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Buckling, flutter and vibration analyses of beams by integral equation formulations

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Abstract

This paper presents a model for the investigation of buckling, flutter and vibration analyses of beams using the integral equation formulation. A mathematical formulation based on Euler–Bernoulli beam theory is presented for beams with variable sections on elastic foundations and subjected to lateral excitation, conservative and non-conservative loads. Using the boundary element method and radial basis functions, the equation of motion is reduced to an algebraic system related to internal and boundary unknowns. Eigenvalue problems related to buckling and vibrations are formulated and numerically solved. Buckling loads, natural frequencies and associated eigenmodes are computed. The corresponding slope, bending and shear forces can be directly obtained. The load-frequency dependence is investigated for various elastic foundations and the divergence critical loads are evidenced. Under non-conservative loads, a dynamic stability analysis is illustrated numerically based on the coalescence of eigenfrequencies. The flutter load and instability regions with respect to the elastic concentrated and distributed foundations are identified. Using the eigenmodes, numerically computed, non-linear vibrations of beams are investigated based on one mode analysis. The presented model is quite general and the obtained numerical results are in agreement with available data.

Keywords: Beam; Buckling; Flutter; Vibration; Integral equation; BEM; Radial basis functions

1. Introduction

Slender structural components such as beams and columns constitute basic parts of many complex engineering structures. Buckling, flutter and vibrations are the main forms of instability of these structures. The accurate prediction of static and dynamic critical loads and responses are very significant in practice and it relies on a better design of light weight structures which can be safely used in the pre-buckling and post buckling ranges. Amongst the numerical methods available for thin structure problems, the finite element method is undoubtedly the most versatile. The only problem with this method is that its formulation is quite laborious and it takes a large amount of computer storage. A powerful alternative method based on integral equations is the Boundary Element Method. One of the main reason of the rapid development of the BEM is a possibility of reduced dimensionality of the problem, which leads to a reduced set of equations and a smaller amount of data required for the computation. Using the fundamental solution

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corresponding to the exact solution of a part of the problem, the inappropriate terms are moved to the right-hand side of the governing equation and considered as a fictitious source density. For buckling and vibrations of beams under elastic foundations, domain integrals are necessary in the formulation. Thus, the main advantage of dimensionality reduction is eliminated. But, the use of dual reciprocity method (DRM), introduced by Nardini and Brebbia [1], permits to combine the dimensionality reduction advantage with a simple fundamental solution and to formulate the problem on boundary unknowns only. A comprehensive literature review of the DRM and multiple reciprocity method (MRM) as applied to elastodynamics can be found in the review paper of Beskos [2]. Details and applications to various engineering problems are clearly presented in the book of Partridge et al. [3]. The eigenvalue analyses of Helmhotz equation using the DRM and MRM have been discussed by Kamiya et al. [4]. Combining the MRM and singular value decomposition method, the rod vibration problem has been analyzed by Chang et al. [5]. The solution of plate bending problems by MRM has been formulated by Sladek and Sladek [6]. Using DRM and differential quadrature method, the longitudinal vibrations of plates and membranes are investigated by Tanaka and Chen [7]. For bending problems of inhomogeneous Euler–Bernoulli beams, an investigation is carried out by Rong et al. [8]. Based on Timoshenko beams theory and a quadrature method, the dynamic behaviors of beams have been analyzed by Schanz and Antes [9]. An extension to beams with arbitrary cross-section has been developed by Sapountzakis [10] and to the non-linear dynamic analysis of beams with variable cross-section has been done by Katsikadelis and Tsiatas [11]. Vibrations of beams with variable sections using BEM and radial basis functions are analyzed in [12]. However, no general modeling based on integral equation formulations concerning buckling, vibration and flutter analyses of beams has been established in previous works. This paper intends to provide a compact formulation for these behaviors and to numerically investigate the static and dynamic instabilities of beams.

The force acting on a beam can be divided into conservative and non-conservative forces. Generally, the instability of a beam under a conservative force is characterized by the divergence which occurs when one of the natural frequencies falls to zero. This critical solution corresponds to buckling load and can be directly investigated with static analysis. The buckling analysis of beams has been studied by many authors and is treated in almost any textbook on mechanics of solids [13]. For a non-conservative system, it has been shown by Ziegler [14] that the usual Euler method and minimum potential energy methods (static methods) are inadequate to predict their instability and that a dynamic method must be employed. The instability analysis of beams under non-conservative force characterized by the flutter which occurs when two of the natural frequencies coincide became complex conjugate can be done only dynamically. The presence of non-conservative loads makes the system of equations mathematically non-self-adjoint and the corresponding eigenvalue problem is dictated by a non-symmetric matrix. This problem has been studied by many authors and various numerical procedures have been set up for its solution. A comprehensive discussion of this subject based on analytical procedures can be found in the books [15–17]. More recently, the flutter and divergence instability of beams and plates subjected to non-conservative loads are analyzed by Zuo and Schreyer [18] by solving the resulting characteristic equations. Similarly, the flutter and internal damping effects on the dynamic stability of rods with intermediate spring support and with relocatable lamped mass under follower loads have been largely investigated by Lee [19–21]. The influence of the tangency coefficient of follower load and the elastically restrained boundary conditions on the elastic instability of beams has been discussed by Lee and Hsu [22]. Enhancing flutter and buckling capacity of beams by using piezoelectric layer is presented by Wang and Quek [23]. Based on the finite element method, the stability and instability of cantilever elastic beams subjected to a follower force have been investigated by Gasparini et al. [24] and by Ryu and Sugiyama [25]. Using the static approach, the divergence instability of thin walled beams in pre-buckling and post-buckling ranges has been recently analyzed by Mohri et al. [26]. Based on the Ramm finite elements and a perturbation method, the load-frequency dependence has been investigated for arches and shells with large rotations by Boutyour et al. [27]. The critical loads and the stability and instability regions are evidenced for largely deformed shells with the smaller eigenfrequencies. The divergence and flutter instability are generally analyzed by analytical method or by finite element methods. To the best knowledge of the authors, there is no available compact formulation and results based on the integral equation formulation for buckling, flutter and vibration analyses of thin structures.

In this paper, a mathematical modeling based on the integral equations for buckling, flutter and vibration analyses of beams is presented. The Euler–Bernoulli beam theory is used and the governing equation is formulated for beams on elastic foundation and subjected to conservative and non-conservative loads. The radial basis functions are used and the required matrices are explicitly presented for various boundary conditions and loads. The eigenvalue problems corresponding to buckling, vibration and flutter are explicitly formulated. The displacement, slope, bending and shear forces can be directly obtained. The buckling and vibration modes and the load-frequency dependence are presented for various boundary conditions and elastic foundations. For axial tangential follower forces, the flutter load is evidenced for uniform and concentrated elastic foundations based on the coalescence criterion. The flutter load variations with respect to the position and amplitude of the foundation and the flutter zone are investigated. The non-linear vibration analyses of beams based on one mode numerically computed are elaborated. Some benchmark tests are investigated ratifying the effectiveness of the presented methodological approach.

2. Basic beam equations

Let us consider a slender beam of length L with a variable cross-section. The Euler–Bernoulli beam formulation based on the assumption that both shear deformation and rotational inertia of the cross-section are negligible is used. The axial displacement will be neglected and the equation of motion is formulated using the transverse displacement only. Based on the integral equation formulation, buckling, flutter and transverse vibrations of beams will be formulated. The governing partial differential equation of motion of beams on elastic foundations and subjected to axial compression and lateral excitation (Fig. 1) is formulated by

$$\frac{\partial^2}{\partial z^2}\left(EI(z)\frac{\partial^2 V(z,t)}{\partial z^2}\right) + \rho(z)S(z)\frac{\partial^2 V(z,t)}{\partial t^2} + \lambda\frac{\partial^2 V(z,t)}{\partial z^2} + \kappa(z)V(z,t) = p(z,t) \quad (1)$$

where V is the transverse displacement, E, I, S and ρ are Young's modulus, inertia, the area and the mass density respectively. $\kappa(z)$ is the elastic foundation, λ is the axial compression, $p(z,t)$ is the lateral excitation and z is the axial coordinate. Assuming harmonic motion, the free vibration problem of axially loaded beam is given by

$$\frac{\partial^2}{\partial z^2}\left(EI(z)\frac{\partial^2 V}{\partial z^2}\right) - \rho S(z)\omega^2 V + \lambda\frac{\partial^2 V}{\partial z^2} + \kappa(z)V = 0 \quad (2)$$

For homogeneous beams with a variable section, the parameters E, I, S and ρ can be assumed in the following form:

$$EI(z) = E(0)I(0)K_1(z)$$
$$\rho S(z) = \rho(0)S(0)K_2(z) \quad (3)$$

where K_1 and K_2 are functions of the axial coordinate z. Using non-dimensional parameters, Eq. (2) can be read as

$$\frac{\partial^2}{\partial x^2}\left(K_1(x)\frac{\partial^2 W}{\partial x^2}\right) - \omega^{*2}K_2(x)W + \lambda^*\frac{\partial^2 W}{\partial x^2} + \kappa^*(x)W = 0 \quad (4)$$

where

$$\omega^{*2} = \rho\frac{S(0)L^4\omega^2}{EI(0)}, \quad \lambda^* = \lambda\frac{L^2}{EI(0)}, \quad \kappa^* = \kappa\frac{L^3}{EI(0)},$$
$$W = \frac{V}{R}, \quad R = \sqrt{\frac{I}{S}} \quad \text{and} \quad x = \frac{z}{L}$$

in which R is the radius of gyration of the beam and $0 \leqslant x \leqslant 1$.

Eq. (4) may be solved by the finite element method or analytically for standard boundary conditions. The aim of this paper is the development of an integral equation formulation for numerical solution of (4) and the investigation of the static and dynamic instability analyses of beams based on the resulting formulation.

3. Integral equation formulation

The fundamental solution of (4) is hard to be explicitly determined due to variable coefficients $K_1(x)$ and $K_2(x)$ even if only buckling ($\omega^* = 0$) or free vibration ($\lambda^* = 0$) problem is considered. For simplified cases, Bessel functions may be used but will lead to some numerical difficulties at standard integral equation formulation of solving the resulting boundary value problem [3–5]. As the domain integrals are inevitable due to the excitation and load, a simple fundamental solution will be used and the resulting domain integrals will be treated by the dual reciprocity method. Let us denote W^* the fundamental solution of the following problem:

$$\frac{\partial^2}{\partial x^2}\left(K_1(x)\frac{\partial^2 W^*(x,s)}{\partial x^2}\right) = \delta(x,s) \quad (5)$$

where δ is the Dirac function and s is the source point. This fundamental solution will be used and the differential Eq. (4) will be transformed into an integral equation. Following the boundary element method procedure

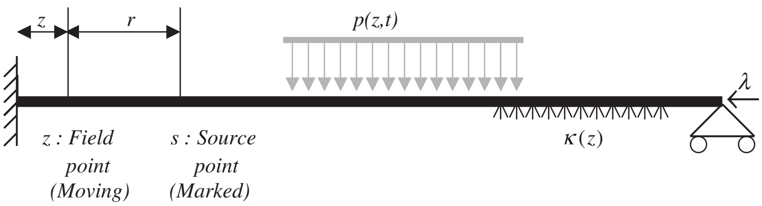

Fig. 1. C–S beam subjected to axial force λ, elastic foundation κ and lateral excitation $p(z,t)$.

[1–12], the resulting integral equation will be reduced to algebraic equation.

As well known in the bending problem of beams, the following variables have physical meanings and may be also known at boundaries:

$$\theta(x) = \frac{\partial W}{\partial x}, \quad M(x) = -K_1(x)\frac{\partial^2 W}{\partial x^2} \quad \text{and} \quad Q(x) = \frac{\partial M}{\partial x} \tag{6}$$

where $\theta(x)$ is the slope, $M(x)$ is the bending moment and $Q(x)$ is the shear force related to the derivatives of the deflection W. Multiplying Eq. (4) by W^* and integrating from 0 to 1, one obtains:

$$\begin{aligned}\int_0^1 \frac{\partial^2}{\partial x^2}\left(K_1(x)\frac{\partial^2 W}{\partial x^2}(x)\right)W^*(s,x)\mathrm{d}x &= \omega^{*2}\int_0^1 K_2(x)W(x)W^*(s,x)\mathrm{d}x \\ &- \lambda^*\int_0^1 \frac{\partial^2 W}{\partial x^2}(x)W^*(s,x)\mathrm{d}x \\ &- \int_0^1 \kappa^*(x)W(x)W^*(s,x)\mathrm{d}x\end{aligned} \tag{7}$$

Integrating by parts four times, the first term of (7) becomes

$$\begin{cases}\int_0^1 \frac{\partial^2}{\partial x^2}\left(K_1(x)\frac{\partial^2 W}{\partial x^2}(x)\right)W^*(s,x)\mathrm{d}x = W(s) + A(s) \\ A(s) = \left[-W^*(s,x)Q(x) + \frac{\partial W^*}{\partial x}(s,x)M(x)\right. \\ \qquad \left. + K_1\frac{\partial^2 W^*}{\partial x^2}(s,x)\theta(x) - \frac{\partial}{\partial x}\left(K_1\frac{\partial^2 W^*}{\partial x^2}\right)(s,x)W(x)\right]_0^1\end{cases} \tag{8}$$

In the right-hand side of Eq. (7), three domain integrals have to be evaluated. Making use of radial basis functions, thus avoiding the additional task of domain integration, these domain integrals are transformed into boundary values. Let us assume that for these integrals, the function $W(x)$ is assumed to be:

$$W(x) = \sum_{j=1}^{n+2} \alpha_j f_j(x) \tag{9}$$

where f_j are radial basis functions, 'n' is the number of interior points, α_j are unknown coefficients [3,4]. Given f_j defines two other functions g_j and h_j which satisfy the following equations:

$$\frac{\mathrm{d}^4 g_j}{\mathrm{d}x^4}(x) = f_j(x) \quad \text{and} \quad \frac{\mathrm{d}^4 h_j}{\mathrm{d}x^4}(x) = K_2(x)f_j(x) \tag{10}$$

More details about W^*, f_j, g_j and h_j, used in this analysis, are given in the Appendix A. Making use of these transformations, it is now possible to evaluate the integral formulation (7) using boundary values only. Based on the decomposition (9) and Eq. (10), the three domain integral in the r.h.s. of (7) are transformed into boundary values as follows:

$$\begin{cases}\int_0^1 K_2(x)W(x)W^*(s,x)\mathrm{d}x = \sum_{j=1}^{n+2}\alpha_j B_j(s) \\ B_j(s) = h_j(s) + \left[W^*(s,x)\frac{\mathrm{d}^3 h_j}{\mathrm{d}x^3}(x)\right. \\ \qquad - \frac{\partial W^*}{\partial x}(s,x)\frac{\mathrm{d}^2 h_j}{\mathrm{d}x^2}(x) + \frac{\partial^2 W^*}{\partial x^2}(s,x)\frac{\mathrm{d}h_j}{\mathrm{d}x}(x) \\ \qquad \left. - \frac{\partial^3 W^*}{\partial x^3}(s,x)h_j(x)\right]_0^1\end{cases} \tag{11a}$$

$$\begin{cases}\int_0^1 \frac{\partial^2 W}{\partial x^2}(x)W^*(s,x)\mathrm{d}x = \sum_{j=1}^{n+2}\alpha_j C_j(s) \\ C_j(s) = \frac{\mathrm{d}^2 g_j}{\mathrm{d}x^2}(s) + \left[W^*(s,x)\frac{\mathrm{d}^5 g_j}{\mathrm{d}x^5}(x)\right. \\ \qquad - \frac{\partial W^*}{\partial x}(s,x)\frac{\mathrm{d}^4 g_j}{\mathrm{d}x^4}(x) + \frac{\partial^2 W^*}{\partial x^2}(s,x)\frac{\mathrm{d}^3 g_j}{\mathrm{d}x^3}(x) \\ \qquad \left. - \frac{\partial^3 W^*}{\partial x^3}(s,x)\frac{\mathrm{d}^2 g_j}{\mathrm{d}x^2}(x)\right]_0^1\end{cases} \tag{11b}$$

For a uniform elastic foundation κ^*

$$\begin{cases}\int_0^1 \kappa^*(x)W(x)W^*(s,x)\mathrm{d}x = \kappa^*\sum_{j=1}^{n+2}\alpha_j D_j(s) \\ D_j(s) = g_j(s) + \left[W^*(s,x)\frac{\mathrm{d}^3 g_j}{\mathrm{d}x^3}(x)\right. \\ \qquad - \frac{\partial W^*}{\partial x}(s,x)\frac{\mathrm{d}^2 g_j}{\mathrm{d}x^2}(x) + \frac{\partial^2 W^*}{\partial x^2}(s,x)\frac{\mathrm{d}g_j}{\mathrm{d}x}(x) \\ \qquad \left. - \frac{\partial^3 W^*}{\partial x^3}(s,x)g_j(x)\right]_0^1\end{cases} \tag{11c}$$

For a concentrated elastic foundation κ^* at point

$$\begin{cases}\int_0^1 \kappa^*(x)W(x)W^*(s,x)\mathrm{d}x = \kappa^*\sum_{j=1}^{n+2}\alpha_j D_j(s) \\ D_j(s) = 0 \quad j \neq L, \\ D_L(s) = W(x_L)W^*(s,x_L), \quad \alpha_L = 1\end{cases} \tag{11d}$$

Based on Eqs. (8) and (11), the integral formulation (7) is reduced to the following algebraic equation at interior points:

$$W(s)+A(s)=\omega^{*2}\sum_{j=1}^{n+2}\alpha_jB_j(s)+\lambda^*\sum_{j=1}^{n+2}\alpha_jC_j(s)+\kappa^*\sum_{j=1}^{n+2}\alpha_jD_j(s) \quad (12)$$

where A, B, C and D will be explicitly given later.

The boundary conditions of the beam may be classified as simply-supported, clamped, free or more general as elastically supported edges. In order to present a general formulation for various boundary conditions, more equations than (12) related to θ, M and Q are needed. They are obtained by derivatives of Eq. (12) according to the variable 's'. For a compact equation representation, the following notations are introduced:

$$\widehat{E}(s)=\frac{\partial E}{\partial s}(s),\quad \widehat{\widehat{E}}(s)=K_1(s)\frac{\partial\widehat{E}}{\partial s}(s)\quad\text{and}\quad \widehat{\widehat{\widehat{E}}}(s)=\frac{\partial\widehat{\widehat{E}}}{\partial s}(s) \quad (13)$$

where E may be A, B, C or $E=D$.

Finally, one obtains for a uniform elastic foundation the following algebraic system:

$$\begin{cases} W(s)+A(s)=\omega^{*2}\sum_{j=1}^{n+2}\alpha_jB_j(s)+\lambda^*\sum_{j=1}^{n+2}\alpha_jC_j(s)+\kappa^*\sum_{j=1}^{n+2}\alpha_jD_j(s)\\ \theta(s)+\widehat{A}(s)=\omega^{*2}\sum_{j=1}^{n+2}\alpha_j\widehat{B}_j(s)+\lambda^*\sum_{j=1}^{n+2}\alpha_j\widehat{C}_j(s)+\kappa^*\sum_{j=1}^{n+2}\alpha_j\widehat{D}_j(s)\\ -M(s)+\widehat{\widehat{A}}(s)=\omega^{*2}\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{B}}_j(s)+\lambda^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{C}}_j(s)+\kappa^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{D}}_j(s)\\ -Q(s)+\widehat{\widehat{\widehat{A}}}(s)=\omega^{*2}\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{\widehat{B}}}_j(s)+\lambda^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{\widehat{C}}}_j(s)+\kappa^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{\widehat{D}}}_j(s) \end{cases} \quad (14a–d)$$

These equations give analytical solution representations with respect to the interior variable s. For a numerical solution, a discretization of (14) and the consideration of boundary conditions are needed.

4. Matrix formulations

After discretization of Eq. (14), one can write

$$\begin{cases} W_i+A_i=\omega^{*2}\sum_{j=1}^{n+2}\alpha_jB_{ij}+\lambda^*\sum_{j=1}^{n+2}\alpha_jC_{ij}+\kappa^*\sum_{j=1}^{n+2}\alpha_jD_{ij}\\ \theta_i+\widehat{A}_i=\omega^{*2}\sum_{j=1}^{n+2}\alpha_j\widehat{B}_{ij}+\lambda^*\sum_{j=1}^{n+2}\alpha_j\widehat{C}_{ij}+\kappa^*\sum_{j=1}^{n+2}\alpha_j\widehat{D}_{ij}\\ -M_i+\widehat{\widehat{A}}_i=\omega^{*2}\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{B}}_{ij}+\lambda^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{C}}_{ij}+\kappa^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{D}}_{ij}\\ -Q_i+\widehat{\widehat{\widehat{A}}}_i=\omega^{*2}\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{\widehat{B}}}_{ij}+\lambda^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{\widehat{C}}}_{ij}+\kappa^*\sum_{j=1}^{n+2}\alpha_j\widehat{\widehat{\widehat{D}}}_{ij} \end{cases} \quad (15a–d)$$

in which $i=1$ and $i=n+2$ correspond to beam ends and $i=2$ to $i=n+1$ correspond to interior points which may correspond to uniform or non-uniform discretisation. Let us recall that we have $(n+4)$ unknowns, (n) interior and four unknowns related to the assumed boundary conditions. Eq. (15a) leads to $(n+2)$ equations and two extra equations are then needed. Eqs. (15b) or (15c) may be used to complete the system for S–S, S–C, C–C and other boundary conditions. A combination of Eqs. (15a)–(15d) can also be used to solve the system for more general boundary conditions.

The present formulation is quite general and is given for beams with variable cross-sections. As the aim of this paper is the development of a simple and general formulation leading to a unique approach for buckling, flutter, transverse vibrations and interaction with elastic foundation, beams with constant sections are considered. For the dynamic analyses of beams with variable cross-sections, more theoretical developments with respect to the associated fundamental solution and radials basis functions are needed. For the sake of clearness, let us rewrite (15) in a matrix form and give more details about introduced matrices for specified boundary conditions.

Assuming that the beam is clamped–simply-supported as presented in Fig. 2 ($W_1=0$, $\theta_1=0$, $W_{n+2}=0$ and $M_{n+2}=0$) and introducing the following notations:

$$A_1(s)=-\frac{\partial W^*}{\partial x}(s,0)M_1,\quad A_2(s)=\frac{|1-s|}{2}\theta_{n+2},$$
$$A_3(s)=W^*(s,0)Q_1,\quad A_4(s)=-W^*(s,1)Q_{n+2}$$
$$\{W\}=\{W_2,W_3,\ldots,W_n,W_{n+1}\}\quad\text{and}$$
$$\{T\}=\{M_1,\theta_{n+2},Q_1,Q_{n+2}\}$$

The vector $\{W\}$ represents the deflection unknowns at interior point and $\{T\}$ represents the boundary unknowns. For the considered boundary conditions and using previous notations, the algebraic system (15) can be written in the following matrix form:

$$\begin{bmatrix} I & A\\ O & A_0\end{bmatrix}\begin{Bmatrix}\{W\}\\ \{T\}\end{Bmatrix}=\left(\omega^{*2}\begin{bmatrix}B\\ B_0\end{bmatrix}+\lambda^*\begin{bmatrix}C\\ C_0\end{bmatrix}+\kappa^*\begin{bmatrix}D\\ D_0\end{bmatrix}\right)F\{W\} \quad (16)$$

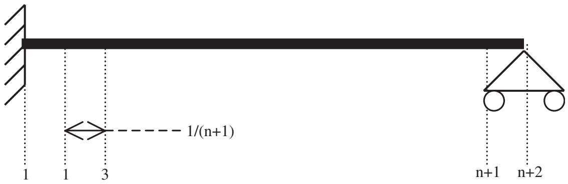

Fig. 2. Uniform discretization of a simply-supported and clamped beam. $W_1 = W_{n+2} = 0$, $\theta_1 = 0$ and $M_{n+2} = 0$.

where I $(n \times n)$, and O $(4 \times n)$ are the identity and zero matrix, respectively.

A : matrix $(n \times 4)$, $A_{ik} = A_k(s_i)$, for $k = 1, 2, 3$ or 4
$s_i = (i-1)/(n+1)$ and $i = 2$ to $n+1$

A_0 : matrix (4×4), $A_{01k} = A_k(0)$, $A_{02k} = A_k\ (1)$,
$A_{03k} = \hat{\hat{A}}_k(0)$, $A_{04k} = \hat{\hat{A}}_k(1)$, for $k = 1$ to 4

B : matrix $(n \times (n+2))$, $B_{ij} = B_j(s_i)$,
$i = 2$ to $n+1$ and $j = 1$ to $n+2$

B_0 : matrix $(4 \times (n+2))$, $B_{01j} = B_j(0)$, $B_{02j} = B_j(1)$,
$B_{03j} = \hat{\hat{B}}_j(0)$, $B_{04j} = \hat{\hat{B}}_j(1)$, $j = 1$ to $n+2$

C : matrix $(n \times (n+2))$, $C_{ij} = C_j(s_i)$,
$i = 2$ to $n+1$ and $j = 1$ to $n+2$

C_0 : matrix $(4 \times (n+2))$, $C_{01j} = C(0)$, $C_{02j} = C_j(1)$,
$C_{03j} = \hat{\hat{C}}_j(0)$, $C_{04j} = \hat{\hat{C}}_j(1)$, $j = 1$ to $n+2$

D : matrix $(n \times (n+2))$, $D_{ij} = D_j(s_i)$,
$i = 2$ to $n+1$ and $j = 1$ to $n+2$

D_0 : matrix $(4 \times (n+2))$, $D_{01j} = D_j(0)$, $D_{02j} = D_j(1)$,
$D_{03j} = \hat{\hat{C}}_j(0)$, $D_{04j} = \hat{\hat{D}}_j(1)$, $j = 1$ to $n+2$

F_1 : matrix $((n+2) \times (n+2))$ of radial function matrix
$F1_{ij} = f_j\left(\frac{(i-1)}{(n+1)}\right)$, $i = 1$ to $n+2$ and $j = 1$ to $n+2$

F : matrix $((n+2) \times (n)), F_{ij} = F1^{-1}_{i(j+1)}$,
$i = 1$ to $n+2$ and $j = 1$ to n
because for clamped simply-supported boundary conditions $W_1 = W_{n+2} = 0$

Eq. (16) presents an algebraic system on the deflection at unknown interior points $\{W\}$ and unknowns at boundaries represented by $\{T\}$. This system is rewritten as:

$$\begin{cases} A_0\{T\} = (\omega^{*2}B_0 + \lambda^* C_0 + \kappa^* D_0 - O)F\{W\} \\ A\{T\} = (\omega^{*2}B + \lambda^* C + \kappa^* D - I)F\{W\} \end{cases} \quad (17a,b)$$

Let us recall that the vectors $\{W\}$ and $\{T\}$ and the previous matrices depend on the boundary conditions considered. Matrices I and O may be also changed according to boundary conditions and the same notations are kept for a general representation. In Appendix B, details about vectors and matrices for simply-supported, clamped–clamped and clamped–free are given. The solutions of the considered problems are obtained by numerically solving Eq. (16). This system is reduced to an eigenvalue problem in deflection vector only as follows:

$$\begin{cases} [X] * \{W\} = [Y] * \{W\} \\ X = (I - AA_0^{-1}O) - \omega^{*^2}(AA_0^{-1}B_0 - B)F \\ Y = (\lambda^*(C - AA_0^{-1}C_0) + \kappa^*(D - AA_0^{-1}D_0))F \end{cases} \quad (18a–c)$$

The unknowns at boundaries can be easily computed by the following algebraic equation:

$$\{T\} = A_0^{-1}[(\omega^{*2}B_0 + \lambda^* C_0 + \kappa^* D_0)F - O]\{W\} \quad (19)$$

This permits to break down the matrix problem (16) into an eigenvalue problem and an algebraic one. Matrices $[X]$ and $[Y]$ can be easily formulated for each considered problem.

For specified boundary conditions, load and foundations, matrices in (18) and (19) have first to be computed following the developments presented in Appendix B. For numerical solutions, a computing program in MATLAB has been developed. The MATLAB environment is exploited for a standard use of the presented formulation. Formulations (18) and (19) are quite general and allow one to investigate buckling, vibrations and combination of them leading to load-frequency dependence for beams on various types of elastic foundations.

4.1. Buckling problem

The buckling problem may be formulated by omitting the frequency parameter in (18) and (19). The critical buckling loads and associated eigenmodes can be determined for various types of boundary conditions and elastic foundations κ^* by solving the following eigenvalue problem:

$$\begin{cases} [X]\{W\} = \frac{1}{\lambda^*}\{W\} \\ X = \left[(I - AA_0^{-1}O) - \kappa^*(D - AA_0^{-1}D_0)F\right]^{-1} \\ \qquad \times (C - AA_0^{-1}C_0)F \end{cases} \quad (20a,b)$$

This allows the investigation of numerical critical buckling loads and associated eigenmodes at interior points. The corresponding slope, moment and shear force can be numerically computed using (15). The unknowns at boundaries are computed by the following algebraic equation:

$$\{T\} = A_0^{-1}[(\lambda^* C_0 + \kappa^* D_0)F - O]\{W\} \tag{20c}$$

4.2. Linear vibration problem

The numerical solution of Eq. (18) permits, on one hand, to study the linear vibration behaviors of beams by omitting the load parameter. On the other hand, the linear vibration analysis of beams under an axial compression also can be investigated by numerically solving the following eigenvalue problem for each fixed load parameter:

$$\begin{cases} [X]\{W\} = \frac{1}{\omega^{*2}}\{W\} \\ X = \left[I - AA_0^{-1}O + \lambda^*(C - AA_0^{-1}C_0)F \right. \\ \qquad \left. + \kappa^*(D - AA_0^{-1}D_0)F\right]^{-1}(A_0^{-1}B_0 - B)F \end{cases} \tag{21}$$

The unknowns at boundaries can be computed by the algebraic Eq. (18). The load-frequency dependence can be investigated and the divergence stability may be analyzed for various types of elastic foundations.

4.3. Flutter problem

Let recall that the matrix X is load dependent. With adjusted matrices, in accordance with the considered boundary conditions, Eq. (21) can be used for conservative and non-conservative loads. When the applied load is a non-conservative follower force, the frequencies can be either real or complex. Therefore, at divergence instability, the lowest frequency vanishes, as for the conservatives system or two frequencies can approach each other, coalesce and then become complex conjugate. This corresponds to flutter instability and the load at the two frequencies coincide is defined as the flutter load. This study is intended to extend the previous analysis based on boundary integral formulation to the stability of a cantilevered beam subjected to a tangential follower force at the free end (Beck's problem) under elastic foundation. The assumed boundary conditions for a C–F beam on an elastic foundation at the free end is:

$$W_1 = 0, \quad \theta_1 = 0, \quad M_{n+2} = 0 \quad \text{and} \quad Q_{n+2} = -\kappa^* \cdot W_{n+2} \tag{22}$$

The dynamic stability analysis can be performed by Eq. (21) with adjusted matrices taking into account the assumed boundary conditions as presented in Appendix B. The load-frequency dependences and the flutter load corresponding to coalescence of two natural frequencies can be investigated. The mode, moment and shear force corresponding to critical frequency or to the flutter load can be directly computed. The control of the linear and non-linear flutter may be performed based on the numerically obtained modes and will be an extension of this work.

4.4. Nonlinear vibration problem

One of the main objectives of the present work is to establish a multi-modal formulation based on boundary element method for non-linear vibration and post-buckling analyses of beams. The numerical solution of the linear vibration problem (21) permits one to get the natural frequencies and associated eigenmodes. Using the obtained eigenmodes, a multimodal formulation can be developed for beams with various boundary conditions and elastic foundations. Based on harmonic balance method, a semi analytical method has been presented for nonlinear free and forced vibrations of beams [28,29]. In that work, analytical beam modes, available for classical boundary conditions, are used. In the present work, the numerically obtained modes will be used and any boundary condition and elastic foundation may be inserted. The nonlinear harmonic response may be computed in the following form:

$$W(x,t) = \cos(\omega t)\sum_{i=1}^{n} a_i w_i(x) \tag{23}$$

where $w_i(x)$ are the computed eigenmodes and $\{A\}^t = \{a_1, a_2, \ldots, a_n\}$ is the corresponding amplitude vector. Following the formulation presented in [28,29], the nonlinear forced vibration of beams can be analyzed using the following a dimensional and multidimensional formulation:

$$([K^*] - \omega^{*2}[M^*])\{A\} + \frac{3}{2}[B^*(A)]\{A\} = \{F^*\} \tag{24}$$

where $[M^*]$, $[K^*]$ and $[B^*(A)]$ are the mass matrix, the linear and nonlinear rigidity matrices respectively. $\{F^*\}$ is the column vector of generalized transverse excitations. The multimodal analysis will be investigated in the next work. In this work, we limit ourselves to one mode analysis and the response is assumed to be

$V(z,t) = Ra_1 w_1(x)\cos(\omega t)$

in which R is the radius of gyration and a_1 the amplitude corresponding to the first mode. The 1-D nonlinear frequency response function is then given by [28]

$$\left(\frac{\omega^*}{\omega_L^*}\right)^2 = 1 + \frac{3}{2}\frac{b_{1111}^*}{k_{11}^*}a_1^2 - \frac{1}{k_{11}^*}\frac{f_1^*}{a_1} \tag{25}$$

where

$$\omega_{L^2}^* = k_{11}^*/m_{11}^*; \quad m_{11}^* = \int_0^1 (w_1(x))^2 \mathrm{d}x$$

$$k_{11}^* = \int_0^1 \left(\frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \kappa^* \int_0^1 (w_1(x))^2 \mathrm{d}x$$

in the case of a uniform elastic foundation

$$k_{11}^* = \int_0^1 \left(\frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \kappa^* (w_1(x_0))^2$$

in the case of a concentrated elastic foundation

$$b_{1111}^* = \left(\frac{1}{2}\int_0^1 \left(\frac{\mathrm{d}w_1(x)}{(\mathrm{d}x)}\right)^2 \mathrm{d}x\right)^2$$

For a concentrated harmonic force of amplitude Fc^c at x_0, f_1^* is given by

$$f_1^* = F^c \frac{L^3}{EIR} w_1(x_0) = F_{0r}^c \ w_1(x_0) \tag{26}$$

The coefficients m_{11}^*, k_{11}^*, b_{1111}^* and $w_1(x_0)$ are evaluated using the first mode shape numerically obtained by the present analysis. The nonlinear frequency–amplitude dependence for beams under various boundary conditions and elastic foundations for free and forced vibration can be easily analyzed based on Eq. (25).

5. Numerical results

The mathematical formulation is presented for beams with variable section under various elastic foundation and boundary conditions and subjected to various types of loads. In this paper, the analysis is limited to isotropic beams with constant sections. The extension to variable sections, anisotropic beams and static and dynamic multimodal analyses of beams will be investigated in the next work. The critical buckling loads and the natural frequencies and the corresponding eigenmodes are determined by solving the resulting eigenvalue problem at concatenation points. The deflection, the slope, the bending moment and the shear force can be also investigated at interior and boundaries of the beam. A large number of natural frequencies or buckling loads and associated eigenmodes can be numerically computed. The accuracy of the result depends on the number of interiors points chosen. Tests of convergence of the buckling and vibration analyses are investigated. Many beam tests are analyzed and only some benchmark ones are presented in order to demonstrate the effectiveness of the developed approach.

5.1. Buckling

The obtained results depend on the number of internal points considered. The convergence to the analytical solution is tested and results are presented. For the sake of brevity, only few tested case are presented. The convergence of the solution with respect to interior points ‘*n*’ is presented in Tables 1 and 2. It can be seen clearly from these tables that 60 interior points is largely enough for accurate results. Results obtained with the present model for critical buckling loads are favorably compared to analytical ones in case of constant sections. Results are presented only for clamped–simply supported and clamped–free beams. For buckling of beams on a concentrated elastic foundation, the critical load changes according to the amplitude and position of the foundation. Numerical results obtained by 60 interior points for S–S beams are presented in Table 3 and are favorably compared to the available analytical ones [13]. The effect of the concentrated foundation at $X_c = 0.75$ on the fifth first modes are presented in Fig. 3 for $\kappa^* = 150$ The same study is done for a C–F beam ($X_c = 1$) leading to the same behavior as presented in Fig. 4. The concentrated elastic foundation effect on the first and higher buckling modes are clearly shown.

5.2. Linear vibrations

The vibration analyses are investigated by numerically solving Eq. (20) and a large number of vibration modes can be obtained. The fifth first free vibration modes are presented in Figs. 5 and 6 for C–S and C–F beams on concentrated elastic foundations at $X_c = 0.75$ and at $X_c = 1$ respectively. The convergence of fifth five natural frequencies with respect to the number of interior points is presented in Tables 4 and 5. Again, a few number of interior points is enough for a good prediction of eigenfrequencies. For beams submitted to an axial compression and on an elastic foundation, the natural frequency ratio $\omega^{*2}/\omega_1^{*2}$ according to the axial

Table 1
The fifth first buckling loads of a C–S beam obtained by the present model for various numbers of internal points *n*

Order	λ^*						Analytical solution [13]
	$n = 10$	$n = 20$	$n = 40$	$n = 60$	$n = 80$	$n = 100$	
1	20.47	20.27	20.21	20.20	20.20	20.19	20.19
2	62.17	60.36	59.86	59.76	59.72	59.71	59.68
3	128.90	121.59	119.60	119.22	119.08	119.02	118.90
4	225.91	205.36	199.81	198.74	198.36	198.18	197.90
5	360.00	313.51	300.94	298.53	297.67	297.27	296.60

Table 2
The fifth first buckling loads of a C–F beam obtained by the present model for various numbers of internal points n

Order	λ^*						Analytical solution [13]
	$n=10$	$n=20$	$n=40$	$n=60$	$n=80$	$n=100$	
1	2.46	2.47	2.47	2.47	2.47	2.47	2.47
2	22.85	22.38	22.25	22.23	22.22	22.21	22.21
3	62.99	62.04	61.78	61.73	61.71	61.70	61.69
4	135.37	124.73	121.89	121.35	121.16	121.07	120.90
5	220.07	205.33	201.28	200.50	200.22	200.09	199.86

Table 3
Comparison between critical buckling load ratios obtained by the present model and analytical ones of a simply-supported beam on concentrated elastic foundations for different positions X_L and amplitudes of κ^* ($n=60$)

Position X_L	$\lambda^*(\kappa^*)/\lambda^*(\kappa^*=0)$		$\lambda^*(\kappa^*=0)=\pi^2$							
	$\kappa^*=20$		$\kappa^*=40$		$\kappa^*=60$		$\kappa^*=80$		$\kappa^*=100$	
	Analytic [13]	Present model	Analytic [13]	Present model	Analytic [13]	Present model	Analytic [13]	Present model	Analytic [13]	Present model
0	1	1.0002	1	1.0002	1	1.0002	1	1.0002	1	1.0002
0.1	1.038	1.043	1.076	1.085	1.111	1.125	1.146	1.163	1.179	1.2
0.2	1.137	1.142	1.262	1.273	1.378	1.393	1.483	1.503	1.578	1.602
0.3	1.259	1.263	1.498	1.506	1.714	1.726	1.905	1.921	2.072	2.091
0.4	1.365	1.366	1.714	1.718	2.043	2.049	2.348	2.353	2.612	2.623
0.5	1.408	1.408	1.809	1.810	2.203	2.205	2.590	2.592	2.967	2.971

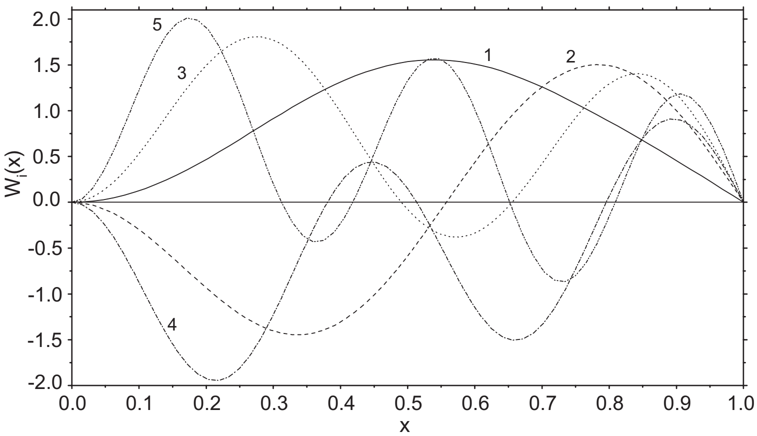

Fig. 3. The fifth first buckling mode shapes, mass normalized, of a clamped–simply supported beam on a concentrated foundation $\kappa^*=150$ at $X_c=0.75$ ($n=60$).

compression ratio λ^*/λ_1^* is presented in Figs. 7 and 8 $(\omega_1^*=\omega^*(\lambda^*=0,\kappa^*=0),\lambda_1^*=\lambda^*(\kappa^*=0))$. One can observe that the first natural frequency vanishes at critical buckling load (divergence). If one adds to axial compression the elastic foundation κ^* the natural frequencies change significantly as clearly shown in Figs. 7 and 8. The load-frequency dependence can be easily obtained for uniform and concentrated elastic foundations at any desired interior point in the pre-buckling region. For nonlinear pre-buckling and the post-buckling regions, a nonlinear analysis is needed [26,27].

5.3. Flutter analysis

For a clamped–free beam loaded by a tangential follower force (Beck's problem), the flutter phenomenon is

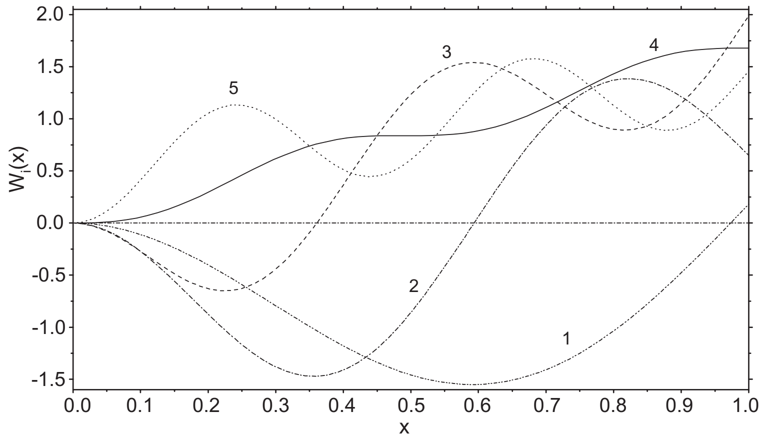

Fig. 4. The fifth first buckling mode shapes, mass normalized, of a clamped–free beam on a concentrated foundation $\kappa^* = 150$ at $X_c = 1$ ($n = 60$).

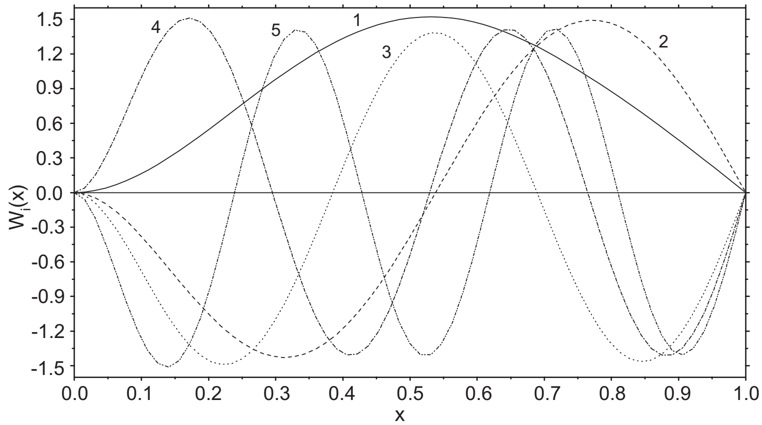

Fig. 5. The fifth first vibration mode shapes, mass normalized, of a clamped–simply supported beam on a concentrated foundation at $\kappa^* = 150$ at $X_c = 0.75$ ($n = 60$).

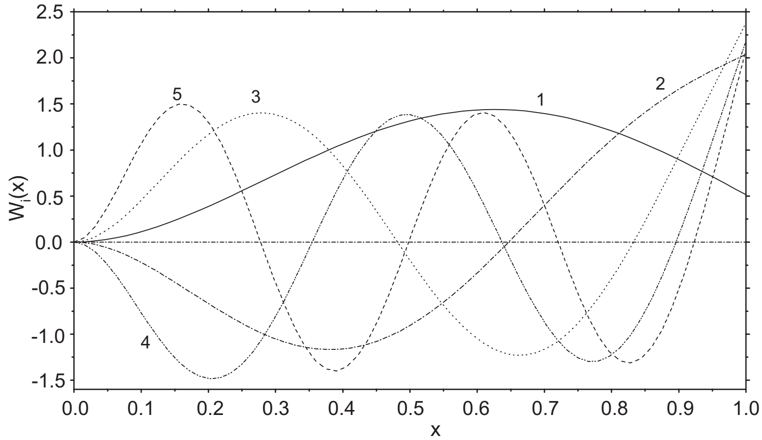

Fig. 6. The fifth first vibration mode shapes, mass normalized, of a clamped–free beam on a concentrated foundation at $\kappa^* = 150$ at $X_c = 1$ ($n = 60$).

Table 4
The fifth first natural frequencies of a C–S beam obtained by the present model for various numbers of internal points n

Order	ω^*						Analytical solution [30]
	$n = 10$	$n = 20$	$n = 40$	$n = 60$	$n = 80$	$n = 100$	
1	239.61	238.23	237.86	237.78	237.76	237.74	237.81
2	2 571.20	2 516.80	2 501.80	2 498.89	2 497.85	2 497.36	2 497.07
3	11 593.84	11 062.42	10 918.36	10 890.49	10 890.49	10 875.93	10 866.83
4	35 599.54	32 786.83	32 041.05	31 897.70	31 897.70	31 822.94	31 780.09
5	88 198.82	77 668.23	74 944.78	74 425.67	74 425.67	74 155.50	74 000.84

Table 5
The fifth first natural frequencies of a C–F beam obtained by the present model for various numbers of internal points n

Order	ω^*						Analytical solution [30]
	$n = 10$	$n = 20$	$n = 40$	$n = 60$	$n = 80$	$n = 100$	
1	12.355	12.360	12.362	12.362	12.362	12.362	12.362
2	489.95	486.74	485.83	485.66	485.60	485.57	485.52
3	3 927.73	3 839.64	3 815.21	3 810.46	3 808.77	3 807.97	3 806.62
4	15 636.15	14 892.46	14 689.09	14 649.68	14 635.65	14 629.09	14 617.45
5	44 875.56	41 257.42	40 284.90	40 097.59	40 030.97	39 999.85	39 943.81

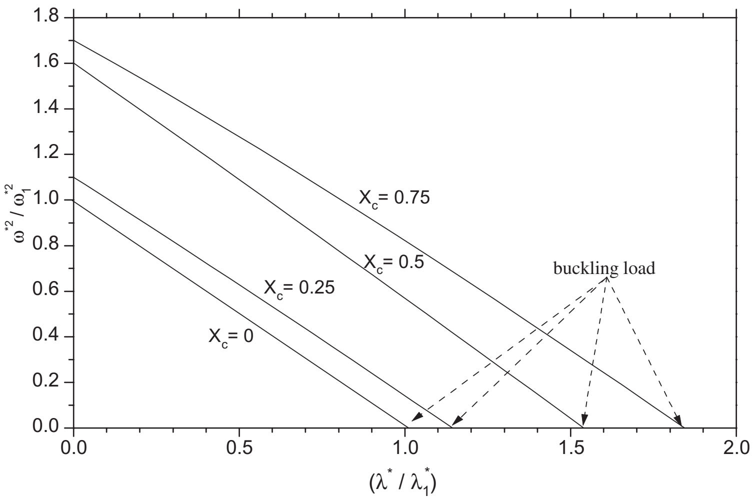

Fig. 7. Frequency ratio ($\omega^{*2}/\omega_1^{*2}$) with respect to axial compression ratio (λ^*/λ_1^*) of C–S beam at various positions of a concentrated elastic foundation $\kappa^* = 50$ ($n = 60$).

investigated. In the present analysis, the coalescence criterion is used. The numerically obtained flutter load with 60 internal points is $\lambda^* = 20.0625$ ($\lambda^* = 20.05$[19–21]). At this load, the first and the second eigenfrequencies coincide ($\omega_1^{*2} = \omega_2^{*2} = 121.46$) and become complex conjugate after the flutter load. In Fig. 9 are presented the fifth first vibration modes and shows a perfect coincidence between the first and the second mode. The third, the fourth and the fifth eigenmodes are real. Fig. 10 shows the load-frequency curves for a uniform elastic foundation with various amplitudes κ^*. It can be seen that the flutter load is amplitude-foundation independent and the Beck's solution [15–21] is obtained. The case of concentrated foundations is presented in Fig. 11 and shows a higher dependence between the flutter load and the position of the concentrated foundation. For this test, some more information is supplied in order to explain its particular behavior. The flutter load variation with respect to the position and the amplitude of the concentrated elastic foundations is presented in Fig. 12.

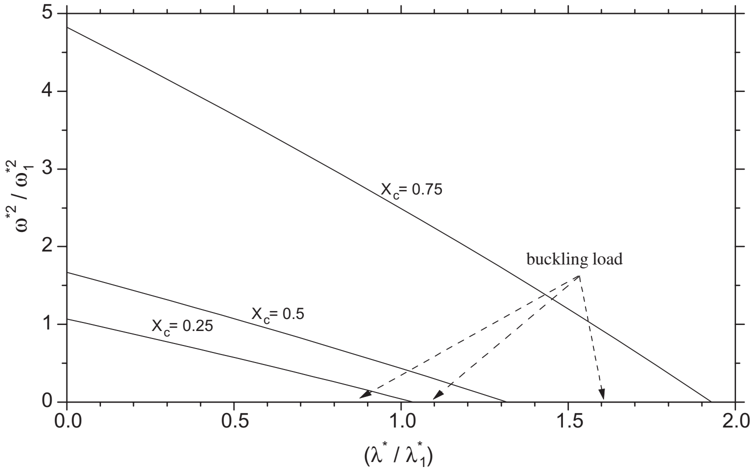

Fig. 8. Frequency ratio ($\omega^{*2}/\omega_1^{*2}$) with respect to axial compression ratio (λ^*/λ_1^*) of C–F beam at various positions of a concentrated elastic foundation $\kappa^* = 20$ ($n = 60$).

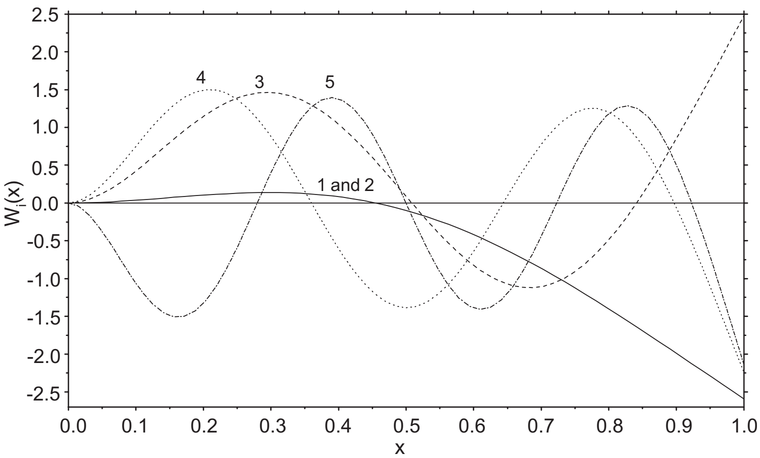

Fig. 9. The fifth first vibration mode shapes at flutter load of a clamped–free beam subjected to a tangential follower force at the free end, $\lambda^*_{(\text{flutter})} = 20.0625$ ($n = 60$).

It is clearly shown that for $0 < \kappa^* < 35$ the flutter may happen at every position X_c and the flutter load varies slowly from Beck's solution and the variation increases for $X_c > 0.5$. In the present analysis, for $X_c = 1$, the flutter limit correspond to $\kappa^*_{\text{lim}} = 35$, which is favorably compared to the results given in [23] ($\kappa^*_{\text{lim}} \approx 36$). For $\kappa^* > 35$, the flutter load is strongly position dependent and there is no flutter (divergence) for some positions. For $\kappa^* < 200$, the position $X_c \approx 0.77$ leads to the smallest value of the flutter load and the beam may flutter at a very low value. For $\kappa^* > 500$, the flutter position zone is largely reduced and the flutter load increases highly from Beck's solution. The transition from flutter zone to divergence zone according to the amplitude and the position of the concentrated elastic foundation is presented in Fig. 13.

5.4. Nonlinear vibrations

The main extension of this work is the development of the multimodal analyses of post-buckling, nonlinear vibration and nonlinear flutter of beams based on the numerically computed eigenmodes. For nonlinear vibration, this study is limited to the 1-mode analysis and the nonlinear amplitude-frequency (24) is used. For the sake of brevity, only some benchmark cases

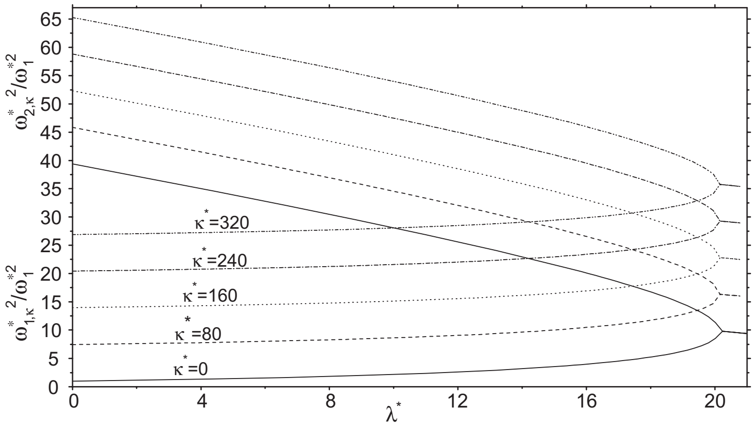

Fig. 10. Frequency ratios $(\omega_{1,\kappa}^{*2}/\omega_1^{*2})$ and $(\omega_{2,\kappa}^{*2}/\omega_1^{*2})$ with respect to axial tangential follower force λ^* at the free end of a clamped–free beam on various uniform elastic foundations κ^* $(\omega_1^{*^2} = 12.36)$ $(n = 60)$.

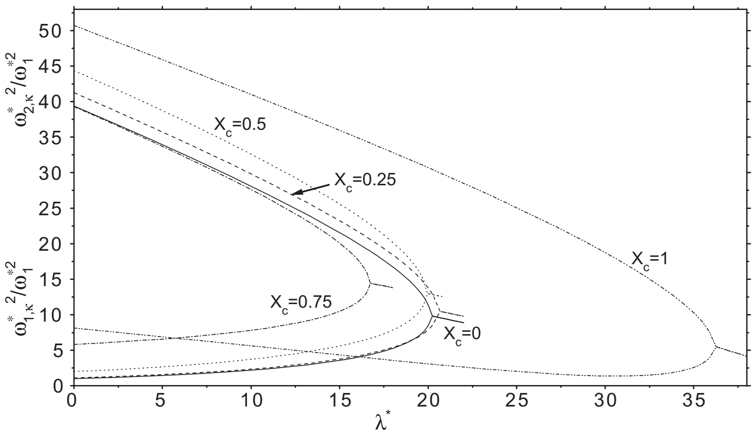

Fig. 11. Frequency ratios $(\omega_{1,\kappa}^{*2}/\omega_1^{*2})$ and $(\omega_{2,\kappa}^{*2}/\omega_1^{*2})$ with respect to axial tangential follower force λ^* at the free end of a clamped–free beam at various positions X_c of concentrated elastic foundations; $\kappa^* = 20$ $(n = 60)$.

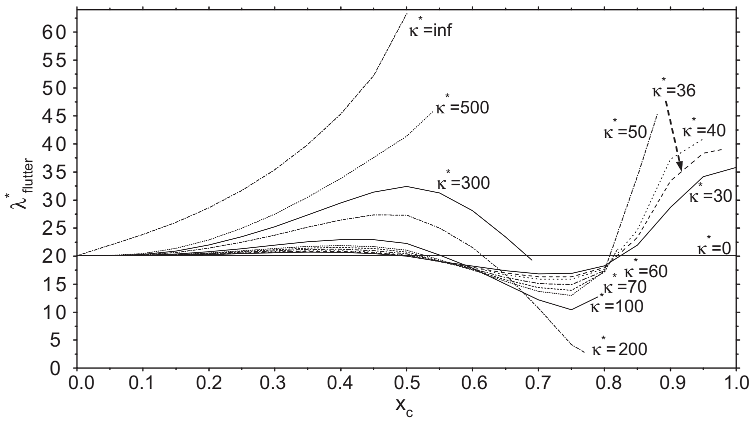

Fig. 12. $\lambda^*_{\text{flutter}}$ for a clamped–free beam according to the position of the concentrated elastic foundation X_c for different κ^*, $\kappa^* = 0$–500 and $\kappa^* = \text{inf}$ $(n = 60)$.

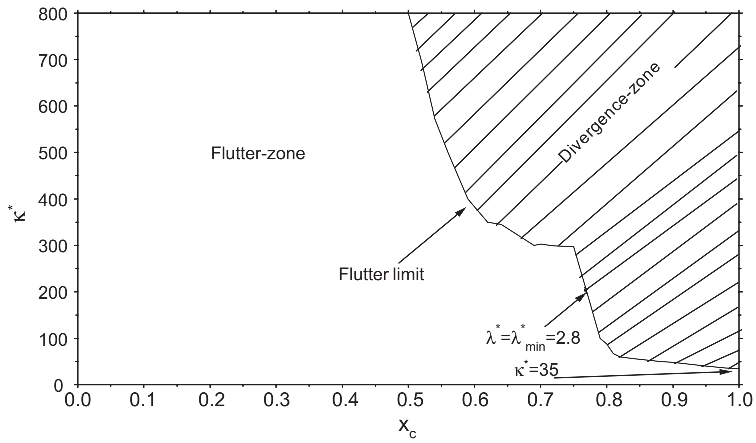

Fig. 13. Flutter limit and zone with respect to the amplitude and the position of concentrated foundation ($n = 60$).

Table 6
Frequency ratio (ω^*/ω_L^*) at various amplitudes of the nonlinear free vibration of a C–C beam under various concentrated elastic foundations at $X_c = 0.75$ ($n = 60$)

a_1	Analytic [28]	(ω^*/ω_L^*) Present model			
	$\kappa^* = 0$	$\kappa^* = 0$	$\kappa^* = 300$	$\kappa^* = 600$	$\kappa^* = \text{inf}$
0.2	1.000899	1.000897	1.000711	1.000675	1.000321
0.4	1.003590	1.003585	1.002843	1.002699	1.001282
1	1.022231	1.022197	1.017636	1.016749	1.007985
1.5	1.049357	1.049284	1.039259	1.037305	1.017879
2	1.086197	1.086070	1.068798	1.065417	1.031570
2.5	1.131801	1.131612	1.105617	1.100507	1.048911
3	1.185159	1.184899	1.149017	1.141930	1.069724
3.5	1.245275	1.244937	1.198283	1.189027	1.093810

Table 7
Frequency ratio (ω^*/ω_L^*) at various amplitudes of the nonlinear free vibration of a C–S beam under various concentrated elastic foundations at $X_c = 0.75$ ($n = 60$)

a_1	Analytic [28]	(ω^*/ω_L^*) Present model			
	$\kappa^* = 0$	$\kappa^* = 0$	$\kappa^* = 300$	$\kappa^* = 600$	$\kappa^* = \text{inf}$
0.2	1.002001	1.001999	1.000685	1.000594	1.000252
0.4	1.007980	1.007975	1.002738	1.002375	1.001008
1	1.017868	1.017850	1.016989	1.014755	1.006283
1.5	1.048881	1.048849	1.037834	1.032902	1.014082
2	1.106951	1.106884	1.066333	1.057786	1.024902
2.5	1.183471	1.183360	1.101891	1.088944	1.038646
3	1.275125	1.274963	1.143852	1.125856	1.055202
3.5	1.378898	1.378682	1.191538	1.167976	1.074440

are presented such as C–C and C–S beams. The nonlinear frequencies with respect to mode-amplitudes for various concentrated elastic foundation amplitudes at $X_c = 0.75$ are presented in Tables 6 and 7. The rigidity effect added by the elastic foundation on the nonlinear free vibration is analyzed. The backbone and resonance curves are presented in Fig. 14 for a C–S beam on concentrated elastic foundations at ($X_c = 0.75$) and subjected to a lateral concentrated harmonic excitation at the beam center. Increasing the foundation amplitude leads to reducing the nonlinear effect.

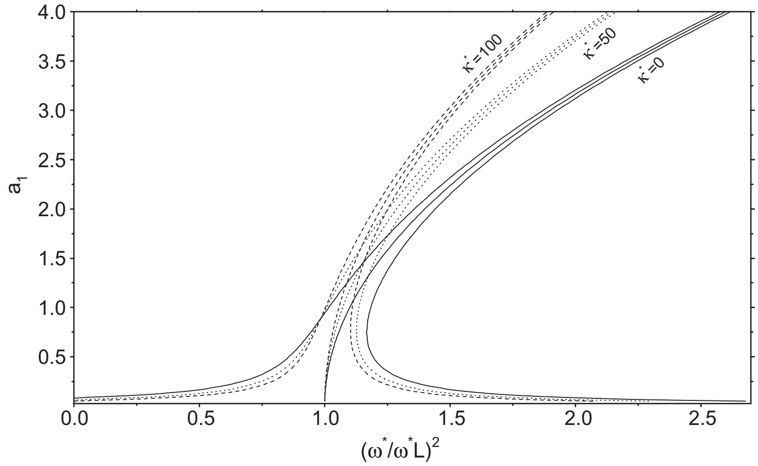

Fig. 14. Resonance curves of forced vibrations of a C–S beam under harmonic concentrated force at the center of the beam (for $c = 20$) for different values of concentrated elastic foundations at $X_c = 0.75$ ($n = 60$).

6. Conclusion

A methodological approach based on integral equation formulations for buckling, flutter and vibration analyses of beams is presented in simple and compact forms. Critical buckling loads, natural frequencies, flutter loads and load-frequency dependences are investigated for beams on various elastic foundations and boundary conditions. Tests of convergence with respect to interior points are carried out and showed that 60 points are largely enough for good accuracies of eigenvalues and eigenmodes. Nonlinear free and forced vibrations of beams based on one mode analysis are investigated. The presented model is quite general and all obtained results are in agreement with available data. Multimodal analyses based on the computed eigenmodes for post-buckling, nonlinear vibrations and nonlinear flutter will be investigated in next work.

Acknowledgements

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Appendix A

In this study, we limit ourselves to isotropic elastic beams with a constant section. In this case, the rigidity and mass functions $K_1(x)$ and $K_2(x)$ are constant ($K_1(x) = K_2(x) = 1$).

The fundamental solution W^* used in this analysis corresponds to $\frac{\partial^4 W^*}{\partial x^4}(x,s) = \delta(x,s)$ and is

$$W^*(x,s) = \frac{|x-s|^3}{12}$$

Several types of radial basis functions $f_j(x)$ are tested and the general form is:

$$f_j(x) = 1 + \alpha r_j + \beta r_j^2 + \gamma r_j^3 \quad \text{where } r_j = |x - x_j|$$

The other functions g_j and h_j are calculate using the following expression:

$$\frac{d^4 g_j}{dx^4}(x) = f_j(x) \quad \text{and} \quad \frac{d^4 h_j}{dx^4}(x) = K_2(x) f_j(x)$$

$$g_j(x) = h_j(x) = \frac{r_j^4}{24} + \alpha \frac{r_j^5}{120} + \beta \frac{r_j^6}{360} + \gamma \frac{r_j^7}{840}$$

where α, β and γ are the chosen constants [3,4].

Appendix B. Matrices corresponding to specified boundary conditions

The system of Eq. (15) is put in the following matrix form vectors and matrices used will be specified for each boundary condition considered

$$\begin{bmatrix} I & A \\ 0 & A_0 \end{bmatrix} \begin{Bmatrix} \{W\} \\ \{T\} \end{Bmatrix} = \left(\omega^{*2} \begin{bmatrix} B \\ B_0 \end{bmatrix} + \lambda^* \begin{bmatrix} C \\ C_0 \end{bmatrix} + K^* \begin{bmatrix} D \\ D_0 \end{bmatrix} \right) F\{W\}$$

Following the notations given in Fig. 2, let us put:

B-1 S–S beam: $W_1 = 0$, $M_1 = 0$, $W_{n+2} = 0$ and $M_{n+2} = 0$

$$A_1(s) = -\frac{|s|}{2}\theta_1, \quad A_2(s) = \frac{|1-s|}{2}\theta_{n+2},$$
$$A_3(s) = W^*(s,0)Q_1, \quad A_4(s) = -W^*(s,1)Q_{n+2}$$

$\{W\} = \{W_2, W_3, \ldots, W_n, W_{n+1}\}$ and
$\{T\} = \{\theta_1, \theta_{n+2}, Q_1, Q_{n+2}\}$

B-2 C–C beam: $W_1 = 0$, $\theta_1 = 0$, $W_{n+2} = 0$ and $\theta_{n+2} = 0$

$$A_1(s) = -\frac{\partial W^*}{\partial x}(s,0)M_1, \quad A_2(s) = \frac{\partial W^*}{\partial x}(s,1)M_{n+2},$$
$$A_3(s) = W^*(s,0)Q_1, \quad A_4(s) = -W^*(s,1)Q_{n+2}$$

$\{W\} = \{W_2, W_3, \ldots, W_n, W_{n+1}\}$ and
$\{T\} = \{M_1, M_{n+2}, Q_1, Q_{n+2}\}$

The other matrices are $I, O, A, A_0, B, B_0, C, C_0, D, D_0, F1, F$ as already defined for a C–S beam.

B-3 C–F beam with a follower tangential force: $W_1 = 0$, $\theta_1 = 0$, $M_{n+2} = 0$ and $Q_{n+2} = -\kappa^* W_{n+2}$

$$A_1(s) = -\frac{\partial W^*}{\partial x}(s,0), M_1, \quad A_2(s) = \frac{|1-s|}{2}\theta_{n+2},$$
$$A_3(s) = W^*(s,0)Q_1$$

$\{W\} = \{W_2, W_3, \ldots, W_{n+1}, W_{n+2}\}$ and
$\{T\} = \{M_1, \theta_{n+2}, Q_1\}$

I : matrix $((n+1)\times(n+1))$, $I_{i,i} = 1$,
$I_{i,n+1} = \kappa^* W^*(s,1) - 0.5$, for $i = 1$ to n;
$I_{n+1,n+1} = \kappa^* W^*(1,1) + 0.5$ the other terms are null

O : matrix $(3\times(n+1))$, $O_{1,n+1} = \kappa^* W^*(0,1) - 0.5$,
$O_{2,n+1} = \kappa^* \dfrac{\partial^2 W^*(s,1)}{\partial s^2}(0,1)$ and the other terms are null

A : matrix $((n+1)\times 3)$, $A_{ik} = A_k(s_i)$, for $k =$ 1–3, $i = 2$ to $n+2$ and $s_i = (i-1)/(n+1)$

A_0 : matrix (3×3), $A_{01k} = A_k(0)$, $A_{02k} = \widehat{\widehat{A}}_k(0)$,
$A_{03k} = \widehat{\widehat{A}}(1)$, for $k =$ 1–3

B : matrix $((n+1)\times(n+2))$, $B_{ij} = B_j(s_i)$, $i = 2$ to $n+2$ and $j = 1$ to $n+2$

B_0 : matrix $(3\times(n+2))$, $B_{01j} = B_j(0)$,
$B_{02j} = \widehat{\widehat{B}}_j(0)$, $B_{03j} = \widehat{\widehat{B}}_j(1)$, $j = 1$ to $n+2$

C : matrix $((n+1)\times(n+2))$, $C_{ij} = C_j(s_i)$, $i = 2$ to $n+2$ and $j = 1$ to $n+2$

C_0 : matrix $(3\times(n+2))$, $C_{01j} = C(0)$,
$C_{02j} = \widehat{\widehat{C}}_j(0)$, $C_{03j} = \widehat{\widehat{C}}_j(1)$, $j = 1$ to $n+2$

D : matrix $((n+1)\times(n+2))$, $D_{ij} = D_j(s_i)$, $i = 2$ to $n+2$ and $j = 1$ to $n+2$

D_0 : matrix $(3\times(n+2))$, $D_{01j} = D_j(0)$,
$D_{02j} = \widehat{\widehat{D}}_j(0)$, $D_{03j} = \widehat{\widehat{D}}_j(1)$, $j = 1$ to $n+2$

$F1$: matrix $((n+2)\times(n+2))$ of radial function matrix
$F1_{ij} = f_j\left(\dfrac{(i-1)}{(n+1)}\right)$, $i = 1$ to $n+2$ and $j = 1$ to $n+2$

F : matrix $((n+2)\times(n+1))F_{ij} = F1^{-1}_{i(j+1)}$,
$F_{i(n+2)} = 0$, $i = 1$ to $n+2$ and $j = 1$ to $n+1$
because for clamped–free boundary conditions $W_1 = 0$

B-4 C–F beam with a conservative axial force. $W_1 = 0$, $\theta_1 = 0$, $M_{n+2} = 0$ and $Q_{n+2} = -\kappa^*$, $W_{n+2} + \lambda^*\theta_{n+2}$

$$A_1(s) = -\frac{\partial W^*}{\partial x}(s,0)M_1, \quad A_2(s) = W^*(s,0)Q_1$$

$\{W\} = \{W_2, W_3, \ldots, W_{n+1}, W_{n+2}, \theta_{n+2}\}$ and
$\{T\} = \{M_1, Q_1\}$

I : matrix$((n+2)\times(n+2))$,

$$\begin{cases} I_{i,i} = 1, \quad I_{i,n+1} = \kappa^* W^*(s,1), -0.5, \\ I_{i,n+2} = \frac{|1-s|}{2}, \quad \text{for } i = 1 \text{ to } n \\ I_{n+1,n+1} = 0.5 \\ I_{n+2,n+1} = \kappa^* W^*(0,1) - 0.5 \\ I_{n+2,n+2} = 0.5 \end{cases}$$

and the other terms are null.

O : matrix $(2\times(n\ +2))$, $O_{1,n+1} = \kappa^* \dfrac{\partial^2 W^*(s,1)}{\partial s^2}(0,1)$
and the other terms are null

A : matrix $((n+2)\times 2)$, $A_{ik} = A_k(s_i)$, for $k = 1$ to 2, $i = 2$ to $n+2$ and $s_i = (i-1)/(n+1)$

A_0 : matrix (2×2), $A_{01k} = \widehat{\widehat{A}}_k(0)$,
$A_{02k} = \widehat{\widehat{A}}_k(1)$, for $k =$ 1–2

B : matrix $((n+2)\times(n+2))$, $B_{ij} = B_j(s_i)$, $i = 2$ to $n+2$ and $j = 1$ to $n+2$

B_0 : matrix $(2\times(n+2))$, $B_{01j} = \widehat{\widehat{B}}_j(0)$,
$B_{02j} = \widehat{\widehat{B}}_j(1)$, $j = 1$ to $n+2$

C : matrix $((n+2)\times(n+2))$, $C_{ij}=C_j(s_i)$, $i=2$ to $n+2$ and $j=1$ to $n+2$

C_0 : matrix $(2\times(n+2))$, $C_{01j}=\widehat{\widehat{C}}_j(0)$,

$C_{02j}=\widehat{\widehat{C}}_j(1)$, $j=1$ to $n+2$

D : matrix $((n+2)\times(n+2))$, $D_{ij}=D_j(s_i)$, $i=2$ to $n+2$ and $j=1$ to $n+2$

D_0 : matrix $(2\times(n+2))$, $D_{01j}=\widehat{\widehat{D}}_j(0)$,

$D_{02j}=\widehat{\widehat{D}}_j(1)$, $j=1$ to $n+2$

$F1$: matrix $((n+2)\times(n+2))$ of radial function matrix $F1_{ij}=f_j\left(\frac{(i-1)}{(n+1)}\right)$, $i=1$ to $n+2$ and $j=1$ to $n+2$

F : matrix $((n+2)\times(n+2))$, $F_{ij}=F1^{-1}_{i(j+1)}$, $i=1$ to $n+2$ and $j=1$ to $n+1$ because for clamped–free boundary conditions $W_1=0$. The other terms are null.

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