

# The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains

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## Abstract

In this paper a new boundary method for Helmholtz eigenproblems in simply and multiply connected domains is presented. The method is based on mathematically modelling the physical response of a system to external excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. So, contrary to the traditional scheme, the method described does not involve evaluation of determinants of linear systems. The solution of an eigenvalue problem is reduced to a sequence of inhomogeneous problems with the differential operator studied. The method shows a high precision in simply and multiply connected domains. The results of the numerical experiments justifying the method are presented.

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## 1. Introduction

In this paper we deal with the following eigenvalue problem:

$$\nabla^2 w + k^2 w = 0, \quad \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \quad (1)$$

$$w = 0, \quad \mathbf{x} \in \partial\Omega. \quad (2)$$

Here  $\Omega$  is a domain of interest with boundary  $\partial\Omega$ . The problem is to find such real  $k$  for which there exist non-null functions  $w$  verifying (1), (2). As a mechanical or acoustic application, this corresponds to recovering the resonance frequencies of a system. Such problems often arise in engineering applications. Boundary methods such as the boundary elements method (BEM) [1,2] and the method of fundamental solutions (MFS) [3–5] are the fastest and most powerful tools in this field.

In the framework of the boundary methods a general approach to solving this problem is as follows. First, using an integral representation of  $w$  in the BEM, or an approximation over fundamental solutions in MFS, one gets an homogeneous

linear system with matrix elements depending on  $k$ :

$$\hat{\mathbf{A}}(k)\mathbf{x} = 0$$

The determinant of the matrix must be zero to obtain the nontrivial solution, i.e.

$$\det[\hat{\mathbf{A}}(k)] = 0$$

This equation must be solved numerically to get the eigenvalues. This technique is described in [6–11] with more details. In the three latest papers there is a complete bibliography on the subject considered.

The method presented in this article is based on a fundamentally different idea. This is a mathematical model of physical measurements when the resonance frequencies of a system are determined by the amplitude of response to some external excitation. As a result, instead of (1), (2) we solve a sequence of *inhomogeneous* problems

$$\nabla^2 w + k^2 w = f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \quad (3)$$

$$w = 0, \quad \mathbf{x} \in \partial\Omega, \quad (4)$$

where  $f$  describes some source placed outside the solution domain. Let  $F(k)$  be some norm of the solution  $w$ . This function of  $k$  has maximums at the eigenvalues and, under some conditions described below, can be used for their determining.

The outline of this paper is as follows: for the sake of simplicity we begin by describing the 1D case in Section 2.

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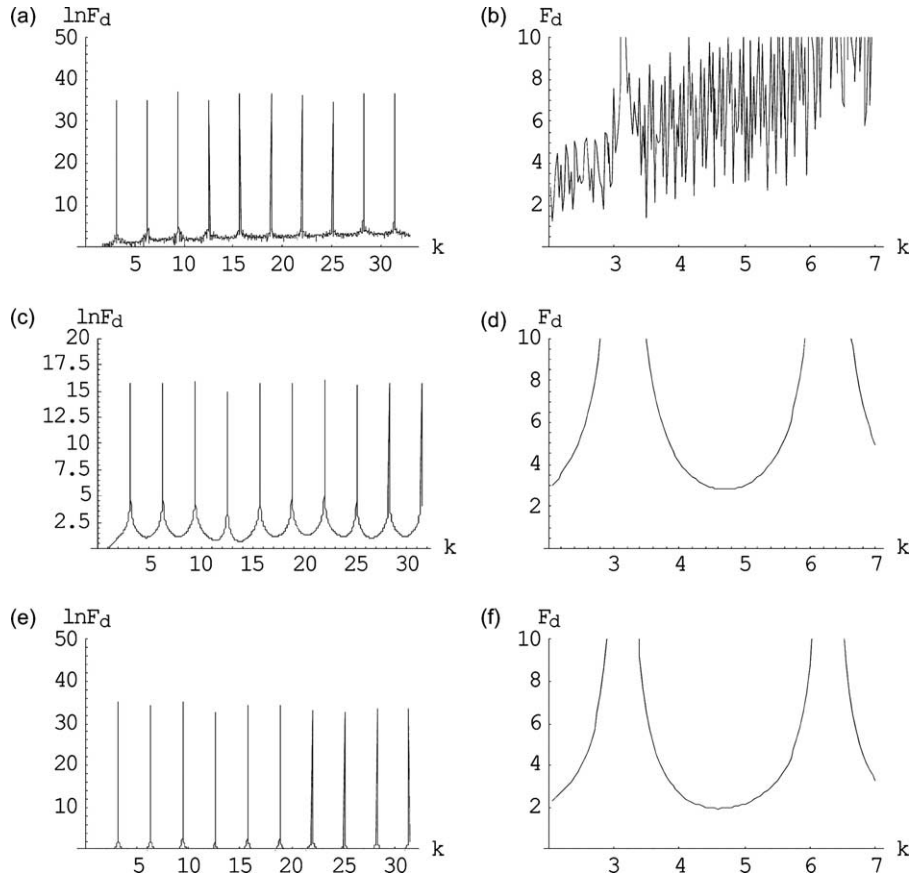


Fig. 1. The resonance curve; (a) and b) without regularization; (c) and (d) regularization by an additional friction term:  $\varepsilon = 10^{-6}$ ; (e) and (f) regularization by shifting of the wave numbers:  $\Delta k = 1$ .

In Section 3, we present the algorithm of MFS in application to the problem considered in the 2D case. In Section 4, we present numerical examples to illustrate the method presented for simply and multiple connected domains. In particular, the case of doubly connected region with the inner region of vanishing maximal dimension which is important for technical applications is considered here.

**2. The one dimensional case**

To illustrate the method presented let us consider a homogeneous string with fixed endpoints  $x=0$  and  $x=1$ . After the process of scaling one can write the wave equation in the form [12]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(1, t) = 0 \tag{5}$$

Considering the free harmonic vibrations  $u(x, t) = e^{-ikt} w(x)$  we get the following 1D Sturm–Liouville problem on the interval  $[0, 1]$ :

$$\frac{d^2 w}{dx^2} + k^2 w = 0, \quad w(0) = w(1) = 0 \tag{6}$$

The well known solution is:  $k_n = n\pi$   $w_n = \sin(n\pi x)$   $n = 1, 2, \dots, \infty$ .

Following the boundary technique, let us consider the fundamental solutions

$$\Psi(x, \xi, k) = \frac{1}{2ki} \exp(ik|x - \xi|), \tag{7}$$

which satisfy the equation everywhere except the singular point  $x = \xi$ . A general solution of the homogeneous equation can be written in the form:

$$w = q_1 \Psi(x, \xi_1, k) + q_2 \Psi(x, \xi_2, k).$$

Here  $\xi_1, \xi_2$  are two source points placed outside the solution domain  $[0, 1]$ ;  $q_1, q_2$  are free parameters. Then, using the boundary conditions  $w(0) = w(1) = 0$ , one gets a  $2 \times 2$  homogeneous linear system:

$$\hat{\mathbf{A}}(k) \mathbf{q} = 0$$

Then the critical wave numbers  $k_n$  can be determined from the condition:

$$\det[\hat{\mathbf{A}}(k)] = 0$$

Table 1  
The resonance curve  $F(k)$  near the eigenvalue  $k_1 = \pi$

$K$	3.079	3.110	3.142	3.173	3.204	3.236
$F(k)$	$7 \times 10^{-16}$	$3 \times 10^{-14}$	0.7	$3 \times 10^{-14}$	$1 \times 10^{-14}$	$7 \times 10^{-15}$

Table 2  
One dimensional eigenproblem. The relative errors in calculations of the eigenvalues.  $\varepsilon$ -procedure

$k_i^{(ex)}$	$\varepsilon=0.1$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-6}$
$\pi$	$1.3 \times 10^{-4}$	$1.3 \times 10^{-8}$	$1.7 \times 10^{-12}$
$2\pi$	$3.2 \times 10^{-5}$	$3.1 \times 10^{-9}$	$1.6 \times 10^{-12}$
$3\pi$	$1.4 \times 10^{-5}$	$1.4 \times 10^{-9}$	$1.5 \times 10^{-12}$
$4\pi$	$7.9 \times 10^{-6}$	$7.9 \times 10^{-10}$	$9.7 \times 10^{-13}$
$5\pi$	$5.1 \times 10^{-6}$	$5.0 \times 10^{-10}$	$9.0 \times 10^{-13}$
$6\pi$	$3.5 \times 10^{-6}$	$3.5 \times 10^{-10}$	$5.8 \times 10^{-13}$
$7\pi$	$2.6 \times 10^{-6}$	$2.6 \times 10^{-10}$	$9.2 \times 10^{-13}$
$8\pi$	$2.0 \times 10^{-6}$	$2.0 \times 10^{-10}$	$1.8 \times 10^{-13}$
$9\pi$	$1.6 \times 10^{-6}$	$1.6 \times 10^{-10}$	$5.3 \times 10^{-13}$
$10\pi$	$1.3 \times 10^{-6}$	$1.3 \times 10^{-10}$	$1.2 \times 10^{-12}$

This is the 1D scheme of applying the method of fundamental solutions to eigenvalue problems. Note that the components of the matrix  $\hat{A}(k)$  are complex numbers, so the determinant is a complex value too. In multidimensional cases such computations are time consuming and not simple.

As it is mentioned above, the method suggested is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined using an increase of amplitude of oscillations near these frequencies. So, instead of the homogeneous equation we solve the inhomogeneous one:

$$\frac{d^2w}{dx^2} + k^2w = f(x). \tag{8}$$

The general solution can be written in the form:

$$w = q_1\Psi(x, \xi_1, k) + q_2\Psi(x, \xi_2, k) + w_p. \tag{9}$$

When the excitation is performed by the point source with the same wave number  $k$  which is placed at the point  $\xi_0$  outside the solution domain, then  $f(x) = \delta(x - \xi_0)$  and the particular solution is

$$w_p = \Psi(x, \xi_0, k) = \frac{1}{2ki} \exp(ik|x - \xi_0|). \tag{10}$$

Using again the same homogeneous boundary conditions  $w(0) = w(1) = 0$  now we get an *inhomogeneous* linear system for each  $k$ .

Table 3  
One dimensional eigenproblem. The relative errors in calculation of the eigenvalues  $k$ -procedure

$k_i^{(ex)}$	$\Delta k=0.1$	$\Delta k=1$	$\Delta k=10$
$\pi$	$1.4 \times 10^{-11}$	$9.1 \times 10^{-12}$	$7.8 \times 10^{-12}$
$2\pi$	$5.8 \times 10^{-13}$	$3.5 \times 10^{-12}$	$5.5 \times 10^{-12}$
$3\pi$	$6.4 \times 10^{-12}$	$1.3 \times 10^{-12}$	$3.5 \times 10^{-12}$
$4\pi$	$3.3 \times 10^{-13}$	$2.8 \times 10^{-12}$	$2.3 \times 10^{-12}$
$5\pi$	$5.3 \times 10^{-12}$	$3.5 \times 10^{-12}$	$5.9 \times 10^{-13}$
$6\pi$	$1.8 \times 10^{-12}$	$7.7 \times 10^{-13}$	$5.6 \times 10^{-12}$
$7\pi$	$1.4 \times 10^{-12}$	$2.6 \times 10^{-12}$	$1.6 \times 10^{-12}$
$8\pi$	$1.5 \times 10^{-12}$	$1.7 \times 10^{-12}$	$8.4 \times 10^{-13}$
$9\pi$	$3.2 \times 10^{-12}$	$5.9 \times 10^{-13}$	$1.2 \times 10^{-13}$
$10\pi$	$1.2 \times 10^{-12}$	$2.5 \times 10^{-12}$	$6.3 \times 10^{-13}$

Let us introduce the norm of the solution as

$$F(k) = \sqrt{\frac{1}{N_t} \sum_{l=1}^{N_t} |w(x_{t,l})|^2}, \quad F_d(k) = F(k)/F(1), \tag{11}$$

where  $F_d$  is the relative value,  $k_0 = 1$  is a reference wave number and the  $N_t$  testing points  $x_{t,l}$  are randomly distributed in  $[0,1]$ . In all the calculations presented in this section we use  $N_t = 5$ . Note that the method is not very sensitive to the number of points  $x_{t,l}$ . But what is more important, they should be placed in an irregular way, i.e. where the eigenmode being investigated has non-zero values. E.g. the mode  $w_n = \sin(n\pi x)$  is equal to zero when  $x = 1/n$ . This norm function characterizes the value of the response of the system to the outer excitation. Note that the right hand side  $f$  corresponding to (10) equals to zero identically inside  $[0,1]$  and BVP (8) has a unique solution  $w = 0$  for all  $k$  except  $k = k_n$ —eigenvalues when the solution is not unique.

In Fig. 1a, the value  $\ln F_d$  as a function of the wave number  $k$  is depicted. The graph contains large sharp peaks at the positions of eigenvalues. Generally speaking, this resonance curve can be used to determine the eigenvalues in the same way as  $\det \hat{A}(k)$  in the technique described above. However, the graph  $F_d(k)$  is a non-smooth one, as it is shown in Fig. 1b, with more details. This can be explained by the following reasons. Let us consider Eq. (9) with  $w_p$  given in (10). We assume that  $\xi_1 < 0, \xi_2 > 1$  and  $\xi_0 > 1$ . It is easy to show that there exists the *exact solution*  $q_1 = 0, q_2 = -e^{ik(\xi_0 - \xi_2)}$  and so the total solution  $w(x) = 0$  for  $x \in [0,1]$ . This result can be illustrated by the data placed in Table 1 where we present behaviour of  $F(k)$  in a neighbourhood of the critical wave number  $k_1 = \pi$ . So, here we have  $F(k)$  which is equal to zero with machine precision accuracy when  $k$  is far from eigenvalues;  $F(k)$  grows considerably in a neighbourhood of the eigenvalues when the linear system becomes almost degenerated. Hence,  $F(k)$  is, in fact, a pseudorandom variable of  $k$ .

Now we describe two regularizing procedures which give a smooth resonance curve. The first procedure consists in

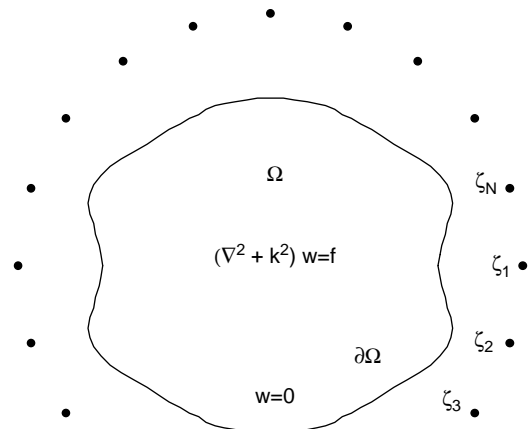


Fig. 2. Geometry configuration of a simply connected domain.

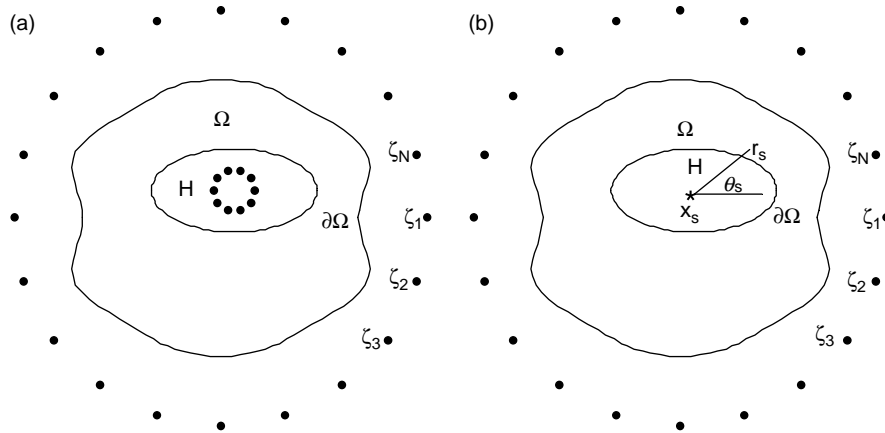


Fig. 3. Geometry configuration of a doubly connected domain.

introducing an additional friction term in the governing equation. Let us consider the following equation which describes vibration of an homogeneous string with a friction [12]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial u}{\partial t}, \quad u(0, t) = u(1, t) = 0, \quad (12)$$

where  $\varepsilon$  is a small parameter. As a result, instead of (6) we get:

$$\frac{d^2 w}{dx^2} + (k^2 + i\varepsilon k)w = 0, \quad w(0) = w(1) = 0 \quad (13)$$

with the fundamental solutions

$$\Psi(x, \xi, k, \varepsilon) = \frac{1}{2\chi i} \exp(i\chi|x - \xi|), \quad \chi = \sqrt{k^2 + i\varepsilon k}. \quad (14)$$

Here the branch  $Re \chi \geq 0$  is taken.

Now the system:

$$w(0) = 0, \quad w(1) = 0 \quad (15)$$

with  $w_p$  given in (10) has a non-zero solution for all real  $k$ . The resonance curve corresponding to  $\varepsilon = 10^{-6}$  is depicted in Fig. 1c and d. Now this is a smooth curve with separated maximums at the positions of eigenvalues

The second regularizing technique is the following. Let us introduce the constant shift  $\Delta k$  between the wave numbers of the exciting source and the studied mode, i.e. instead of (10), let us take the particular solution in the form:

$$w_p = \Psi(x, \xi_0, k + \Delta k) = \frac{1}{2(k + \Delta k)i} \exp(i(k + \Delta k)|x - \xi_0|). \quad (16)$$

Now the linear system  $w(0) = w(1) = 0$  has non-zero solutions for all  $k$  except the eigenvalues  $k_n$  when the system becomes degenerate. However, due to iterative procedure of solution and rounding errors we never solve the system with the exact  $k_n$ . And we observe degeneration of the system as a considerable growth of the solution in a neighbourhood of the eigenvalues. The resonance curve corresponding to  $\Delta k = 1$  is depicted in Fig. 1 e and f. As with the first procedure we get a smooth resonance curve.

Below we will name these techniques as  $\varepsilon$ -procedure and  $k$ -procedure.

We use algorithm A throughout the paper to find the eigenvalues. Let us look for the eigenvalues on the interval  $[a, b]$  Then:

Algorithm A

- step 0: Choose the step size  $h > 0$ ;
- if  $F(a) > F(a+h)$  goto step 5;
- step 1:  $k_1 = a$ ;  $F1 = F(k_1)$ ;
- step 2:  $k_2 = k_1 + h$ ;  $F2 = F(k_2)$ ; if  $k_2 > b$  stop;
- step 3: if  $F2 > F1$  then  $[F1 = F2; k_1 = k_2]$ ; goto step 2;
- step 4: find the maximum point  $k_m$  of  $F(k)$  on  $[k_2 - 2h, k_2]$ ;
- step 5:  $k_1 = a$ ;  $F1 = F(k_1)$ ;
- step 6:  $k_2 = k_1 + h$ ;  $F2 = F(k_2)$ ;
- If  $k_2 > b$  stop;
- step 7: if  $F2 < F1$  then  $[F1 = F2; k_1 = k_2]$ ; goto step 6;
- else goto step 2.

Note that any univariate optimization procedure can be used at step 4. In particular, we applied Brent's method based on a combination of parabolic interpolation and bisection of the function near to the extremum (see [13], [14]). The step is taken  $h = 0.01$  through the paper except the cases specified. The data

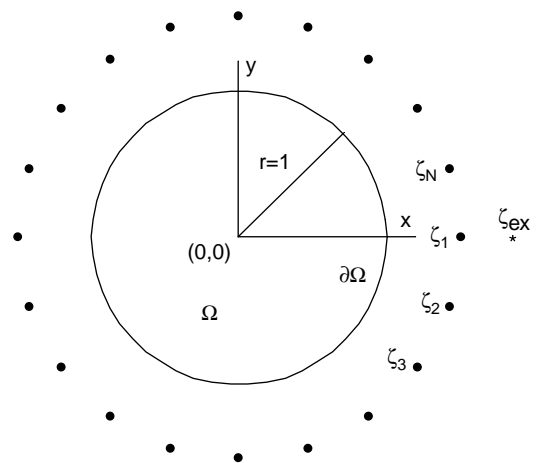


Fig. 4. Geometry of the eigenproblem in a circular domain.

Table 4  
Circle with radius  $r=1$  with Dirichlet boundary condition. The relative errors in calculations of the first 10 eigenvalues.  $\varepsilon$ -procedure with  $\varepsilon=10^{-6}$

$l$	$k_l^{(ex)}$	$N=10$	$N=15$	$N=20$	$N=25$	$N=30$
1	2.4048255577	–	$4.3 \times 10^{-11}$	$2.1 \times 10^{-11}$	$1.3 \times 10^{-12}$	$1.3 \times 10^{-12}$
2	3.8317059702	–	–	$5.1 \times 10^{-11}$	$2.2 \times 10^{-12}$	$1.6 \times 10^{-11}$
3	5.1356223072	–	–	$9.0 \times 10^{-10}$	$1.0 \times 10^{-9}$	$1.0 \times 10^{-9}$
4	5.5200781103	–	–	–	$1.4 \times 10^{-12}$	$2.2 \times 10^{-12}$
5	6.3801619049	–	–	–	$1.4 \times 10^{-9}$	$1.4 \times 10^{-9}$
6	7.0155866698	–	–	–	$1.7 \times 10^{-11}$	$1.9 \times 10^{-12}$
7	7.5883424466	–	–	–	$4.6 \times 10^{-10}$	$1.6 \times 10^{-9}$
8	8.4172441621	–	–	–	–	$2.6 \times 10^{-9}$
9	8.6537279129	–	–	–	–	$2.3 \times 10^{-11}$
10	8.7714838419	–	–	–	–	$3.0 \times 10^{-9}$

placed in Table 2 are obtained using the  $\varepsilon$ -procedure with  $\varepsilon=0.1, 10^{-3}, 10^{-6}$ . Other parameters are:  $\xi_1=-0.5, \xi_2=1.5, \xi_0=5$ . Here we place the relative errors

$$e_r = \frac{|k_i - k_i^{(ex)}|}{k_i^{(ex)}} \tag{17}$$

in the calculation of the first ten eigenvalues. Some results of the calculations we got using the k-procedure are presented in Table 3. The values  $\xi_1, \xi_2, \xi_0$  are the same as above.

**3. The two dimensional case**

In this section we are dealing with the problem which is an analog of the one considered above. Following the method presented, we consider the inhomogeneous boundary value problem:

$$\nabla^2 w + k^2 w = f(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathcal{R}^2, \tag{18}$$

$$w = 0, \quad \mathbf{x} \in \partial\Omega. \tag{19}$$

Let  $w_p$  be the particular solution corresponding to  $f$ . Then, applying MFS technique an approximate solution is looked for in the form of a linear combination:

$$w(\mathbf{x}|\mathbf{q}) = w_p(\mathbf{x}) + \sum_{n=1}^N q_n \Phi_n(\mathbf{x}), \tag{20}$$

where the trial functions  $\Phi_n(\mathbf{x})$  satisfy the homogeneous PDE. The free parameters  $q_n$  should be chosen to satisfy the boundary condition  $w(\mathbf{x}|\mathbf{q})=0, \mathbf{x} \in \partial\Omega$ . In particular the unknowns  $q_n$  are taken as a solution of the minimization problem:

$$\min_{\mathbf{q}} \sum_{i=1}^{N_c} \left\{ w_p(\mathbf{x}_i) + \sum_{n=1}^N q_n \Phi_n(\mathbf{x}_i) \right\}^2 \tag{21}$$

Here the points  $\mathbf{x}_i, i=1, \dots, N_c$  are distributed uniformly on the boundary. We take  $N_c$  approximately twice as large as the number of free parameters  $N$ . The problem is solved by the standard least squares procedure. Note that we get (21) as a result of discretization of the integral condition:

$$\min_w \int_{\partial\Omega} [w(\mathbf{x}|\mathbf{q})]^2 ds.$$

More details of this technique can be found, e.g. in [3–5].

For 2D Helmholtz operator  $\nabla^2 + k^2$  the following set of fundamental solutions can be used as the trial functions:

$$\Phi_n(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x} - \zeta_n|). \tag{22}$$

Here  $H_0^{(1)}(z)$ , denotes the Hankel functions of the first kind and zero order. The source points  $\zeta_n = (\xi_n, \eta_n), n=1, 2, \dots, N$  are placed outside the solution domain as it is depicted in Fig. 2. This is the so-called Kupradze basis [15]. As a particular solution corresponding to the exciting source we take the same fundamental solution

$$w_p(\mathbf{x}) = \Phi_{ex}(\mathbf{x}, \zeta_{ex}, k) \equiv H_0^{(1)}(k|\mathbf{x} - \zeta_{ex}|) \tag{23}$$

with  $\zeta_{ex}$  placed outside the solution domain.

When we deal with problems in multiply connected domains, the same trial functions can be used. And the source points should be placed also inside each hole as it is depicted in Fig. 3 a).

As an alternative approach one can use the special trial functions associated with each hole:

$$\Psi_{s,1}(\mathbf{x}) = H_0^{(1)}(kr_s), \quad \Psi_{s,2n+1}(\mathbf{x}) = H_n^{(1)}(kr_s) \cos n\theta_s, \tag{24}$$

$$\Psi_{s,2n}(\mathbf{x}) = H_n^{(1)}(kr_s) \sin n\theta_s.$$

Here  $r_s = |\mathbf{x} - \mathbf{x}_s|, \theta_s$  is the local polar coordinate system with the origin at the point  $\mathbf{x}_s$  of multipoles location (see Fig. 3b)).

Table 5  
Circular domain with the radius  $r=1$ . Dirichlet condition. The relative errors in calculations of the eigenvalues.  $\varepsilon$ -procedure with varying  $\varepsilon, N=25$

$l$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-8}$
1	$6.4 \times 10^{-6}$	$6.0 \times 10^{-10}$	$7.3 \times 10^{-12}$	$4.9 \times 10^{-11}$
2	$2.4 \times 10^{-6}$	$1.9 \times 10^{-10}$	$2.0 \times 10^{-11}$	$4.3 \times 10^{-11}$
3	$3.2 \times 10^{-6}$	$1.4 \times 10^{-9}$	$1.0 \times 10^{-9}$	–
4	$9.0 \times 10^{-7}$	$1.6 \times 10^{-10}$	$1.3 \times 10^{-11}$	–
5	$1.1 \times 10^{-6}$	$1.6 \times 10^{-9}$	$1.4 \times 10^{-9}$	–
6	$6.5 \times 10^{-7}$	$1.5 \times 10^{-10}$	–	–
7	$4.9 \times 10^{-7}$	$4.8 \times 10^{-10}$	–	–
8	$2.7 \times 10^{-6}$	$1.1 \times 10^{-9}$	–	–
9	$4.9 \times 10^{-7}$	$5.9 \times 10^{-9}$	–	–
10	$5.2 \times 10^{-6}$	–	–	–

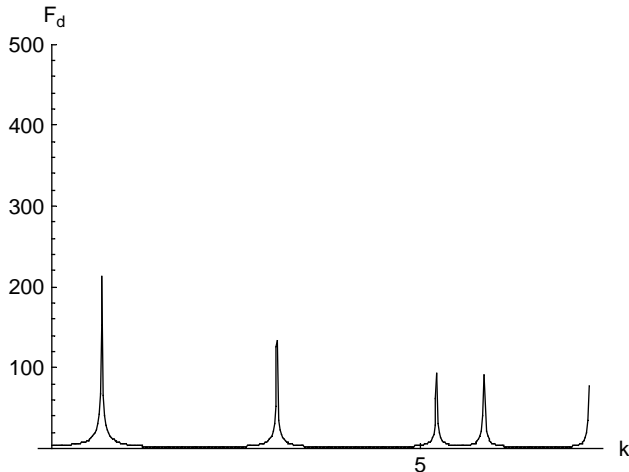


Fig. 5. Circular membrane with the radius 1. Dirichlet conditions.  $\varepsilon$ —procedure with  $\varepsilon = 10^{-2}$ .

This is so-called Vekua basis [16,17] or multipole expansion. It is proven that every regular solution of the 2D Helmholtz equation in a domain with holes can be approximated with any desired accuracy by linear combinations of such functions if the origin  $\mathbf{x}_s$  of a multipole is inside every hole. In this case instead of (20) we use:

$$w(\mathbf{x}|\mathbf{q}, \mathbf{p}_s) = w_p(\mathbf{x}) + \sum_{n=1}^N q_n \Phi_n(\mathbf{x}) + \sum_{s=1}^S \sum_{m=1}^M p_{s,m} \Psi_{s,m}(\mathbf{x}), \quad (25)$$

$$\mathbf{q} = (q_n)_{n=1}^N, \quad \mathbf{p}_s = (p_{s,m})_{m=1}^M,$$

where  $S$  is the number of holes and  $M$  is the number of terms in each multipole expansion.

When the  $\varepsilon$ -smoothing procedure is applied, then according to the method presented we consider the problem:

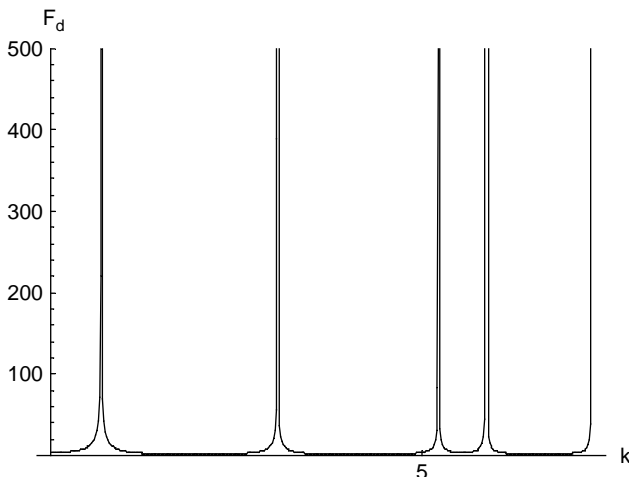


Fig. 6. Circular membrane with the radius 1. Dirichlet conditions.  $\varepsilon$  - procedure with  $\varepsilon = 10^{-4}$ .

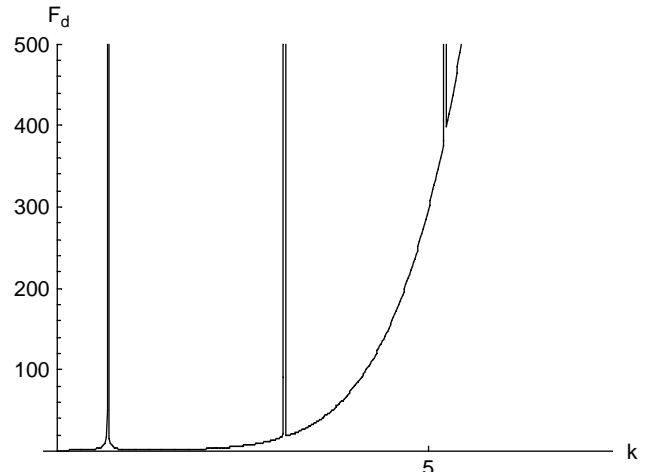


Fig. 7. Circular membrane with the radius 1. Dirichlet conditions.  $\varepsilon$ —procedure with  $\varepsilon = 10^{-8}$ .

$$\nabla^2 w + (k^2 + i\varepsilon k)w = 0, \quad \mathbf{x} \in \Omega, \quad w(\mathbf{x}) = -w_p(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \quad (26)$$

with some small  $\varepsilon > 0$ . Note that since we shift the spectra of the operator from the real axis the problem has a unique nonzero solution for all real  $k$ . The trial functions (22) should be also modified:

$$\Phi_n(\mathbf{x}) = H_0^{(1)}(\chi|\mathbf{x} - \zeta_n|), \quad \chi(k, \varepsilon) = \sqrt{k^2 + i\varepsilon k}. \quad (27)$$

Applying the  $k$ -procedure we modify the particular solution which should be taken in the form:

$$w_p(\mathbf{x}) = \Phi_{ex}(\mathbf{x}, \tilde{k}) \equiv H_0^{(1)}(\tilde{k}|\mathbf{x} - \zeta_{ex}|), \quad \tilde{k} = k + \Delta k. \quad (28)$$

Note that we do not need an explicit expression for the exciting source because we use the corresponding particular solution only.

#### 4. Numerical examples

In this section, numerical experiments are carried out to examine the method presented and to compare the two regularizing techniques introduced above. The data mainly are taken from [8–11]. The Dirichlet condition is used on all the boundaries through this section. In all the numerical examples considered in this section the resonance curve  $F(k)$  is computed using  $N_t = 15$  testing points  $\mathbf{x}_{t,i} \in \Omega$  (see (11)). They are distributed inside  $\Omega$  with the help of RNUF generator of pseudorandom numbers from the Microsoft IMSL Library. Note that in all the calculations presented in this section all the values are taken in the dimensionless form. Examples are the eigenvalues  $k$  placed in the tables are reduced to the dimensionless form with the help of some typical dimension  $a$  of the problem considered.

##### 4.1. Simply connected domains

*Example 1* A circular domain with the radius  $r = 1$  subjected to Dirichlet boundary condition is considered as it is depicted

Table 6  
Circle with radius  $r=1$  with Dirichlet boundary condition. The relative errors in calculations of the first ten eigenvalues.  $k$ -procedure with  $\Delta k=1$

$i$	$k_i^{(ex)}$	$N=15$	$N=20$	$N=25$	$N=30$	$N=35$
1	2.4048255577	$8.9 \times 10^{-9}$	$8.9 \times 10^{-9}$	$8.9 \times 10^{-9}$	$8.9 \times 10^{-9}$	$8.9 \times 10^{-9}$
2	3.8317059702	$6.5 \times 10^{-6}$	$1.1 \times 10^{-7}$	$5.2 \times 10^{-9}$	$3.3 \times 10^{-9}$	$3.5 \times 10^{-9}$
3	5.1356223072	$5.3 \times 10^{-5}$	$5.4 \times 10^{-7}$	$7.8 \times 10^{-9}$	$5.1 \times 10^{-10}$	$5.0 \times 10^{-10}$
4	5.5200781103	$3.6 \times 10^{-9}$	$3.2 \times 10^{-9}$	$2.9 \times 10^{-9}$	$2.8 \times 10^{-9}$	$2.7 \times 10^{-9}$
5	6.3801619049	$8.2 \times 10^{-4}$	$3.2 \times 10^{-6}$	$4.4 \times 10^{-8}$	$4.5 \times 10^{-9}$	$4.5 \times 10^{-9}$
6	7.0155866698	$9.0 \times 10^{-2}$	$6.6 \times 10^{-7}$	$1.3 \times 10^{-8}$	$5.4 \times 10^{-9}$	$4.5 \times 10^{-9}$
7	7.5883424466	$7.6 \times 10^{-2}$	$3.7 \times 10^{-5}$	$2.0 \times 10^{-7}$	$5.1 \times 10^{-9}$	$3.4 \times 10^{-9}$
8	8.4172441621	0.1	$1.0 \times 10^{-5}$	$5.6 \times 10^{-8}$	$1.8 \times 10^{-9}$	$2.5 \times 10^{-9}$
9	8.6537279129	0.1	$1.2 \times 10^{-11}$	$1.8 \times 10^{-11}$	$1.5 \times 10^{-12}$	$1.5 \times 10^{-12}$
10	8.7714838419	$4.3 \times 10^{-2}$	$5.8 \times 10^{-4}$	$1.8 \times 10^{-6}$	$1.7 \times 10^{-8}$	$3.1 \times 10^{-9}$

in Fig. 4. The exact eigenvalues  $k_i^{(ex)}$  are the roots of the equation  $J_n(k)=0$ , where  $J_n$  is the Bessel function of the first kind and of order  $n$ .

First, we apply the  $\varepsilon$ - procedure. The exciting source corresponding to particular solution (23) is placed at the position  $\zeta_{ex}=(5,0)$ ; the source points of the MFS trial functions (22) are located on the circle with the radius  $R=2$ . The results shown in Table 4 correspond to  $\varepsilon=10^{-6}$ . The calculations were performed with different  $N$ —the number of the free parameters in linear combination (20). The line—in a cell indicates that the solution process failed with these parameters.

The role of the parameter  $\varepsilon$  is shown in Table 5. Here we fix  $N=25$  and change  $\varepsilon$ . The regularizing parameter  $\varepsilon$  coarsens the system. For a large  $\varepsilon$  we can calculate all the eigenvalues  $k_i, i=1, \dots, 10$  but the accuracy is not very high. When  $\varepsilon$  decreases, the accuracy in determining  $k_i$  increases but it fails for large  $i$ . Figs. 5–7 correspond to the data placed in the table. For  $\varepsilon=10^{-2}$  the resonance peaks are spread because of the friction. When  $\varepsilon$  decreases the peaks become sharper and more narrow. Besides for  $\varepsilon=10^{-8}$  the peaks corresponding to  $k_i, i>2$  are placed on the sharply rising part of the resonance curve. As a result the algorithm A ‘jumps over’ the eigenvalues and one should decrease the step parameter  $h$  to capture the maxima. As it is shown in Table 5, for  $\varepsilon=10^{-8}$  the algorithm finds  $k_1$  and  $k_2$  with  $h=0.01$ . When  $h$  is reduced to 0.001 then the algorithm also gives the eigenvalues  $k_3$  and  $k_4$ . To get  $k_i, i=1, \dots, 10$  one should take  $h=0.0001$ . However, the algorithm becomes highly expensive in the CPU time.

The data placed in Table 6 correspond to the same problem with the application of the second regularization procedure with the shift  $\Delta k=1$ . The method is not sensitive to the value of the shift. We got approximately the same results for all  $\Delta k$  in the interval  $[0.1, 10]$ .

Example 2 Next, we consider the case when  $\Omega$  is the unit square with the same Dirichlet boundary condition (see Fig. 8). This problem has an analytical solution:  $k^{(ex)} = \pi\sqrt{i^2 + j^2}, i, j=1, 2, \dots, \infty$ . In Table 7, we present the results of applying both regularizing techniques. The exciting source is placed at the position  $\zeta_{ex}=(5,5)$ .

Comparing these two regularizing techniques it should be noted that they provide approximately the same accuracy. However, the regularization with the help of an additional friction term leads to calculations of Bessel functions with a complex argument and is computationally demanding.

4.2. doubly connected domain

Example 3 In this subsection, first, we consider an annular case (Fig. 9). The inner and outer radii of an annular domain are  $r_1=0.5$  and  $r_2=2$  correspondingly. The singular points are distributed at the circles with the radii  $a=5$  (outside the domain) and  $b=0.3$  (inside the hole). The numbers of the singular points on each auxiliary contour is equal to  $N$ . The exciting source is placed at the position  $\zeta_{ex}=(10,10)$ . In Table 8 we place the relative errors (17) in calculation of the first ten eigenvalues of the problem described. The values  $k_i^{(ex)}$  are obtained numerically as roots of the equation:

$$J_n(r_1 k)Y_n(r_2 k) - J_n(r_2 k)Y_n(r_1 k) = 0 \tag{29}$$

The results correspond to the  $k$ -procedure with the shift  $\Delta k=1$ . Here,  $J_n$  and  $Y_n$  are the Bessel functions of the first and second kinds and of order  $n$ .

Example 4 An eccentric case is considered as the second example. The radii of the inner and outer circular boundaries are the same. But the center of the hole is placed at  $(-0.5,0)$ . The eigenvalues are calculated with a different number of the free parameters  $N$ . The results are placed in Table 9. It appears that the values of  $k_i$  placed in the last column of Table 9 have converged.

Example 5 In this case we considered the doubly connected domain with the outer boundary of the square with the side 4

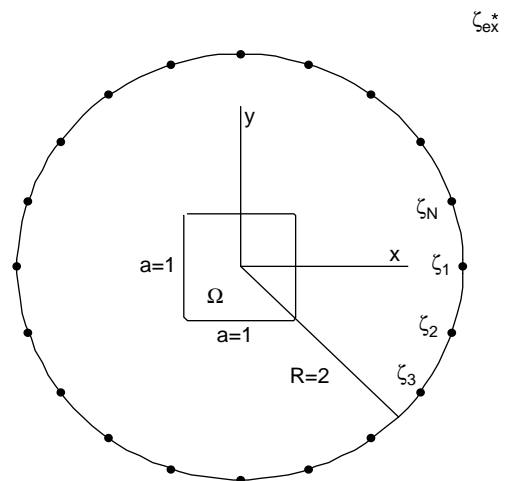


Fig. 8. Geometry of the eigenproblem in the square.

Table 7

Unit square with Dirichlet boundary condition. The relative errors in calculation of the first ten eigenvalues. The two types of the regularizing procedure. Parameters:  $\varepsilon = 10^{-6}$  and  $\Delta k = 1$ .

$i$	$k_i^{(ex)}$	Friction $\varepsilon = 10^{-6}$		Shift $\Delta k = 1$	
		$N = 30$	$N = 40$	$N = 30$	$N = 40$
1	4.4428829382	$6.1 \times 10^{-12}$	$6.0 \times 10^{-12}$	$2.0 \times 10^{-8}$	$1.9 \times 10^{-8}$
2	7.0248147310	$7.7 \times 10^{-11}$	$5.8 \times 10^{-12}$	$1.2 \times 10^{-9}$	$1.2 \times 10^{-11}$
3	8.8857658763	$1.0 \times 10^{-7}$	$2.0 \times 10^{-12}$	$1.2 \times 10^{-7}$	$1.4 \times 10^{-11}$
4	9.9345882658	$7.5 \times 10^{-11}$	$3.4 \times 10^{-12}$	$9.4 \times 10^{-9}$	$4.6 \times 10^{-10}$
5	11.3271733991	$3.5 \times 10^{-7}$	$8.0 \times 10^{-9}$	$3.4 \times 10^{-7}$	$7.0 \times 10^{-10}$
6	12.9531183434	$6.3 \times 10^{-6}$	$7.6 \times 10^{-9}$	$6.4 \times 10^{-6}$	$3.1 \times 10^{-9}$
7	13.3286488145	$2.6 \times 10^{-5}$	$6.5 \times 10^{-8}$	$2.5 \times 10^{-5}$	$6.2 \times 10^{-8}$
8	14.0496294621	$3.3 \times 10^{-6}$	$1.1 \times 10^{-7}$	$3.4 \times 10^{-6}$	$2.2 \times 10^{-9}$
9	15.7079632679	$3.5 \times 10^{-5}$	$3.9 \times 10^{-7}$	$3.5 \times 10^{-5}$	$3.2 \times 10^{-7}$
10	16.0190422444	$4.1 \times 10^{-5}$	$1.7 \times 10^{-6}$	$4.0 \times 10^{-5}$	$1.4 \times 10^{-6}$

and the inner boundary of the circle with the radius 0.5. The centre of the circular hole is placed at  $(-0.5, 0)$ . The eigenvalues are calculated with a different number of the free parameters  $N$ . The step of the algorithm A is taken  $h = 0.005$ . The results are placed in Table 10.

*Example 6* In this example, doubly connected region with the inner region of vanishing maximal dimension is concerned. The geometry of the problem is the same as in Example 3. However, here we consider the case of very small inner holes. In particular, we take  $r_1 = 10^{-1}, 10^{-2}, 10^{-3}$  with the same fixed  $r_2 = 2$ . Now the Kupradze type basis functions (22) are unfit to approximate solution in a neighbourhood of the hole because when the singular points, say  $\zeta_i, \zeta_j$  of two sources are very close, the corresponding functions  $\Phi_i(\mathbf{x}) \Phi_j(\mathbf{x})$  become indistinguishable and the collocation matrix has two identical columns. Here, we use a combined basis which includes the trial functions (22) with the singular points placed on an auxiliary circular contour outside the solution domain and a multipole expansion with the origin at the center of the hole. Thus, we look for an approximate solution in the form:

$$w(\mathbf{x}|\mathbf{q}, \mathbf{p}) = w_p(\mathbf{x}) + \sum_{n=1}^N q_n \Phi_n(\mathbf{x}) + \sum_{m=1}^M p_m \Psi_m(\mathbf{x}).$$

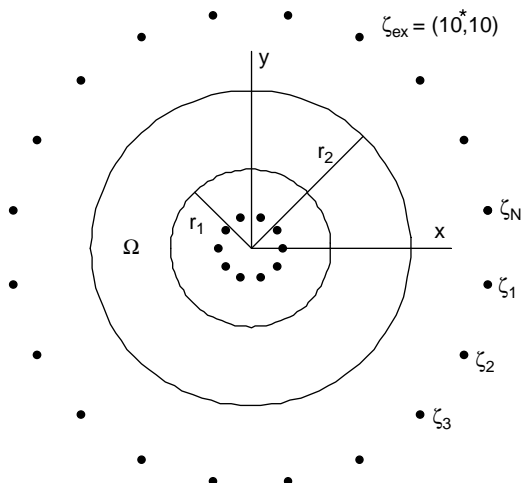


Fig. 9. Geometry configuration and distribution of the sources in the annular eigenproblem.

The data presented in Table 11 correspond to the number of sources on the outer auxiliary circular contour  $N = 50$ . The number of terms in multipole expansion  $M$  varies from  $M = 11 (r_1 = 10^{-1})$  to  $M = 5 (r_1 = 10^{-3})$ . The exciting source is placed at the position  $\zeta_{ex} = (10, 10)$ . We use the  $k$ -procedure with the shift  $\Delta k = 1$ .

We would like to draw the readers' attention to the fact that the method presented can separate the very close eigenvalues:  $k_4^{(ex)} = 3.1900833197$  and  $k_5^{(ex)} = 3.2126996563$  (see data correspond to  $r_1 = 10^{-1}$ ). However, here the value  $h$  in the algorithm placed in Section 2 is taken  $h = 0.001$ . The detailed discussion of the application of Vekua basis for Helmholtz equation can be found in [17].

### 5. Concluding remarks

In this paper, a new boundary method for the Helmholtz eigenproblem is proposed. This is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined using the growth of amplitude of oscillations near these frequencies. The calculation of the eigenvalue problem is reduced to a sequence of inhomogeneous problems with the differential operator studied. The method shows a high precision in simply and doubly connected domains.

Note that the method is not very sensitive to the position of the external source. When the distance from the exciting source

Table 8

Two concentric circles with radiuses  $r_1 = 0.5$  and  $r_2 = 2$  with Dirichlet boundary condition. The relative errors in calculation of the first ten eigenvalues.  $k$ -procedure with  $\Delta k = 1$

$i$	$k_i^{(ex)}$	$N = 20$	$N = 30$	$N = 40$
1	2.0488427696	$1.1 \times 10^{-9}$	$2.4 \times 10^{-9}$	$1.3 \times 10^{-9}$
2	2.2237527968	$2.5 \times 10^{-9}$	$8.4 \times 10^{-9}$	$8.4 \times 10^{-9}$
3	2.6599339351	$4.5 \times 10^{-8}$	$3.8 \times 10^{-10}$	$3.4 \times 10^{-10}$
4	3.2132363558	$7.7 \times 10^{-7}$	$7.1 \times 10^{-10}$	$7.3 \times 10^{-10}$
5	3.7992037166	$2.5 \times 10^{-5}$	$1.6 \times 10^{-9}$	$1.3 \times 10^{-9}$
6	4.1618868190	$2.5 \times 10^{-9}$	$2.5 \times 10^{-9}$	$2.5 \times 10^{-9}$
7	4.2684607541	$2.6 \times 10^{-6}$	$5.5 \times 10^{-9}$	$5.6 \times 10^{-9}$
8	4.3867216590	$4.9 \times 10^{-4}$	$6.2 \times 10^{-9}$	$5.3 \times 10^{-9}$
9	4.5721805095	$2.7 \times 10^{-5}$	$2.7 \times 10^{-9}$	$2.7 \times 10^{-9}$
10	4.9682310314	$4.3 \times 10^{-3}$	$2.3 \times 10^{-8}$	$2.1 \times 10^{-11}$



Table 9  
Two eccentric circles with Dirichlet boundary condition. The first ten eigenvalues computed using different number of free parameters  $N$ .  $k$ -procedure with  $\Delta k=1$

$i$	$N=20$	$N=25$	$N=30$	$N=35$	$N=40$
1	1.73613039	1.73613035	1.73613035	1.73613035	1.73613035
2	2.13198028	2.13198023	2.13198023	2.13198023	2.13198023
3	2.46105944	2.46106033	2.46106033	2.46106034	2.46106033
4	2.76965197	2.76965147	2.76965146	2.76965146	2.76965146
5	2.96340393	2.96340534	2.96340536	2.96340536	2.96340536
6	3.31786162	3.31785332	3.31785316	3.31785316	3.31785316
7	3.37637124	3.37637601	3.37637598	3.37637598	3.37637598
8	3.85682143	3.85676705	3.85676681	3.85676682	3.85676682
9	4.27722493	4.27707102	4.27707123	4.27707122	4.27707123
10	4.41796066	4.42109506	4.42109921	4.42109929	4.42109928

Table 10  
Circle in square with Dirichlet boundary condition. The first six eigenvalues computed using different number of free parameters  $N$ .  $k$ -procedure with  $\Delta k=1$

$i$	$N=25$	$N=30$	$N=35$	$N=40$
1	1.58799	1.58800	1.58802	1.58803
2	1.92825	1.92875	1.92875	1.92879
3	2.21677	2.21732	2.21746	2.21744
4	2.35166	2.35273	2.35276	2.35288
5	2.64250	2.64363	2.64324	2.64333
6	2.89311	2.89405	2.89450	2.89446

Table 11  
Circle with a small hole. Dirichlet boundary condition. The outer radius:  $r_2=2$  The relative errors in calculation of the first ten eigenvalues.  $k$ -procedure with  $\Delta k=1$

$i$	$r_1=0.1$ $N=50$ , $M=11$		$r_1=0.01$ $N=50$ , $M=7$		$r_1=0.001$ $N=50$ , $M=5$	
	$k_i^{(ex)}$	$e_r$	$k_i^{(ex)}$	$e_r$	$k_i^{(ex)}$	$e_r$
1	1.5322036536	$1.9 \times 10^{-8}$	1.3709447159	$2.5 \times 10^{-8}$	1.3148533741	$2.0 \times 10^{-8}$
2	1.9301625755	$5.8 \times 10^{-9}$	1.9160005377	$5.4 \times 10^{-9}$	1.9158544900	$5.4 \times 10^{-9}$
3	2.5680354360	$1.6 \times 10^{-9}$	2.5678112121	$1.6 \times 10^{-9}$	2.5678111892	$1.5 \times 10^{-9}$
4	3.1900833197	$1.3 \times 10^{-11}$	2.9632630840	$5.3 \times 10^{-9}$	2.8883437835	$2.8 \times 10^{-9}$
5	3.2126996563	$7.4 \times 10^{-9}$	3.1900809955	$2.9 \times 10^{-12}$	3.1900809955	$1.1 \times 10^{-10}$
6	3.5522743165	$3.7 \times 10^{-10}$	3.5082790790	$2.3 \times 10^{-12}$	3.5077982552	$3.0 \times 10^{-11}$
7	3.7941712382	$1.2 \times 10^{-11}$	3.7941712738	$1.0 \times 10^{-9}$	3.7941712738	$1.2 \times 10^{-11}$
8	4.2101115868	$9.0 \times 10^{-12}$	4.2086222910	$7.6 \times 10^{-12}$	4.2086221329	$5.9 \times 10^{-12}$
9	4.3857419081	$4.4 \times 10^{-12}$	4.3857419733	$1.1 \times 10^{-11}$	4.3857419733	$1.2 \times 10^{-12}$
10	4.8805392651	$1.0 \times 10^{-11}$	4.5543927267	$1.3 \times 10^{-9}$	4.4650868082	$3.6 \times 10^{-10}$

Table 12  
A circle with the radius 1. The relative errors in calculations of the first 3 eigenvalues.  $k$  - procedure with  $\Delta k=0.001$

$i$	$R=1.1$	$R=2$	$R=5$
1	$6 \times 10^{-5}$	$6 \times 10^{-10}$	$6 \times 10^{-10}$
2	$3 \times 10^{-5}$	$3 \times 10^{-10}$	$3 \times 10^{-10}$
3	$3 \times 10^{-5}$	$2 \times 10^{-9}$	$2 \times 10^{-9}$

Table 13  
Circular domain with Dirichlet conditions. The number of the source points  $N=30$ .  $\varepsilon$ -procedure

$i$	$\varepsilon=10^{-1}$		$\varepsilon=10^{-4}$		$\varepsilon=10^{-6}$	
	$e_r$	$F(k_i)$	$e_r$	$F(k_i)$	$e_r$	$F(k_i)$
1	$4 \times 10^{-4}$	0.701	$4 \times 10^{-10}$	0.701	$5 \times 10^{-12}$	0.701
2	$2 \times 10^{-4}$	0.652	$1 \times 10^{-10}$	0.654	$6 \times 10^{-11}$	0.654
3	$9 \times 10^{-5}$	0.509	$9 \times 10^{-10}$	0.516	$1 \times 10^{-9}$	0.516

to the excited domain is too large then there are difficulties in calculation of the Bessel functions. So the sources of excitation are placed at distances of 5–10 times the diameter of a domain from the domain itself. Besides the particular solution in the form of the plane wave  $w_p = \exp[i(x \cos \theta + y \sin \theta)]$  can also be used. In fact this means that the exciting source is removed to infinity. The MFS sources are located at distances of 2–5 times the diameter of a domain from its boundary. In this case the results are the same. When the sources are placed close to the boundary this decreases the accuracy. The data placed in Table 12 correspond to the eigenvalue problem for the circle with the radius 1 with Dirichlet boundary conditions. The MFS sources are placed on the circle with the radius  $R$ . Here we place the relative errors (17) in the calculation of the first 3 eigenvalues.

Comparing the method with the technique based on computations of the determinant of the system, the following circumstances should be taken into account. Since the MFS is highly ill conditioned, the determinant is very small. Indeed, let us consider again the same eigenvalue problem which is described in Example 1, i.e. Helmholtz equation in the circle with the radius 1 and Dirichlet boundary condition. We take the number of the sources  $N$  equal to the number of the collocation points on the boundary. Thus, we get a square matrix of the problem  $A(k, N)$  and can calculate the determinant  $|\det A(k, N)|$ . Placing the sources on the circle with radius 2 and taking  $k = 1$  we get:  $|\det A(1, 20)| = 3 \times 10^{-47}$ ,  $|\det A(1, 30)| = 4 \times 10^{-117}$ ,  $|\det A(1, 40)| = 3 \times 10^{-217}$ . The wave number  $k = 1$  is not an eigenvalue of the problem. This is the ‘background’ value between extremums and one looks for the minima of  $|\det A(k, N)|$  on such background. So, using this technique one operates with values of the order  $\sim 10^{-50} - 10^{-500}$ , see [11, 18] for more detailed information.

At the same time let us calculate the norm function  $F(k)$  which is used to obtain the eigenvalues in the method presented. We present the values of  $F(k)$  when  $k$  is close to eigenvalue in Table 13. Here the number of the sources is fixed  $N = 30$  and the smoothing parameter  $\varepsilon$  is varied. Here,  $e_r$  is the relative error (17) in determining of the approximated eigenvalue  $k_i$  and  $F(k_i)$  denotes the value of the norm function at this approximated eigenvalue. So, in the framework of the method presented we always deal with the values, which can be handled on PC with a single precision. The method is easy to program and not expensive in the CPU time. All the calculations presented in the paper were performed using 366 MHz PC. For example, it takes approximately 3 min. to calculate the first 5 eigenvalues in Example 1 using the  $\varepsilon$ -procedure.

Although the method is demonstrated for 1D and 2D Dirichlet problems, the idea can be extended quite simply to

the 3D case and other boundary conditions. It seems possible to extend this technique to eigenproblems with other differential equation, e.g. to problems of plates and shells vibration. This will be the subject of further investigations.

## References

- [1] Brebbia CA. The boundary element method for engineers. London: Pentech Press; 1978.
- [2] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques—theory and applications in engineering. Berlin: Springer; 1984.
- [3] Fairweather G, Karageorghis A. The method of fundamental solutions for elliptic boundary value problems. Adv Comp Math 1998;9:69–95.
- [4] Golberg MA, Chen CS. Discrete projection methods for integral equations. Southampton: Computational Mechanics Publications; 1997.
- [5] Golberg MA, Chen CS. The method of fundamental solutions for potential, Helmholtz and diffusion problems. In: Golberg MA, editor. Boundary inetgral methods—numerical and mathematical aspects. Computational Mechanics Publications; 1998. p. 103–76.
- [6] Kondapalli PS, Shippy DJ. Analysis of acoustic scattering in fluids and solids by the method of fundamental solutions. J Acoust Soc Am 1992;91:1844–54.
- [7] Karageorghis A. The method of fundamental solutions for the calculation of the eigenvalues of the Helmholtz equation. Appl Math Lett 2001;14:837–42.
- [8] Chen JT, Lin JH, Kuo SR, Chyuan SW. Boundary element analysis for the Helmholtz eigenvalues problems with a multiply connected domain. Proc R Soc Lond A 2001;457:2521–46.
- [9] Chen JT, Liu LW, Hong HK. Spurious and true eigensolutions of Helmholtz BIEs and BEMs for a multiply connected problem. Proc R Soc Lond A 2003;459:1897–924.
- [10] Chen JT, Chang MH, Chen KH, Lin SR. The boundary collocation method with meshless concept for acoustic eigenanalysis of two-dimensional cavities using radial basis function. J Sound Vibr 2002;257:667–711.
- [11] Chen JT, Chen IL, Lee YT. Eigensolutions of multiply connected membranes using the method of fundamental solutions. Eng Anal Bound Elem 2005;29:166–74.
- [12] Morse PM, Feshbach H. Methods of theoretical physics, vol. 1–2. New York: McGraw-Hill; 1953.
- [13] Press WH, Teukolsky SA, Vetterling WT, Flannery BP. Numerical recipes in C++. 2nd ed. Cambridge: Cambridge University Press; 2002.
- [14] Brent RP. Algorithms for minimization without derivatives. Englewood Cliffs, NJ: Prentice-Hall; 1973.
- [15] Kupradze VD. On approximate solution of problems in mathematical physics. Russian Math Surv 1967;22:58–108.
- [16] Vekua IN. New methods for solving elliptic equations. Amsterdam: North-Holland; 1967.
- [17] Hafner C. The generalized multipole technique for computational electromagnetics. Boston: Artech House Books; 1990.
- [18] Alves CJS, Antunes PRS. The method of fundamental solutions applied to the calculation of eigenfrequencies and eigenmodes of 2D simply connected shapes. CMC: Computers, Materials & Continua 2005.