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Research note

General solutions and fundamental solutions of varied orders to the vibrational thin, the Berger, and the Winkler plates

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Abstract

In this note, we derive the general and the fundamental solutions of varied orders of vibrational thin plate, Berger plate, and Winkler plate. These solutions are of important use in the multiple reciprocity BEM, dual reciprocity BEM, boundary particle method, boundary knot method, and a variety of radial basis function techniques.

Keywords: Fundamental solution; General solution; Multiple reciprocity method; Radial basis function

1. Introduction

In recent years, the multiple reciprocity boundary element method (MR-BEM) [1] has attracted increasing attention due to its striking advantage being a truly boundary-only method for a variety of inhomogeneous problems. To the authors' best knowledge, the method may be the only BEM technique which does not require in general any inner nodes to calculate inhomogeneous problems. The MR-BEM approximates the particular solution by a sum of high-order homogeneous solutions, which are evaluated by using the high-order fundamental solutions. Thus, the high-order fundamental solution plays a central role in this technique. In the literature, the high-order fundamental solutions of the Laplace operator are often chosen to solve various problems. For problems having some particular properties such as periodicity and directional preference, the use of the high-order solutions of other differential operators, however, may be more efficient and stable.

On the other hand, Chen [2] recently developed a truly boundary-only meshfree boundary particle method (BPM), which also evaluates the particular solution via the multiple reciprocity method. The BPM differs from the MR-BEM in that the method uses the high-order general solution instead of the fundamental solutions in the collocation formulation. In addition, a recursive multiple reciprocity scheme is also developed to reduce computing cost dramatically. On the other hand, in recent decade the radial basis functions (RBF) has been found to be a powerful approach to construct truly meshfree numerical techniques and are widely used in the dual reciprocity boundary element method (DR-BEM) [3], the method of fundamental solution (MFS) [4], and the boundary knot method (BKM) [5]. The high-order general and the fundamental solutions of partial differential equations (PDEs) are in fact the RBF and can be used in a variety of RBF-based methods, such as the Kansa method, DR-BEM, MFS, and BKM. This highlights the importance of these operator-dependent kernel solutions.

Besides the well-known high-order fundamental solution of the Laplace operator, Itagaki [6] and Chen [7], respectively, find the explicit high-order fundamental solutions of Helmholtz, modified Helmholtz, and steady convection–diffusion operators. This note is to derive the high-order general and fundamental solutions of vibrational thin plate, Berger plate, and Winkler plate based on Chen [8]. In particular, we also first give the zero-order general solution of the Berger and the Winkler plates. In the following Section 2, we give a brief definition of the general and the fundamental solutions for the sake of completeness. Then the Sections 3–5 present the high-order general

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and fundamental solutions for vibrational thin plate, Berger plate, and Winkler plate, respectively. Finally, in Section 6, we conclude this communication with some remarks.

2. General solution and fundamental solution

Without loss of any generality, the general solution $u^{\#}$ and the fundamental solution u^{*} of a differential operator $L\{\}$ have to satisfy, respectively

$$L\{u^{\#}(r)\} = 0, \tag{1}$$

$$L\{u^{*}(r)\} + \Delta_i = 0, \tag{2}$$

where $r=\|x-x_j\|$, and Δ_i represents the Dirac delta function which goes to infinity at the origin point x_i and is equal to zero elsewhere. In contrast, it is seen from Eq. (1) that the general solution at origin has a limited value rather than zero and infinity. The general solution of a differential operator differs essentially from its corresponding fundamental solution in that the former is non-singular everywhere, while the latter is singular at origin. The general solutions are actually infinitely continuous. It is noted that since the differential operators concerned in this study do not have a preferred direction under isotropic media, their general fundamental solutions only involve the radial distance. Otherwise, some other generalized distance variables will be included in the solution expression, e.g. as in those of the convection–diffusion equation [7].

The solution satisfying Eqs. (1) or (2) is called the zero-order general solution [2] or fundamental solution [1], while the mth order general and fundamental solutions need respectively satisfy

$$L^{m}\{u^{\#}(r)\}\} = 0, \tag{3}$$

$$L^{m}\{u^{*}(r)\} + \Delta_i = 0, \tag{4}$$

where $L^{m}\{\}$ denotes the rth order, operator of $L\{\}$, say $L^{1}\{\}=L\{L\{\}\}$, $L^{2}\{\}=L\{L^{1}\{\}\}$.

3. Thin plate vibration

The operator of thin plate vibration is given by

$$L_{\mathrm{T}}\{u\} = \nabla^{4}u - \lambda^{4}u \tag{5}$$

Its zero-order general and fundamental solutions are known in the literature as

$$u^{\#}_{\mathrm{T0}}(r) = J_0(\lambda r) + I_0(\lambda r), \tag{6}$$

$$u^{*}_{\mathrm{T0}}(r) = Y_0(\lambda r) + K_0(\lambda r), \tag{7}$$

where $J_0(r)$ and $Y_0(r)$ are the zero-order Bessel functions of the first and the second kinds, respectively; and I_0 and K_0 the zero-order modified Bessel function of the first and the second kinds, respectively. Here (6) and (7) omit the constant coefficients. Chen et al. [9] applied the general solution (6) to obtain very accurate solutions of harmonic vibration of thin plates. In most BEM literature, only $Y_0(r)$ in formula (7) is chosen as the fundamental solution. Ref. [10] discusses the essential concept of the complete fundamental solution. For instance, the 2D Laplacian has the essential fundamental solution $-\ln(r)/2\pi$ and the complete fundamental solution $-(\ln(r)+C)/2\pi$, where C is a constant. The standard BEM only uses the former. In this study, we do not touch this issue. The fundamental solution given in this study can be considered a complete fundamental solution.

The operator (5) can be decomposed as

$$\nabla^{4}u - \lambda^{4}u = (\nabla^{2} + \lambda)(\nabla^{2} - \lambda)u. \tag{8}$$

Namely, the operator (5) can be considered a product of the Helmholtz and the modified Helmholtz operators. This is also clearly recognized from its general and fundamental solutions stated in Eqs. (6) and (7). By combining the mth order general and fundamental solutions of the Helmholtz and the modified Helmholtz operators of arbitrary dimensions, we intuitively get the corresponding solutions of thin plate vibration as respectively expressed below

$$u^{\#}_{\mathrm{T}m}(r) = A_m(r\lambda)^{-n/2+1+m}(J_{n/2-1+m}(\lambda r) + I_{n/2-1+m}(\lambda r)), \tag{9}$$

$$u^{\#}_{\mathrm{T}m}(r) = A_m(r\lambda)^{-n/2+1+m}(Y_{n/2-1+m}(\lambda r) + K_{n/2-1+m}(\lambda r)), \tag{10}$$

where $A_m=A_{m-1}/(2m\lambda^2)$, $A_0=1/((\mathrm{n}-2)S_n(1))$; n is the topological dimension of the problem, and $S_n(1)$ the surface size of a n-dimensional unit sphere. By using the mathematical deduction approach, we verify that Eqs. (9) and (10) are indeed the mth order general and fundamental solutions of the thin plate vibration. Namely, the zero-order general solution in (9) is known to be correct. We prove that $L_T\{u^{\#}_m\}$ is only consisted of lower than the mth order general solutions of operator L_{T}. Therefore, the m-order general solution (9) is validated. The same strategy is applied to the fundamental solution. On the other hand, we verified that those higher-order fundamental and general solutions are also established for more than 3-dimensions via computer software 'Maple'.

It is observed from Eq. (10) that since the Bessel functions Y and K have singularity at origin, the high-order fundamental solution of the thin plate vibration operator has the singularity-order of (r^{2-n}) except the 2D case, where the only singularity occurs in the zero-order fundamental solution. For instance, the singularity for the 3D case is always r^{-1} irrespective of the order of the fundamental solution.

4. Winkler plate

The Winkler equation for a plate resting on an elastic foundation is

$$D\nabla^4 u + \kappa^2 u = q, \tag{11}$$

where κ is foundation stiffness, u the deflection subject to an arbitrary lateral load q, and D the bending rigidity of the plate. From a mathematical point of view, we define a general Winkler operator for any dimensions

$$L_{\mathrm{W}}\{u\} = \nabla^4 u + \kappa^2 u. \tag{12}$$

The fundamental solutions of the 2D Winkler plate are given in Katsikadelis and Armenakas [11]

$$u^*_{\mathrm{W}0} = \mathrm{kei}(\sqrt{\kappa r}), \tag{13}$$

where kei represents the modified Kelvin functions of the second kind. Comparing the Winkler operator of a single radial variable with the ordinary differential operator of the Kelvin functions, we find that both are actually equivalent. Therefore, all four-Kelvin functions are the component functions of either the fundamental solution or the general solution of the Winkler operator. With the help of the computer algebraic package 'Maple', we found and proved that the zero-order general solutions of the Winkler operator of two up to 5-dimensions are

$$u^{\#}_{\mathrm{W}0}(r) = (r\sqrt{\kappa})^{-n/2+1}(\mathrm{ber}_{n/2-1}(r\sqrt{\kappa}) + \mathrm{bei}_{n/2-1}(r\sqrt{\kappa})), \tag{14}$$

where n is the dimensionality, ber and bei represent the Kelvin and the modified Kelvin functions of the first kind. It is stressed that we could not verify the above solutions for more than 6-dimensions. There are two possible explanations: (1) the solutions (14) are not applicable for the Winkler operator of more than 5-dimensions, (2) the solutions of the Winkler operator of more than 5-dimensions do not exist. By now this is still an open issue.

Furthermore, we find that the mth order general solution and fundamental solutions of the 2D and 3D Winkler operators can be represented as

$$u^{\#}_{\mathrm{W}m} = A_m(kr)^{-n/2+1+m}(\mathrm{ber}_{n/2}(r\sqrt{k}) + \mathrm{bei}_{n/2}(r\sqrt{k})),\quad n=2,3 \tag{15a}$$

when the order m is an odd integer, and

$$u^{\#}_{\mathrm{W}m} = A_m(kr)^{-n/2+1+m}(\mathrm{ber}_{n/2-1}(r\sqrt{k}) + \mathrm{bei}_{n/2-1}(r\sqrt{k})),\quad n=2,3 \tag{15b}$$

when m is an even integer, where A_m is defined as in Eq. (10). The above formulas (15a,b) do not take effect for the Winkler operators of more than 3-dimensions.

Similarly, by replacing ber and bei by the Kelvin and the modified Kelvin functions of the second kind ker and kei, respectively, we have the mth order fundamental solutions

$$u^*_{\mathrm{W}m} = A_m(kr)^{-n/2+1+m}(\mathrm{ker}_{n/2}(r\sqrt{k}) + \mathrm{kei}_{n/2}(r\sqrt{k})),\quad n=2,3 \tag{16a}$$

when m is an odd integer, and

$$u^*_{\mathrm{W}m} = A_m(kr)^{-n/2+1+m}(\mathrm{ker}_{n/2-1}(r\sqrt{k}) + \mathrm{kei}_{n/2-1}(r\sqrt{k})),\quad n=2,3 \tag{16b}$$

when m is an even integer. It is noted that in the case of $m=0$, Katsikadelis and Armenakas [11] choose the kei part of the fundamental solution (16b) as their zero-order fundamental solution.

5. Berger plate

Under the Berger hypothesis, which assumes the plate has not in-plane movement at the boundary, the Berger plate equation is derived as a linearized model of the well-known von Karman equations for non-linear deflection of plates under large loading. The Berger equation is given by

$$\nabla^4 u - \mu^2\nabla^2 u = f, \tag{17}$$

where f is the outer force inflicting on the plate. The fundamental solution of the 2D Berger operator is expressed as [12]

$$u^*_{\mathrm{B}0}(r) = -\frac{1}{2\pi\mu^2}(\ln(r) + K_0(\mu r)), \tag{18}$$

where K_0 denotes the modified Bessel function of the second kind of the zero-order. Clearly, the Berger operator can be split as

$$\nabla^4 u - \mu^2\nabla^2 u = \nabla^2(\nabla^2 - \mu^2)u. \tag{19}$$

(19) shows that the Berger operator is the product of the Laplace and the modified Helmholtz operator. This fact is also reflected in its fundamental solution (18) which equals a sum of the fundamental solutions of the 2D Laplace and the modified Helmholtz operators. Thus, the higher-order fundamental solution of the Berger equation is a sum of the higher-order fundamental solutions of the Laplace and the Helmholtz operators. The higher-order fundamental solutions of the Laplace operator [1] are known as

$$u^*_{\mathrm{L}m} = \begin{cases} \dfrac{1}{2\pi} r^{2m}(C_m \ln r - B_m), & 2D \quad \text{problems} \\ \dfrac{1}{4\pi}\dfrac{1}{(2m)!} r^{2m-1}, & 3D \quad \text{problems} \end{cases} \tag{20}$$

where $C_0=1$, $B_0=0$;

$$C_{m+1} = \frac{C_m}{4(m+1)^2},\quad B_{m+1} = \frac{1}{4(m+1)^2}\left(\frac{C_m}{m+1} + B_m\right).$$

Following the strategy in Section 2, it is straightforward to write out the mth order fundamental solution of the Berger operator of arbitrary dimensions as

$$u^*_{\mathrm{B}m}(r) = u^*_{\mathrm{L}m}(r) + A_m(\mu r)^{-n/2+1+m}K_{n/2-1+m}(\mu r). \tag{20a}$$

However, the general solution of the Laplace operator is a constant. Therefore, the corresponding general solution of the Berger operator is

$$u^{\#}_{\mathrm{B}m}(r) = A_m(1 + (\mu r)^{-n/2+1+m} I_{n/2-1+m}(\mu r)). \tag{20b}$$

6. Some remarks

The composite operator is considered an operator, which is the product of a few other PDE operators of different types, e.g. the thin plate vibration operator being the product of the Laplace and the Helmholtz operators. In Section 5, we see the Berger operator is also a composite operator of the Laplace and the modified Helmholtz operators, and their fundamental and general solutions of varied orders are a sum of the solutions of the corresponding component operators.

In Ref. [13], there are ample examples of the composite operator. To find analytical particular solution of the Laplace and the Helmholtz-type operators, Cheng [14] adopted the same approach used in the study to derive the fundamental solution of composite operators of the Laplace and the Helmholtz operators, some of which appear closely like the Berger plate operator. The goal of Cheng [14] is to facilitate the DR-BEM solution of inhomogeneous problems via the RBF technique. As such, the results of this study can be used in a variety of the RBF techniques for PDEs.

The reason that we use the special function, i.e. the Bessel functions of varied types, in the above-given general and fundamental solutions is to unify the expression. It should be pointed out that we could greatly simplify these mathematical expressions via the sine, cosine, and exponential functions.

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