5) as

$$\boldsymbol{F}'(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) \begin{pmatrix} \hat{t}_{n+1} - \hat{t}_{n,h} \\ \hat{\kappa}_{n+1} - \hat{\kappa}_{n,h} \end{pmatrix} + \boldsymbol{F}(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) = \boldsymbol{0}.$$

Hence,

$$\begin{aligned}
\boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_{n+1}-\hat{t}_{n+1} \\ \kappa_{n+1}-\hat{\kappa}_{n+1}\end{pmatrix} &= \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_{n+1}-t_n \\ \kappa_{n+1}-\kappa_n\end{pmatrix} + \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_n-\hat{t}_{n,h} \\ \kappa_n-\hat{\kappa}_{n,h}\end{pmatrix} \\
&\quad + \left[\boldsymbol{F}'(t_n,\kappa_n)-\boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h}-\hat{t}_{n+1} \\ \hat{\kappa}_{n,h}-\hat{\kappa}_{n+1}\end{pmatrix} \\
&\quad + \boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\begin{pmatrix} \hat{t}_{n,h}-\hat{t}_{n+1} \\ \hat{\kappa}_{n,h}-\hat{\kappa}_{n+1}\end{pmatrix} \\
&= \boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n) + \boldsymbol{F}'(t_n,\kappa_n)\begin{pmatrix} t_n-\hat{t}_{n,h} \\ \kappa_n-\hat{\kappa}_{n,h}\end{pmatrix} \\
&\quad + \left[\boldsymbol{F}'(t_n,\kappa_n)-\boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h}-\hat{t}_{n+1} \\ \hat{\kappa}_{n,h}-\hat{\kappa}_{n+1}\end{pmatrix}.
\end{aligned}$$

Since $\boldsymbol{F}'(t_n,\kappa_n) : H^{-1/2}(\Gamma)\times\mathbb{R} \to H^{1/2}(\Gamma)\times\mathbb{R}$ is assumed to be invertible, see Corollary 2.4, we therefore have

$$\begin{aligned}
\begin{pmatrix} t_{n+1}-\hat{t}_{n+1} \\ \kappa_{n+1}-\hat{\kappa}_{n+1}\end{pmatrix} &= [\boldsymbol{F}'(t_n,\kappa_n)]^{-1}\left[\boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n)\right] + \begin{pmatrix} t_n-\hat{t}_{n,h} \\ \kappa_n-\hat{\kappa}_{n,h}\end{pmatrix} \\
&\quad + [\boldsymbol{F}'(t_n,\kappa_n)]^{-1}\left[\boldsymbol{F}'(t_n,\kappa_n)-\boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h}-\hat{t}_{n+1} \\ \hat{\kappa}_{n,h}-\hat{\kappa}_{n+1}\end{pmatrix}
\end{aligned}$$

and thus

$$\begin{aligned}
&\|(t_{n+1}-\hat{t}_{n+1},\kappa_{n+1}-\hat{\kappa}_{n+1})\|_{H^{-1/2}(\Gamma)\times\mathbb{R}} \le \|(t_n-\hat{t}_{n,h},\kappa_n-\hat{\kappa}_{n,h})\|_{H^{-1/2}(\Gamma)\times\mathbb{R}} \\
&\quad + c_{F'}\left\|\left[\boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n)\right]\right\|_{H^{1/2}(\Gamma)\times\mathbb{R}} \\
&\quad + c_{F'}\left\|\left[\boldsymbol{F}'(t_n,\kappa_n)-\boldsymbol{F}'(\hat{t}_{n,h},\hat{\kappa}_{n,h})\right]\begin{pmatrix} \hat{t}_{n,h}-\hat{t}_{n+1} \\ \hat{\kappa}_{n,h}-\hat{\kappa}_{n+1}\end{pmatrix}\right\|_{H^{1/2}(\Gamma)\times\mathbb{R}},
\end{aligned}$$

where $c_{F'}$ is the boundedness constant of $[\boldsymbol{F}'(t_n,\kappa_n)]^{-1}$, and where $c_{F'}$ depends only on t_*, κ_*, and on ρ. The second term can be written by using the definition (2.4) of $\boldsymbol{F}$ as

$$\begin{aligned}
&\left\|\left[\boldsymbol{F}(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n,\kappa_n)\right]\right\|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \\
&\quad = \|F_1(\hat{t}_{h,h},\hat{\kappa}_{n,h}) - F_1(t_n,\kappa_n)\|^2_{H^{1/2}(\Gamma)} + |F_2(\hat{t}_{n,h},\hat{\kappa}_{n,h}) - F_2(t_n,\kappa_n)|^2 \\
&\quad = \|V_{\hat{\kappa}_{n,h}}\hat{t}_{n,h} - V_{\kappa_n}t_n\|^2_{H^{1/2}(\Gamma)} + |\langle V\hat{t}_{n,h},\hat{t}_{n,h}\rangle_\Gamma - \langle Vt_n,t_n\rangle_\Gamma|^2.
\end{aligned}$$

With the mean value theorem we get

$$
\begin{aligned}
(V_{\kappa_n} - V_{\hat{\kappa}_{n,h}})t_n(x) &= \frac{1}{4\pi} \int_\Gamma \frac{\cos(\kappa_n|x-y|) - \cos(\hat{\kappa}_{n,h}|x-y|)}{|x-y|} t_n(y) ds_y \\
&= (\hat{\kappa}_{n,h} - \kappa_n) \frac{1}{4\pi} \int_\Gamma \sin(\kappa_n^*|x-y|)\, t_n(y) ds_y \\
&= (\hat{\kappa}_{n,h} - \kappa_n)(A_{\kappa_n^*} t_n)(x)
\end{aligned}
$$

and therefore we obtain, by using (2.7),

$$
\begin{aligned}
\| V_{\hat{\kappa}_{n,h}} \hat{t}_{n,h} - V_{\kappa_n} t_n \|_{H^{1/2}(\Gamma)} &= \| V_{\hat{\kappa}_{n,h}} (\hat{t}_{n,h} - t_n) - (V_{\kappa_n} - V_{\hat{\kappa}_{n,h}}) t_n \|_{H^{1/2}(\Gamma)} \\
&\le \| V_{\hat{\kappa}_{n,h}} (\hat{t}_{n,h} - t_n) \|_{H^{1/2}(\Gamma)} + |\hat{\kappa}_{n,h} - \kappa_n| \, \| A_{\kappa_n^*} t_n \|_{H^{1/2}(\Gamma)} \\
&\le c_{V_{\hat{\kappa}_{n,h}}} \| t_n - \hat{t}_{n,h} \|_{H^{-1/2}(\Gamma)} + c_{\kappa_n,\rho} \, |\hat{\kappa}_{n,h} - \kappa_n| \, \| t_n \|_{H^{-1/2}(\Gamma)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|\langle V \hat{t}_{n,h}, \hat{t}_{n,h} \rangle_\Gamma - \langle V t_n, t_n \rangle_\Gamma| &= |\langle V(\hat{t}_{n,h} - t_n), \hat{t}_{n,h} \rangle_\Gamma + \langle V t_n, \hat{t}_{n,h} - t_n \rangle_\Gamma| \\
&\le c_2^V \, \| t_n - \hat{t}_{n,h} \|_{H^{-1/2}(\Gamma)} \left[\| \hat{t}_{n,h} \|_{H^{-1/2}(\Gamma)} + \| t_n \|_{H^{-1/2}(\Gamma)} \right].
\end{aligned}
$$

Hence we conclude

$$
\left\| \left[\boldsymbol{F}(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) - \boldsymbol{F}(t_n, \kappa_n) \right] \right\|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \le c \left[\left\| t_n - \hat{t}_{n,h} \right\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2 \right],
$$

where the constant c depends on (t_*, κ_*), and on ϱ.

Finally, for $(t, \kappa) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ we have

$$
\begin{aligned}
&\left\| \left[\boldsymbol{F}'(t_n, \kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h}, \hat{\kappa}_{n,h}) \right] \begin{pmatrix} t \\ \kappa \end{pmatrix} \right\|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \\
&\quad = \left\| (V_{\kappa_n} - V_{\hat{\kappa}_{n,h}}) t + \kappa (A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h} - A_{\kappa_n} t_n) \right\|^2_{H^{1/2}(\Gamma)} + 4 |\langle V(t_n - \hat{t}_{n,h}), t \rangle_\Gamma|^2,
\end{aligned}
$$

where we can bound

$$
\begin{aligned}
\| (V_{\kappa_n} - V_{\hat{\kappa}_{n,h}}) t \|_{H^{1/2}(\Gamma)} &= |\hat{\kappa}_{n,h} - \kappa_n| \, \| A_{\kappa_n^*} t \|_{H^{1/2}(\Gamma)} \le c_{A_{\kappa_n^*}} \, |\hat{\kappa}_{n,h} - \kappa_n| \, \| t \|_{H^{-1/2}(\Gamma)}, \\
|\langle V(t_n - \hat{t}_{n,h}), t \rangle_\Gamma| &\le c_2^V \, \| t \|_{H^{-1/2}(\Gamma)} \| t_n - \hat{t}_{n,h} \|_{H^{-1/2}(\Gamma)}.
\end{aligned}
$$

For the remaining term we first consider

$$
\begin{aligned}
\| A_{\hat{\kappa}_{n,h}} \hat{t}_{n,h} - A_{\kappa_n} t_n \|_{H^{1/2}(\Gamma)} &\le \| A_{\hat{\kappa}_{n,h}} (\hat{t}_{n,h} - t_n) \|_{H^{1/2}(\Gamma)} + \| (A_{\hat{\kappa}_{n,h}} - A_{\kappa_n}) t_n \|_{H^{1/2}(\Gamma)} \\
&\le c_{A_{\hat{\kappa}_{n,h}}} \, \| \hat{t}_{n,h} - t_n \|_{H^{-1/2}(\Gamma)} + \| (A_{\hat{\kappa}_{n,h}} - A_{\kappa_n}) t_n \|_{H^{1/2}(\Gamma)}.
\end{aligned}
$$

Again by using the mean value theorem with respect to κ_n we have

$$\begin{aligned}(A_{\hat{\kappa}_{n,h}} - A_{\kappa_n})t_n(x) &= \frac{1}{4\pi}\int_\Gamma \left[\sin(\hat{\kappa}_{n,h}|x-y|) - \sin(\kappa_n|x-y|)\right] t_n(y)ds_y \\ &= (\hat{\kappa}_{n,h} - \kappa_n)\frac{1}{4\pi}\int_\Gamma |x-y|\cos(\tilde{\kappa}_n|x-y|)t_n(y)ds_y \\ &= (\hat{\kappa}_{n,h} - \kappa_n)(B_{\tilde{\kappa}_n}t_n)(x)\end{aligned}$$

with $\tilde{\kappa}_n \in (\hat{\kappa}_{n,h}, \kappa_n)$, and where the operator $B_{\tilde{\kappa}_n} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded. This follows as for the Laplace single layer potential, we skip the details. Hence we conclude

$$\begin{aligned}&\left\| \left[\boldsymbol{F}'(t_n, \kappa_n) - \boldsymbol{F}'(\hat{t}_{n,h}, \hat{\kappa}_{n,h})\right] \begin{pmatrix} \hat{t}_{n,h} - \hat{t}_{n+1} \\ \hat{\kappa}_{n,h} - \hat{\kappa}_{n+1} \end{pmatrix} \right\|^2_{H^{1/2}(\Gamma)\times\mathbb{R}} \\ &\le c\left[\|t_n - \hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2\right]\end{aligned}$$

where the constant c depends on (t_*, κ_*), and on ϱ. Therefore we can conclude

$$\|t_{n+1} - \hat{t}_{n+1}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1}|^2 \le c\left[\|t_n - \hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2\right],$$

from which the assertion follows. □

Corollary 3.3 *Let (t_*, κ_*) be a solution of $\boldsymbol{F}(t, \kappa) = 0$ and let the assumptions* (A1) *and* (A2) *be satisfied. Let $(t_{0,h}, \kappa_{0,h}) \in S_h^0(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$ be satisfied where ϱ is appropriately chosen as discussed in Corollary 2.4. Then, for a sufficiently small mesh size $h < h_0$, the Galerkin variational problem* (3.4) *has a unique solution $(\hat{t}_{n+1,h}, \hat{\kappa}_{n+1,h}) \in S_h^0(\Gamma) \times \mathbb{R}$ satisfying the error estimate*

$$\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \le c\left[\varrho^4 + h^3\right] \tag{3.7}$$

when assuming $t_* \in H^1_{pw}(\Gamma)$. Note that the constant c depends on (t_*, κ_*), n, and on ϱ.

Proof Let us define the error e_n as

$$e_n := \|t_n - \hat{t}_{n,h}\|^2_{H^{-1/2}(\Gamma)} + |\kappa_n - \hat{\kappa}_{n,h}|^2.$$

Then, by using the error estimate (3.6), we conclude

$$\begin{aligned} e_{n+1} &\le c \left[e_n + \inf_{w_h \in S_h^0(\Gamma)} \|t_{n+1} - w_h\|^2_{H^{-1/2}(\Gamma)} \right] \\ &\le c \left[e_n + \|t_* - t_{n+1}\|^2_{H^{-1/2}(\Gamma)} + \inf_{w_h \in S_h^0(\Gamma)} \|t_* - w_h\|^2_{H^{-1/2}(\Gamma)} \right] \\ &\le c \left[e_n + c_1 \varrho^4 + c_2 h^3 \|t_*\|^2_{H^1_{\mathrm{pw}}} \right] \\ &\le \widetilde{c} \left[e_n + \varrho^4 + h^3 \right] \end{aligned}$$

when assuming $t_* \in H^1_{\mathrm{pw}}(\Gamma)$. Since for $n = 0$, we set $(\hat{t}_0, \hat{\kappa}_0) = (t_0, \kappa_0) \in H^{-1/2}(\Gamma) \times \mathbb{R}$, and we conclude

$$e_1 \le c \left[\varrho^4 + h^3 \right]$$

when assuming $(t_0, \kappa_0) \in S_h^0(\Gamma) \times \mathbb{R} \cap U_\varrho(t_*, \kappa_*)$. Now the assertion follows by induction. □

When using the Aubin–Nitsche trick, see for example [23], it is possible to derive error estimates in Sobolev spaces with lower Sobolev index. In particular we obtain the error estimate

$$\|t_{n+1} - \hat{t}_{n+1,h}\|^2_{H^{-2}(\Gamma)} + |\kappa_{n+1} - \hat{\kappa}_{n+1,h}|^2 \le c \left[\varrho^4 + h^6 \right] \tag{3.8}$$

when assuming $t_* \in H^1_{\mathrm{pw}}(\Gamma)$. Hence we can expect a cubic convergence rate for the eigenvalues,

$$|\kappa_{n+1} - \hat{\kappa}_{n+1,h}| \le c \left[\varrho^4 + h^6 \right]^{1/2} = \mathcal{O}(h^3). \tag{3.9}$$

4 Numerical results

In this section, we present some numerical results to investigate the behavior of the nonlinear boundary element approach as presented in this paper. In addition, we compare our approximations with the results of a finite element discretization using linear tetrahedral elements. The FEM matrices are generated by Netgen/NGSolve [21] and as eigenvalue solver we use LOBPCG [13] with a two-level preconditioner.

As a model problem we consider the interior Dirichlet eigenvalue problem (1.1) where the domain $\Omega = (0, \frac{1}{2})^3$ is a cube. Hence the eigenvalues are given by

$$\lambda_k = 4\pi^2 \left[k_1^2 + k_2^2 + k_3^2 \right]$$

Table 1 BEM approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue

L	h	N	$\kappa_{1,N}^{\mathrm{BEM}}$	$\lvert\kappa_1 - \kappa_{1,N}^{\mathrm{BEM}}\rvert$	eoc	it
2	2^{-3}	384	10.8768	5.986e-03	–	7
3	2^{-4}	1536	10.8821	6.962e-04	3.1	7
4	2^{-5}	6144	10.8827	8.619e-05	3.0	8

and the associated eigenfunctions are

$$u_k(x) = (\sin 2\pi k_1 x_1)(\sin 2\pi k_2 x_2)(\sin 2\pi k_3 x_3).$$

It turns out that the first eigenvalue ($k_1 = k_2 = k_3 = 1$)

$$\lambda_1 = 12\pi^2, \quad \kappa_1 = 2\sqrt{3}\pi$$

is simple, while the second eigenvalue ($k_1 = 2, k_2 = k_3 = 1$)

$$\lambda_2 = 24\pi^2, \quad \kappa_2 = 2\sqrt{6}\pi$$

is multiple.

For the boundary element discretization the boundary $\Gamma = \partial\Omega$ is decomposed into N uniform triangular boundary elements with mesh size h. The numerical results to approximate the simple eigenvalue $\kappa_1 = \sqrt{\lambda_1}$ are given in Table 1 where it denotes the number of Newton iterations which are performed. As initial values we choose for $\kappa = 3.5$ and for the eigenvector a random vector.Next we consider the case of a multiple eigenvalue, the results to approximate $\kappa_2 = \sqrt{\lambda_2}$ are given in Table 2. As initial values we choose for $\kappa = 13$ and for the eigenvectors random vectors. Note that the numerical results for the BEM approxiamtions of the eigenvalues confirm the theoretical error estimate in (3.9). The numerical results of the finite element discretization to approximate the first and second eigenvalue are listed in Tables 3 and 4, where h is the mesh size of the uniform discretization, and M the number of interior nodes which is equal to the number of degrees of freedom.

The numerical results reflect the different convergence rates of both methods. The convergence rate for the finite element approximations is quadratic compared to the cubic order of the boundary element approximations. Note that the BEM approximations of the coarsest mesh on level $L = 2$ with matrix size $N = 384$ are approximately the same as the FEM approximations of the finest mesh on level $L = 6$ with matrix size $N = 250{,}047$. The disadvantage of the boundary element approach is that in one run of the algorithm just one eigenpair is approximated and that sufficiently good initial values are needed for the convergence to a desired eigenpair.

As in other boundary element approaches [3], the problem of the so–called spurious eigenvalues occurs. These spurious eigenvalues can be filtered out with an a posteriori error control by using the complex valued fundamental solution (1.10) for

Table 2 BEM approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue

L	h	N	$\kappa_{2_1,N}^{\mathrm{BEM}}$	$\lvert\kappa_2 - \kappa_{2_1,N}^{\mathrm{BEM}}\rvert$	eoc	it
2	2^{-3}	384	15.373851	1.7e-02	–	8
3	2^{-4}	1536	15.3887048	1.9e-03	3.1	8
4	2^{-5}	6144	15.39037160	2.3e-04	3.1	9
L	h	N	$\kappa_{2_2,N}^{\mathrm{BEM}}$	$\lvert\kappa_2 - \kappa_{2_2,N}^{\mathrm{BEM}}\rvert$	eoc	it
2	2^{-3}	384	15.37364	1.7e-02	–	10
3	2^{-4}	1536	15.3887060	1.9e-03	3.1	10
4	2^{-5}	6144	15.39037171	2.3e-04	3.1	10
L	h	N	$\kappa_{2_3,N}^{\mathrm{BEM}}$	$\lvert\kappa_2 - \kappa_{2_3,N}^{\mathrm{BEM}}\rvert$	eoc	it
2	2^{-3}	384	15.373876	1.7e-02	–	9
3	2^{-4}	1536	15.3887071	1.9e-03	3.1	9
4	2^{-5}	6144	15.39037180	2.3e-04	3.1	9

Table 3 FEM approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue

L	h	M	$\kappa_{1,M}^{\mathrm{FEM}}$	$\lvert\kappa_1 - \kappa_{1,M}^{\mathrm{FEM}}\rvert$
3	2^{-4}	343	11.3693	4.9e-01
4	2^{-5}	3375	11.0038	1.2e-01
5	2^{-6}	29791	10.9132	3.0e-02
6	2^{-7}	250047	10.8903	7.6e-03

Table 4 FEM approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue

L	h	M	$\kappa_{2_1,M}^{\mathrm{FEM}}$	$\lvert\kappa_2 - \kappa_{2_1,M}^{\mathrm{FEM}}\rvert$	$\kappa_{2_2,M}^{\mathrm{FEM}}$	$\lvert\kappa_2 - \kappa_{2_2,M}^{\mathrm{FEM}}\rvert$	$\kappa_{2_3,M}^{\mathrm{FEM}}$	$\lvert\kappa_2 - \kappa_{2_3,M}^{\mathrm{FEM}}\rvert$
3	2^{-4}	343	16.27	8.8e-01	16.28	8.9e-01	17.59	2.2
4	2^{-5}	3375	15.60	2.1e-01	15.60	2.4e-01	16.12	7.3e-01
5	2^{-6}	29791	15.44	5.1e-02	15.44	5.3e-02	15.63	2.4e-01
6	2^{-7}	250047	15.40	1.3e-02	15.40	1.4e-02	15.47	8.0e-02

an eigensolution (t, κ),

$$\frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}t(y)ds_y = (V_\kappa t)(x) + i\frac{1}{4\pi}\int_\Gamma \frac{\sin(\kappa|x-y|)}{|x-y|}t(y)ds_y = 0.$$

Then the residuum

$$\frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa_h|x-y|}}{|x-y|}t_h(y)ds_y$$

for actual approximations of eigensolutions (t_h, κ_h) is significant smaller than for spurious eigensolutions, and the residuum tends to zero for actual approximations of the eigensolutions if h gets smaller. Note that no spurious eigenvalues occur when an analogous algorithm is used which is based on the complex valued fundamental solution (1.10). But then we have to use complex arithmetics so that the computational complexity is more expensive as for the real-valued version.

5 Conclusions

In this paper we have presented and analyzed a boundary element method for the solution of the interior Dirichlet eigenvalue problem for the Laplace operator. Hereby, the linear eigenvalue problem for the partial differential operator is transformed into a nonlinear eigenvalue problem for an associated boundary integral operator which is solved via a Newton iteration. The discretization by using a Galerkin boundary element method gives a cubic convergence of the approximated eigenvalues. When using fast boundary element methods [17] an almost optimal computational complexity can be obtained. For this, also efficient preconditioned iterative solution methods to solve the Galerkin equation (3.4) are mandatory. As already mentioned in Remark 2.5, a further analysis in the case of multiple eigenvalues is needed.

A crucial issue is to increase the robustness of the presented method. Techniques which use Rayleigh functionals [16,22] are theoretically applicable to the boundary integral eigenvalue problem. However, the required Rayleigh functionals are in our case just locally defined and therefore the convergence regions for the eigenvalues are still local. First numerical experiments where the safe-guarded iteration [16] is applied to the boundary integral eigenvalue problem indicate no larger convergence regions than for the proposed method. Therefore other techniques have to be considered and analyzed in order to increase the robustness.

Finally, we mention that the proposed approach can be used to solve the interior Neumann eigenvalue problem for the Laplace operator, and to solve related eigenvalue problems in linear elastostatics.

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