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THE METHODS OF EXTERNAL EXCITATION FOR ANALYSIS OF ARBITRARILY-SHAPED HOLLOW CONDUCTING WAVEGUIDES

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Abstract—A new numerical technique is proposed for analyzing arbitrary shaped hollow waveguides. The method is based on mathematically modelling of physical response of a system to excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. The results of the numerical experiments justifying the method are presented. The method is validated by circular waveguide, rectangular waveguide an equilateral triangular waveguide. We apply the method for multi connected domains and for waveguides with boundary singularities like the L-shaped waveguide. Good agreements between the simulated and the published results have been obtained. The method does not generate spurious eigenfrequencies.

1. INTRODUCTION

The analysis of the hollow conducting waveguides leads to solution of the eigenvalue problems of the type

$$\nabla^2 w + k^2 w = 0,\ \mathbf{x} \in \Omega \subset \mathcal{R}^2,\ B\,[w] = 0,\ \mathbf{x} \in \partial\Omega. \tag{1}$$

Here Ω is a simply or multiply connected domain with boundary $\partial\Omega$; the boundary operator $B\,[\dots]$ specifies the boundary conditions and is considered to be of the two types: the Dirichlet $B\,[w] = w$ and Neumann $B\,[w] = \partial w/\partial n$ conditions. The eigenvalue problem is to find such real k (the cutoff wavenumbers) for which there exist non-null functions w verifying (1).

The problem (1) is a classical problem of mathematical physics [1] so, there is a variety of methods available in the literature to calculate the cutoff wavenumbers. However, apart from a few analytically solvable cases with simple, regular domains, there is no general solution of this problem. Therefore a large amount of numerical methods has been developed for many practical problems. The usual approach for eigenvalue problems with a positive defined operator is to use the Rayleigh minimal principle. Then, using an approximation for w with a finite number of free parameters, one gets the same problem in a finite-dimensional subspace which can be solved by a standard procedure of linear algebra. In the framework of this approach the polynomial approximations are used in [2, 3] and the trigonometric functions are used in [4]. More recently, the same problems have been studied by Swaminathan et al. [5] using the surface integral equation method. Application of the traditional finite-difference method is presented in [6, 7]. Application of algebraic function approximation in eigenvalue problems of lossless metallic waveguides is presented in [8, 9]. The Fourier and Taylors series expansions are used for analysis longitudinally inhomogeneous waveguides [10, 11]. The method of lines is utilized for eigenvalue problems in arbitrary geometry in [12]. The analysis of waveguides with a complex cross-section is of a great interest for engineering application [13, 14]. To handle such problems the generalized differential quadrature method has been developed and applied for waveguide analysis in [15]. Efficient numerical methods for analysis of arbitrary cross sections waveguide problems have been developed recently [16–18].

The method of fundamental solutions (MFS) is convenient in application to the problem (1). A general approach is as follows. First, using the MFS approximation, one gets a homogeneous linear system $\mathcal{A}(k)\,\mathbf{q} = \mathbf{0}$ with matrix elements depending on the wave number k. To obtain the nontrivial solution the determinant of this matrix must be zero:

$$\det\left[\mathcal{A}(k)\right] = 0. \tag{2}$$

This equation must be investigated analytically or numerically to get the eigenvalues This technique is described in [19–24] in more details. In the three latest papers there is a complete bibliography on the subject considered. Note that the MFS is widely used in electromagnetic calculation under the name of the method of auxiliary sources (MAS) [25, 26]. It belongs to a new class of numerical methods, meshless methods (also called mesh-free methods), that has been developing fast recently [27, 28]. Meshless methods rely on a group of points. This means that the burdensome work of mesh generation is avoided and more accurate description of irregular complex geometries can be achieved.

The method presented in the paper is based on the following quite trivial statement. Let $w_e(\mathbf{x})$ be a smooth enough function defined in the solution domain below named as the *exciting field*. If the *response field* w_r is a solution of the boundary value problem (BVP)

$$\nabla^2 w_r + k^2 w_r = -\nabla^2 w_e - k^2 w_e, \tag{3}$$

$$B[w_r] = -B[w_e], \tag{4}$$

then, the sum $w(\mathbf{x}, k) = w_r + w_e$ satisfies the initial problem (1). Let $F(k)$ be some norm of the solution w. This function of k has extremums at the eigenvalues and, under some conditions described below, can be used for their determining. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the matrix of the linear algebraic system which approximates the BVP. From this point of view the presented approach is similar to the one described in [29], where the degeneracy is measured by the infinitesimal values of the minimal eigenvalue of the stiffness matrix of the problem. Recently this technique has been applied for solving problems of free vibrations of beams, membranes and plates [30–32].

Generally, no conditions are imposed on $w_e(\mathbf{x})$. As a result, one gets the sequence of the inhomogeneous PDEs (3), (4) with a non-null right hand side which can be solved by an appropriate volume method. For example, the FD method and Kanza's method were used in [21, 22]. However, when the exciting field is chosen in such a way that the right hand side of (3) is equal to zero:

$$\nabla^2 w_e + k^2 w_e = 0, \tag{5}$$

then, the response field w_r also satisfies the homogeneous Helmholtz equation

$$\nabla^2 w_r + k^2 w_r = 0, \tag{6}$$

which can be solved by a boundary method. Note that we can take any solution of (5) as the *exciting field*, e.g., we can take it in the form of a travelling wave or as a field of a point source placed outside the solution domain. On the other hand, w_r depends on this choice because it should satisfy the boundary condition (4).

The 2D Helmholtz equation has the known fundamental solutions

$$\Phi(\mathbf{x} - \zeta, k) = H_0^{(1)}(k|\mathbf{x} - \zeta|),$$

where $H_0^{(1)}$ is the Hankel function. This admits of applying very effective meshless numerical techniques to solve (3), (4): the MFS, the boundary knot method [35], the boundary integral method [36].

When applied to the non-singular problems these techniques provide high accuracy of solutions. To handle the eigenvalue problems with boundary singularities, we combine the global approximation of the solution by the Hankel function and the local approximation by the Fourier-Bessel functions near the singular points.

The outline of this paper is as follows: in Section 2, to explain the main algorithm we begin with the simplest case when waveguide has a single connected cross section without boundary singularities.The multiple connected domains are considered in Section 3. In Section 4, we present the extension of the algorithm to problems with boundary singularities. Finally, in Section 5, we give the conclusion.

2. MAIN ALGORITHM

2.1. One-dimensional Case

To illustrate the method presented in the simplest case, let us consider the wave equation in homogeneous medium $\partial_{tt}^2 u = \partial_{xx}^2 u$ with the Dirichlet conditions at the endpoints of the interval $[0, 1]$, i.e., $u(0, t) = u(1, t) = 0$. Considering the time dependence $u(x, t) = e^{-ikt} w(x)$, we get the eigenvalue problem on the interval $[0, 1]$:

$$\frac{d^2 w}{dx^2} + k^2 w = 0, \tag{7}$$

$$w(0) = w(1) = 0, \tag{8}$$

which admits of an analytic solution $k_n = n\pi$.

When applied to (7), (8) the MFS technique is as follows. Let us consider the fundamental solution of (7)

$$\Phi(x - \xi, k) = \frac{1}{2k} \exp(ik|x - \xi|),$$

which satisfies the homogeneous equation everywhere except at the singular point $x = \xi$. A general solution of the homogeneous equation in the interval $[0, 1]$ can be written in the form:

$$w = q_1 \Phi(x - \xi_1, k) + q_2 \Phi(x - \xi_2, k).$$

Here ξ_1, ξ_2 are two source points placed outside the solution domain [0, 1]; q_1, q_2 are free parameters. Using the boundary conditions $w(0) = w(1) = 0$, one gets the linear system:

$$\mathcal{A}(k)\,\mathbf{q} = \begin{cases} q_1 e^{-ik\xi_1} + q_2 e^{ik\xi_2} = 0 \\ q_1 e^{ik(1-\xi_1)} + q_2 e^{ik(\xi_2 - 1)} = 0 \end{cases}$$

The wave numbers k_n can be determined from the condition: $\det[\mathcal{A}(k)] = 0$. After simple transforms we get: $\exp(2ik) = 1$, or $k = n\pi$. Thus, MFS gets the exact solution. Note that in multidimensional cases such computations are time consuming and not so simple. Since the MFS is highly ill conditioned, the determinant is very small. So, using this technique in the 2D case, one operates with values of the order $\sim 10^{-50}$–10^{-500}, see [24, 37] for more detailed information.

According to the method presented in the paper, we take the *response field* w_r as a solution of the BVP:

$$\frac{d^2 w_r}{dx^2} + k^2 w_r = -\frac{d^2 w_e}{dx^2} - k^2 w_e, \tag{9}$$

$$w_r(0) = -w_e(0), \; w_r(1) = -w_e(1), \tag{10}$$

then the sum $w = w_e + w_r$ satisfies the initial BVP (7), (8). The right hand side of (9) can be considered as an external exciting source. In this subsection we take $w_e(x) = e^{ikx}$ and so, the right hand side of (9) is equal to zero. This excitation corresponds to the travelling wave which propagates along the x-axis from the source placed in $-\infty$. Let us introduce the norm of the solution as

$$F(k) = \sqrt{\sum_{n=1}^{N_t} |w(x_n)|^2 / N_t}, \tag{11}$$

where the points x_n are randomly distributed in $[0, 1]$. We also use the dimensionless form of this function: $F_d(k) = F(k)/F(1)$. The function $F(k)$ characterizes the value of the response of the system to the excitation with the wave number k. Varying k, we get the response curve and calculate the eigenvalues as positions of maxima.

However, this initial form of the method is unfit for our goal. Indeed, looking for the response field in the form

$$w_r = A_r \exp(ikx) + B_r \exp(-ikx),$$

we get the linear system for A_r, B_r

$$A_r + B_r = -1, \; A_r \exp(ik) + B_r \exp(-ik) = -\exp(ik). \tag{12}$$

For $k \neq n\pi$ the system has the unique solution $A_r = -1$, $B_r = 0$. Thus, $w = w_e + w_r \equiv 0$ and $F(k) = 0$ with the precision error. In Fig. 1 we place the corresponding response curve to illustrate this situation.

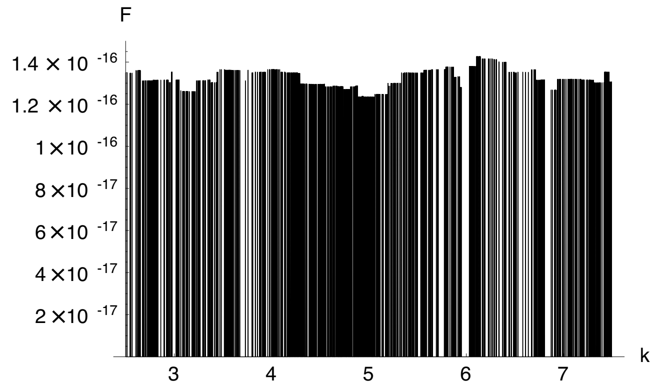

Figure 1. The response curve without regularization.

The two regularizing procedures which give a smooth response curve were proposed in [30–32]. Applying the first one, we substitute BVP (9), (10) as follows:

$$\frac{d^2 w_r}{dx^2} + \left(k^2 + i\varepsilon k\right) w_r = 0,\ w_r\left(0\right) = -w_e\left(0\right),\ w_r\left(1\right) = -w_e\left(1\right), \quad (13)$$

where $\varepsilon > 0$ is a small value. Here we take into account that the right hand side of (9) is equal to zero for $w_e\left(x\right) = e^{ikx}$. From the mathematical point of view this means that we shift the spectra of differential operator from the real axis. On the other hand, from the physical point of view, this means that the wave propagates in a weakly absorbing medium and the initial equation is replaced by the equation $\partial_{tt}^2 u = \partial_{xx}^2 u - \varepsilon \partial_t u$. This wave equation also describes vibrations of the string with friction [1]. Resulting BVP has a unique non zero solution for all real k.

As a result, we get the following system instead of (12):

$$A_r + B_r = -1,\ A_r e^{ik_\varepsilon} + B_r e^{-ik_\varepsilon} = -e^{ik} \quad (14)$$

and $w = w_e + w_r \neq 0$. The dimensionless response curves $F_d\left(k\right)$ depicted in Fig. 2 correspond to $\varepsilon = 10^{-15}$ (left) and $\varepsilon = 10^{-10}$ (right). The value $\varepsilon = 10^{-15}$ is too small to regularize the solution. The response curve $F_d(k)$ has separate maximums at the positions of eigenvalues but is not smooth. The value $\varepsilon = 10^{-10}$ provides a smooth curve.

Another regularizing procedure can be described in the following way. Let us introduce the constant shift Δk between the wave numbers of the exciting source and the studied mode, i.e., we take the exciting field $w_e(x) = w_e(x, k + \Delta k) = \exp[i(k + \Delta k)x]$ and get the linear system

$$A_r + B_r = -1,\ A_r e^{ik} + B_r e^{-ik} = -e^{i(k+\Delta k)}, \tag{15}$$

which provides $w = w_e + w_r \neq 0$. The solution exists for all k except the eigenvalues k_n when the system becomes degenerate. However, due to iterative procedure of solution and rounding errors we never solve the system with the *exact* k_n. We observe degeneration of the system as a considerable growth of the solution in a neighbourhood of the eigenvalues.

The response curves corresponding to $\Delta k = 10^{-15}$ and $\Delta k = 10^{-10}$ are absolutely similar to the curves depicted in Fig. 2 for $\varepsilon = 10^{-15}$ and $\varepsilon = 10^{-10}$. The value $\Delta k = 10^{-15}$ is too small to regularize the solution. But the value $\Delta k = 10^{-10}$ yields a smooth curve.

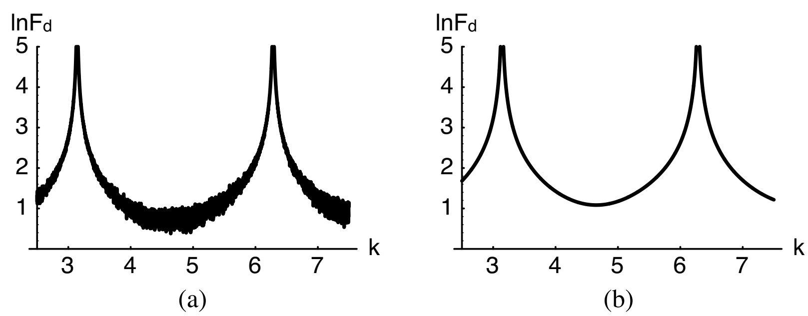

Figure 2. The response curve; ε-procedure with $\varepsilon = 10^{-15}$ — left, $\varepsilon = 10^{-10}$ — right.

When comparing these two procedures, it should be noted that they provide approximately the same precision in the calculations of eigenvalues. However, dealing with a real PDE and using the ε-procedure, we have to perform the calculations with complex variables. The use of the k-procedure provides calculations with real variables only.

Having a smooth response curve, we apply the following simple algorithm. First, we localize these maxima of $F(k)$ on the intervals $[a_i, b_i]$. Next, we solve the univariate optimization problem inside each one. In particular, we apply Brent's method based on a combination of parabolic interpolation and bisection of the function near to the extremum (see [41]).

2.2. Waveguides with Non-singular Boundary

The same technique can be utilized for a homogeneous waveguide analysis of the TM and TE fields satisfying the eigenvalue problem (1). For the TM case, $w = E_z$ and w satisfies the homogeneous Dirichlet boundary condition

$$B[\mathbf{x},\, w] = w = 0, \quad x \in \partial\Omega,$$

where $\partial\Omega$ denotes the conducting boundary of the waveguide wall. For the TE case, $w = H_z$ and w satisfies the homogeneous Neumann boundary condition

$$B[\mathbf{x},\, w] = \partial w/\partial \mathbf{n} = 0,$$

with $\mathbf{n}$ being the unit normal to $\partial\Omega$.

In this subsection we consider waveguides with a smooth and piecewise-smooth boundary $\partial\Omega$ without singular points. We apply the MFS and look for the solution of the Helmholtz equation (6) in the form of the linear combination

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n \Phi(\mathbf{x} - \zeta_n, k), \tag{16}$$

Here q_n are free parameters of the problem and the source points ζ_n are placed outside the solution domain. This is the so-called Kupradze basis [42]. The free parameters are obtained from the boundary condition (4) as a solution of the collocation problem

$$B[w_r(\mathbf{x}_i)] = \sum_{n=1}^{N} q_n B[\Phi(\mathbf{x}_i - \zeta_n, k)] = -B[w_e(\mathbf{x}_i, k)], \ \mathbf{x}_i \in \partial\Omega. \tag{17}$$

The collocation points $\mathbf{x}_i$ are uniformly distributed on the boundary. The number of the collocation points is taken twice as large as the number of unknowns N and the resulting linear system is solved by the procedure of the least squares. Then, having the solution $w_r(\mathbf{x})$ and so, $w(\mathbf{x}) = w_r(\mathbf{x}) + w_e(\mathbf{x})$, we introduce the norm $F(k)$ like (11). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima. We take the exciting field in the form of the travelling wave

$$w_e(\mathbf{x}, k) = \exp[ik(x\cos\upsilon + y\sin\upsilon)], \tag{18}$$

which satisfies (5) for any angle of incidence υ.

Table 1. The relative errors in solution of eigenvalue problem for the circular waveguide with the radius $R = 1$.

TM				TE			
i	$k_i^{(ex)}$	MFS	BIM	i	$k_i^{(ex)}$	MFS	BIM
1	2.404825560	1.1E-09	1.2E-09	1	1.841183786	2.8E-09	2.8E-09
2	3.831705970	2.5E-12	1.4E-11	2	3.054236921	2.4E-09	2.5E-09
3	5.135622307	1.0E-09	1.0E-09	3	3.831705970	6.4E-12	1.5E-11
4	5.520078106	8.4E-10	8.7E-10	4	4.201188938	8.4E-10	8.5E-10
5	6.380161905	1.4E-09	1.4E-09	5	5.317553128	3.6E-10	3.7E-10
6	7.015586677	1.1E-09	1.1E-09	6	5.331442774	1.4E-10	1.4E-10
7	7.588342447	1.6E-09	1.6E-09	7	6.415616391	2.4E-09	2.4E-09
8	8.417244162	2.6E-09	2.6E-09	8	6.706133203	1.3E-09	1.4E-09
9	8.653727935	2.5E-09	2.5E-09	9	7.015586677	1.1E-09	1.1E-09
10	8.771483842	3.0E-09	2.9E-09	10	7.501266166	2.8E-09	2.4E-09

Using the ε-procedure, we replace the matrix terms $\Phi(\mathbf{x}_i - \zeta_n, k)$ by $\Phi(\mathbf{x}_i - \zeta_n, k_\varepsilon)$, $k_\varepsilon = \sqrt{k^2 + i\varepsilon k}$. When the k-procedure is used for smoothing the response curve $F(k)$, the exciting field $w_e(\mathbf{x}_i, k)$ in the right hand side of (17) is replaced by $w_e(\mathbf{x}_i, k + \Delta k)$.

Circular waveguide: The data placed in Table 1 correspond to the circular waveguide with the radius 1 and are obtained using the two different boundary methods:

1) The MFS described above.

2) The dual boundary integral equation method. It is based on the dual boundary integral formulation

$$\int_{\partial\Omega} T(\mathbf{s}, \mathbf{x})\, w(\mathbf{s})\, dl(\mathbf{s}) - \int_{\partial\Omega} U(\mathbf{s}, \mathbf{x}) \frac{\partial w}{\partial n}(\mathbf{s})\, dl(\mathbf{s}) = \begin{cases} 0, & \mathbf{x} \in R^2 \backslash \Omega, \\ 2\pi w(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

where $U(\mathbf{s}, \mathbf{x})$ and $T(\mathbf{s}, \mathbf{x})$ are well-known Green's function and normal derivative respectively:

$$U(\mathbf{s}, \mathbf{x}) = -\frac{1}{2} i\pi H_0^{(1)}(kr), \; T(\mathbf{s}, \mathbf{x}) = -\frac{1}{2} i\pi k H_1^{(1)}(kr) \frac{y_i n_i}{r},$$

where $H_n^{(1)}$ is the nth-order Hankel function of the first kind, $r = |\mathbf{x} - \mathbf{s}|$, $y_i = s_i - x_i$, and n_i is the ith component of the outer normal vector at the boundary point $\mathbf{s}$. For the Dirichlet problem we find the unknown normal derivative $\partial w/\partial n$ as a solution of the integral equation

$$\int_{\partial\Omega} U(\mathbf{s},\mathbf{x}) \frac{\partial w}{\partial n}(\mathbf{s})\, dl(\mathbf{s}) = F(\mathbf{x}) \equiv \int_{\partial\Omega} T(\mathbf{s},\mathbf{x})\, w(\mathbf{s})\, dl(\mathbf{s}),\ \mathbf{x} \in \Gamma,$$

where the auxiliary contour Γ contains the solution domain Ω and does not intersect it's boundary $\partial\Omega$. So, the integrals are not singular. Having the boundary value of the normal derivative $\partial w/\partial n(\mathbf{s})$, we obtain the solution $w(\mathbf{x})$ in the interior of Ω

$$w(\mathbf{x}) = \frac{1}{2\pi} \int_{\partial\Omega} T(\mathbf{s},\mathbf{x})\, w(\mathbf{s})\, dl(\mathbf{s}) - \frac{1}{2\pi} \int_{\partial\Omega} U(\mathbf{s},\mathbf{x}) \frac{\partial w}{\partial n}(\mathbf{s})\, dl(\mathbf{s}).$$

All the integrals are approximated by the finite sums using an appropriate quadrature rule. The similar technique is used in the case of Neumann's boundary condition. The description of this technique with more details can be found in [36] and in the literature presented here.

The calculations presented in Table 1 correspond to the following parameters: MFS utilizes $N = 40$ sources placed on the circle with the radius $R_a = 5$; and using BIM, we take $N = 40$ unknown values of the normal derivative $\partial w/\partial n$. Here we place the relative errors

$$e_r = \frac{|k_i - k_i^{(ex)}|}{k_i^{(ex)}} \tag{19}$$

in the calculation of the first ten eigenvalues.

The exciting field (18) is taken with $\upsilon = 0.25\pi$. The k-procedure is used with $\Delta k = 10^{-6}$. The exact eigenvalues $k_i^{(ex)}$ are the roots of the equation $J_n(k) = 0$ (TE) or $J'_n(k) = 0$ (TM).

Square waveguide. The data placed in Table 2 correspond to the waveguide with the cross section $[-1, +1] \times [-1, +1]$. To solve the Helmholtz equation the MFS is used. The MFS source points are placed on the same circle with the radius $R_a = 5$. We take the same exciting field in the form of the travelling wave with $\upsilon = 0.25\pi$. The k-procedure is used for smoothing with $\Delta k = 10^{-6}$. The left hand part of the table shows the convergence of the TM solution with the growth of N — the number of free parameters in approximation (16). The results of solution in the TE case are shown in the right hand side of the table. One can see that $N = 40$ provides the 9–10 true digits of the first ten eigenvalues. The method gives the eigenvalues without differing the eigenmodes. For example, $k = 1.57079632679489$ corresponds to TE_{10} and TE_{01} modes.

Table 2. Square waveguide. Convergence with the growth of the number of free parameters N.

TM					TE		
i	$k_i^{(ex)}$	N=20	N=30	N=40	i	$k_i^{(ex)}$	N=40
1	2.221441469	2.2214	2.221441	2.221441469	1	1.570796327	1.570796327
2	3.512407366	3.5124	3.512407	3.512407366	2	2.221441469	2.221441469
3	4.442882938	4.4429	4.442883	4.442882938	3	3.141592654	3.141592654
4	4.967294133	4.9673	4.967294	4.967294133	4	3.512407366	3.512407366
5	5.663586700	5.6635	5.663587	5.663586700	5	4.442882938	4.442882938
6	6.476559172	6.4743	6.476559	6.476559172	6	4.712388980	4.712388980
7	6.664324407	6.6636	6.664323	6.664324405	7	4.967294133	4.967294133
8	7.024814731	7.0247	7.024815	7.024814731	8	5.663586699	5.663586686
9	7.853981634	-	7.853984	7.853981635	9	6.283185307	6.283185162
10	8.009521122	-	8.009521	8.009521073	10	6.476559172	6.476559184

Rectangular waveguide. The splitting of the eigenvalues is shown in Table 3. The data correspond to the rectangular waveguide with the cross section $[-1.01, +1.01] \times [-1, +1]$. The parameters of the MFS, exciting and smoothing are the same as above.

Table 3. Rectangular waveguide $[-1.01, +1.01] \times [-1, +1]$. Splitting of the eigenvalues.

TM			TM		
i	$k_i^{(ex)}$	$N = 30$	i	$k_i^{(ex)}$	$N = 30$
1	2.210471590	2.210471590	1	1.555243888	1.555243888
2	3.484614083	3.484614083	2	1.570796327	1.570796327
3	3.505479704	3.505479704	3	2.210471590	2.210471590
4	4.420943181	4.420943181	4	3.110487777	3.110487781
5	4.923053226	4.923053231	5	3.141592654	3.141592639
6	4.962397954	4.962397959	6	3.484614083	3.484614088

Triangular waveguide. The same approach was applied to the waveguide with the cross section in the form of the equilateral triangle with the width equal to 1. The parameters of the MFS, exciting and smoothing are the same as above. The data placed in the last columns ($N = 237$) of Table 4 have the 10–11 true digits.

Table 4. The triangular waveguide. Convergence with the growth of the number of free parameters N.

	TM			TE		
i	N=57	N=107	N=237	N=57	N=107	N=237
1	7.2551975	7.2551974569	7.2551974569	4.1887902	4.1887902048	4.1887902048
2	11.082497	11.0824971761	11.0824971761	7.2551975	7.2551974569	7.2551974569
3	14.510395	14.5103949139	14.5103949139	8.377580	8.3775804095	8.3775804095
4	15.102898	15.1028978655	15.1028978656	11.082497	11.0824971760	11.0824971760
5	18.258514	18.2585131983	18.2585131983	12.566371	12.5663706143	12.5663706144

3. MULTIPLY CONNECTED CROSS SECTION

When we deal with problems in multiply connected domains, the same basis functions $\Phi(\mathbf{x} - \zeta_n, k)$ can be used. And the source points should be placed also inside each hole as it is depicted in Fig. 3(a).

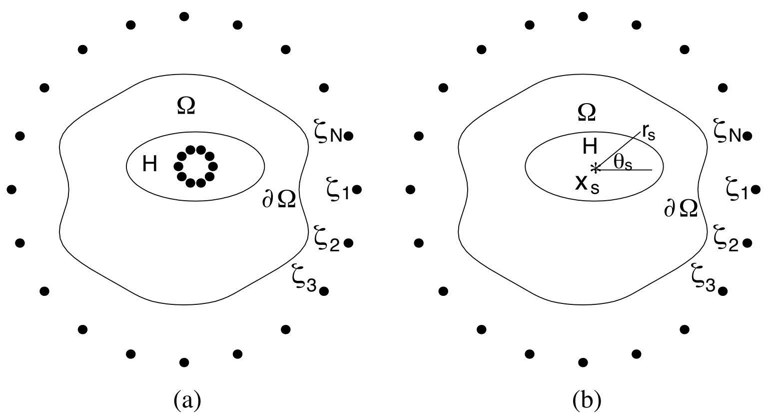

Figure 3. Geometry configuration of a doubly connected domain.

As an alternative approach one can use the special trial functions associated with each hole:

$$\begin{aligned}\Psi_{s,1}(\mathbf{x}, k) &= H_0^{(1)}(kr_s),\\ \Psi_{s,2n+1}(\mathbf{x}, k) &= H_n^{(1)}(kr_s)\cos n\theta_s,\\ \Psi_{s,2n}(\mathbf{x}, k) &= H_n^{(1)}(kr_s)\sin n\theta_s.\end{aligned} \tag{20}$$

Here $r_s = |\mathbf{x} - \mathbf{x}_s|$, θ_s is the local polar coordinate system with the origin at the point $\mathbf{x}_s$ of multipoles location (see Fig. 3(b)).

This is so-called Vekua basis [43, 44] or multipole expansion. It is proven that every regular solution of the 2D Helmholtz equation in a domain with holes can be approximated with any desired accuracy by linear combinations of such functions if the origin $\mathbf{x}_s$ of a multipole is inside every hole. In this case instead of (16) we use:

$$w_r = \sum_{n=1}^{N} q_n \Phi(\mathbf{x}_i - \zeta_n, k) + \sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m} \Psi_{s,m}(\mathbf{x}, k), \tag{21}$$

where $\mathbf{q} = (q_n)_{n=1}^{N}$, $\mathbf{p}_s = (p_{s,m})_{m=1}^{M}$, S is the number of holes and M is the number of terms in each multipole expansion.

Coaxial waveguide. The data shown in Table 5 correspond to the coaxial with the radii $R = 1$ (outer) and $r = 0.5$ (inner). The MFS is utilized as a solver of the Helmholtz equation. The MFS sources are placed on the concentric circles with the radii $R_1 = 5$ — around the cross section, and $R_2 = 0.2$ — inside the central rod cross section. The data correspond to the $N_1 = 60$ MFS sources placed out of the cross section and $N_2 = 30$ sources inside the hole. The parameters of the excitation and smoothing are the same as above. The exact $k_i^{(ex)}$ are the roots of the equations: $J_n(R)\,Y_n(r) = J_n(r)\,Y_n(R)$ (TM) and $J_n'(R)\,Y_n'(r) = J_n'(r)\,Y_n'(R)$ (TE).

Waveguide with two rods inside. The data placed in Table 6 correspond to the waveguide shown in Fig. 4. The domain with a radius $R = 1$ contains two circular inner boundaries with the eccentricity of 0.5 and the radii of $r = 0.5$. The MFS sources are placed on the circle the radius $R_1 = 5$ — around the cross section and on the two circle with the radius $R_2 = 0.15$ inside the cross sections of the rods. The numbers

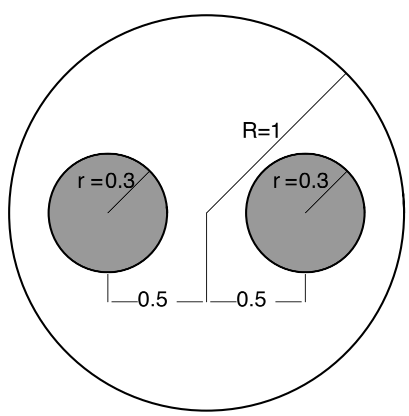

Figure 4. Cylindrical waveguide with two holes in the cross section.

Table 5. Coaxial waveguide. The radii $R = 1$, $r = 0.5$.

TM			TE		
i	$k_i^{(ex)}$	$N_1/N_2 = 60/30$	i	$k_i^{(ex)}$	$N_1/N_2 = 60/30$
1	6.246061839	6.246061839	1	1.354672010	1.354672012
2	6.393156762	6.393156762	2	2.681204287	2.681204294
3	6.813842853	6.813842853	3	3.957754188	3.957754208
4	7.457740136	7.457740136	4	5.175227740	5.175227821
5	8.266730435	8.266730435	5	6.338887082	6.338886815
6	9.190044425	9.190044424	6	6.393156762	6.393156762
7	10.188929924	10.188929921	7	6.564942382	6.564942380
8	11.235707793	11.235707777	8	7.062581616	7.062581614
9	12.311308597	12.311308688	9	7.462157848	7.462157069
10	12.546871428	12.546871428	10	7.840109098	7.840109093

Table 6. Circle with two holes.

TM				TE			
i	$N_1 = 40$	$N_1 = 50$	$N_1 = 60$	i	$N_1 = 40$	$N_1 = 50$	$N_1 = 60$
1	4.49759	4.4976021	4.4976022	1	1.38573	1.3857496	1.3857515
2	4.55883	4.5588357	4.5588355	2	1.76998	1.7699497	1.7699483
3	6.39849	6.3985053	6.3985051	3	2.60074	2.6007376	2.6007364
4	7.09633	7.0963032	7.0963042	4	2.63443	2.6344713	2.6344661
5	7.59309	7.5931077	7.5931064	5	3.66223	3.6621892	3.6621924

of the sources placed around the cross section are taken $N_1 = 40$, 50, 60. The corresponding numbers of the MFS sources placed inside the rods are: $N_2 = 20$, 25, 30. So, the whole numbers of the free parameters are $N = 80$, 100, 120 for the data placed in the columns of the table. We take the same exciting field in the form of the travelling wave with $\upsilon = 0.25\pi$. The k-procedure is used for smoothing the response curve $F(k)$ with $\Delta k = 10^{-6}$.

Coaxial waveguide with a small central rod. Let us consider the coaxial waveguide with a very small central rod. Now the Kupradze type basis functions $\Phi(\mathbf{x} - \zeta, k) = H_0^{(1)}(k\,|\mathbf{x} - \zeta|)$ are unfit for approximating the solution in a neighbourhood of the central circle

Table 7. Coaxial waveguide. The radii $R = 1$, $r = 0.01$.

TM			TE		
i	$k_i^{(ex)}$	k_i	i	$k_i^{(ex)}$	k_i
1	2.8009217551	2.8009217529	1	1.8407980375	1.8407980373
2	3.8328841728	3.8328841727	2	3.0542367965	3.0542367964
3	5.1356230349	5.1356230350	3	3.8328841728	3.8328841729
4	6.0109006903	6.0109006870	4	4.2011889412	4.2011889403
5	6.3801618962	6.3801618962	5	5.3175531261	5.3175487666

because when the singular points, say ζ_i, ζ_j, of two sources are very close, the corresponding functions $\Phi(\mathbf{x} - \zeta_i, k)$, $\Phi(\mathbf{x} - \zeta_j, k)$ become indistinguishable and the collocation matrix has two identical columns. Here we use a combined basis which includes the Kupradze type basis functions with the singular points placed on an auxiliary circular contour outside the solution domain and a multipole expansion (Vekua basis) with the origin inside the central rod. Thus, we look for an approximate solution in the form:

$$w\left(\mathbf{x}|\mathbf{q},\mathbf{p}\right) = \sum_{n=1}^{N} q_n \Phi(\mathbf{x}_i - \zeta_n, k) + \sum_{m=1}^{M} p_m \Psi_m(\mathbf{x}, k).$$

The data presented in Table 7 correspond to the coaxial waveguide with the radii $R = 1$ (outer) and $r = 0.01$ (inner). The number of the sources placed around the cross section is taken $N = 30$, the number of harmonics in the multipole expansion is $M = 7$. So, the calculations presented in the table were performed using 37 free parameters only.

4. SINGULAR PROBLEMS

In this section our study is focused on the case when the cross section Ω of the waveguide has boundary singularities like a reentrant corner, or an abrupt change in the boundary conditions (see Fig. 5). The MFS faces great difficulties when applied to such problems because it utilizes smooth basis functions. Indeed, the functions of the Kupradze basis become very smooth when the singular point is far from the solution domain. To overcome the difficulties, the Trefftz method (TM) [29] and the method of particular solutions (MPS) [38–40] have been developed. These techniques use various particular solutions of the eigenvalue equation which describes the local behaviour of the

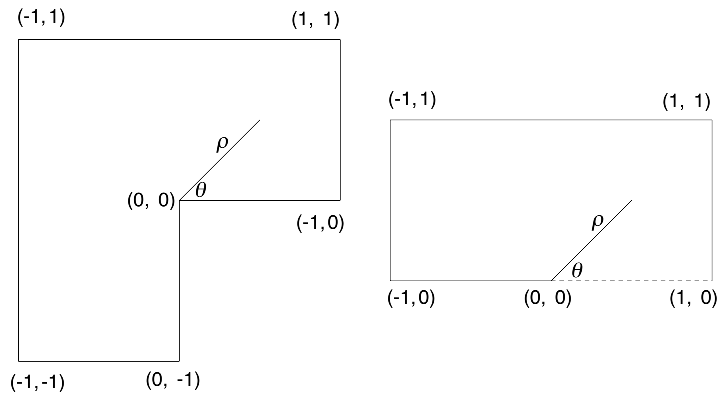

Figure 5. The domains with boundary singularities. In the right part of the figure the solid line corresponds to the Dirichlet boundary condition and the dashed line denotes Neumann's condition. Here (ρ, θ) is the local polar coordinate system with the origin at the singular point.

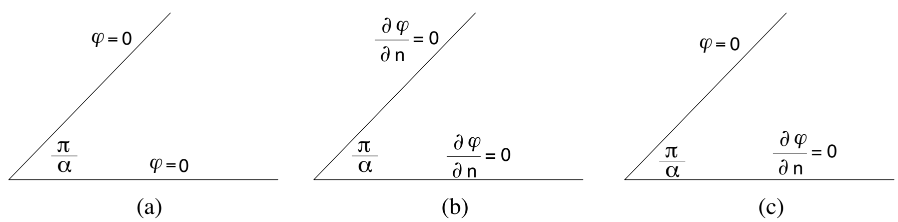

Figure 6. A wedge with interior angle π/α and different boundary conditions along the adjacent line segments: (a) Dirichlet; (b) Neumann; (c) mixed.

eigenfunction near the singular points. It can be shown that the convenient sets of particular solutions near a corner of angle π/α are the functions:

$$\varphi_n(\rho, \theta) = J_{n\alpha}(k\rho) \sin(n\alpha\theta), \quad n = 1, 2, \ldots, \infty, \tag{22}$$

$$\varphi_n(\rho, \theta) = J_{n\alpha}(k\rho) \cos(n\alpha\theta), \quad n = 0, 1, \ldots, \infty, \tag{23}$$

$$\varphi_n(\rho, \theta) = J_{(n-1/2)\alpha}(k\rho) \cos((n - 1/2)\alpha\theta), \quad n = 1, 2, \ldots, \infty \tag{24}$$

for three cases of the boundary conditions shown in Fig. 6. Here (ρ, θ) is the local polar coordinate system with the origin at the singular point. The advantage of these functions (Fourier-Bessel functions) is that not only do they satisfy the governing equation, but they also satisfy the boundary conditions along the adjacent line segments. For more details see the original papers. To extend the technique described in the previous section onto the case of the boundary singularities we use the MFS as a solver of BVP for w_r and look for the response field in the form

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=1}^{M} p_j \varphi_j(\rho, \theta), \tag{25}$$

where the Fourier-Bessel functions $\varphi_j(\rho, \theta)$ correspond to the kind of the singularity. Only the Fourier-Bessel functions (22), (23) and (24) are considered in this paper.

We find the unknowns q_n, p_j as a solution of the collocation problem

$$\begin{aligned} B[w_r(\mathbf{x}_i)] &= \sum_{n=1}^{N} q_n B\left[H_0^{(1)}(k|\mathbf{x}_i - \zeta_n|)\right] + \sum_{j=1}^{M} p_j B[\varphi_j(\rho_i, \theta_i)] \\ &= -B[w_e(\mathbf{x}_i)], \quad \mathbf{x}_i \in \partial\Omega \end{aligned} \tag{26}$$

The collocation points $\mathbf{x}_i \equiv (\rho_i, \theta_i)$ are uniformly distributed on the boundary. The number of the collocation points is taken twice as large as the number of the unknowns $N + M$ and the resulting linear system is solved by the procedure of the least squares. Then, having the solution $w_r(\mathbf{x})$, we introduce the norm $F(k)$ like (11). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima.

Waveguide with L-shaped cross section. Let us consider the eigenvalue problem for L-shaped domain with the Dirichlet boundary conditions (TM). The response field is looked for in the form:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=1}^{M} p_j J_{2j/3}(k\rho) \sin(2j\theta/3)$$

with the Fourier-Bessel functions satisfying the boundary conditions $\varphi_j(\rho, 0) = \varphi_j(\rho, 3\pi/2) = 0$. The data placed in Table 8 are obtained using the k-procedure with $\Delta k = 10^{-6}$. The exciting field is taken with the angle $\upsilon = \pi/4$. The MFS source points are placed on the circle with the radius $R_s = 3$. The data in the last column of the table are taken from [40]. To compare these data with our result we place the squares k_i^2 in the table. It looks like the data corresponding to $N = 120$, $M = 20$ give the eigenvalues with the 10 true digits. To

Table 8. The L-shaped domain. Dirichlet condition. Convergence with the growth of the number of free parameters. The value of k^2 is shown.

i	N/M=40/15	N/M=60/20	N/M=80/20	N/M=100/20	N/M=120/20	B&T
1	9.639726	9.639723840	9.639723841	9.639723842	9.639723841	9.6397238
2	15.197253	15.19725193	15.197251927	15.197251926	15.197251926	5.197252
3	19.739209	19.73920880	19.739208803	19.739208802	19.739208802	19.739209
4	29.521481	29.52148112	29.521481113	29.521481114	29.521481113	29.521481
5	31.912648	31.91263592	31.912635960	31.912635951	31.912635961	31.912636
6	41.474483	41.47450984	41.474509891	41.474509893	41.474509894	41.474510
7	44.948476	44.94848776	44.948487781	44.948487782	44.948487781	–
8	49.348022	49.34802200	49.348022007	49.348022006	49.348022006	–
9	56.709605	56.70960993	56.709609889	56.709609885	56.709609884	–
10	65.376529	65.37653575	65.376535711	65.376535708	65.376535709	–

Table 9. The L-shaped domain. Neumann condition. Convergence with the growth of the number of free parameters. The value of k^2 is shown.

i	N/M=40/10	N/M=40/15	N/M=60/10	N/M=80/10	N/M=100/10	Shu&Chen
1	1.475623	1.475623	1.475622	1.475621813	1.475621793	1.47455
2	3.534032	3.534032	3.534031	3.534031369	3.534031369	3.53409
3	9.869604	9.869604	9.869604	9.869604401	9.869604401	9.86964
4	11.389478	11.389478	11.389479	11.389479388	11.389479386	11.38951
5	12.572380	12.572383	12.572388	12.572386842	12.572386864	12.56893
6	19.739209	19.739209	19.739209	19.739208802	19.739208802	19.73940
7	21.424745	21.424742	21.424734	21.424732805	21.424734381	21.41668
8	23.344379	23.344378	23.344372	23.344372031	23.344372008	23.34424
9	28.490787	28.490807	28.490812	28.490823385	28.490814353	28.48205
10	35.878650	35.878648	35.878631	35.878631521	35.878631920	-

solve the Neumann problem (TE) we use the expansion:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=0}^{M} p_j J_{2j/3}(k\rho)\cos(2j\theta/3)$$

with the Fourier-Bessel functions satisfying the boundary conditions $\partial\varphi_j/\partial n\,(\rho,0) = \partial\varphi_j/\partial n\,(\rho,3\pi/2) = 0$. Thus, only the last term is modified. In Table 9 we test a convergence of the eigenvalues. The results are compared with the results of Shu and Chew [15] obtained by the global method of generalized differential quadrature (GDQ). Note that the L-shaped domain considered in [15] is smaller than the one depicted in Fig. 5. The similarity coefficient 0.635 is taken into account in the data presented in the table.

Rectangular waveguide with an inner rib. The eigenvalue problem for the rectangular waveguide with an inner rib (Fig. 5, right) is the example of problems with an abrupt change in the boundary conditions. We consider TM case (Dirichlet conditions) and set the symmetry conditions $\partial w/\partial y = 0$ along the interval $y = 0$, $0 \leq x \leq 1$. This problem as the eigenvalue problem of a cracked beam is studied in detail in [29]. We look for the MFS solution in the form:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=1}^{M} p_j J_{j-1/2}(k\rho)\cos((j-1/2)\theta)$$

with the Fourier-Bessel functions corresponding to the boundary conditions $\partial\varphi_j/\partial n(\rho, 0) = \varphi_j(\rho, \pi) = 0$. Some results of the calculations are presented in Table 10. Using the package Mathematica, the first two eigenvalues were calculated in [29] with 13 significant digits. They are shown in the last column of the table. One can see that the method presented gives the eigenvalues of the problem with 10 true digits.

Table 10. The eigenvalues of the cracked beam.

i	*N*/*M*=50/15	*N*/*M*=60/10	*N*/*M*=75/10	*N*/*M*=80/10	[17]
1	2.0116970	2.01169717	2.011697117	2.011697117	2.011697117212
2	3.2931526	3.29315261	3.293152635	3.293152635	3.293152635104
3	4.0798642	4.07986416	4.079864129	4.079864128	-
4	4.8863146	4.88631468	4.886314665	4.886314665	-
5	5.2893785	5.28937862	5.289378620	5.289378620	-
6	6.1326889	6.13268898	6.132689008	6.132689010	-
7	6.4719151	6.47191515	6.471915149	6.471915149	-
8	6.8246202	6.82462027	6.824620261	6.824620261	-
9	7.3939713	7.39397129	7.393971287	7.393971287	-
10	7.9781250	7.97812500	7.978125002	7.978125001	-

5. CONCLUSION

In this paper, a numerical technique is proposed for the analysis of various hollow conducting waveguides. This is a mathematical model of the physical measurements when the resonant frequencies of a system are determined by the amplitude of response to some excitation. Varying the wave number k, we get the eigenvalues as positions of maxima of the norm function $F(k)$. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the collocation matrix. The key moment of the algorithm is the use of the special regularizing procedures which provides a smooth response curve and, as a sequence, provides a very high precision in determining eigenfrequencies.

The method can be applied for the analysis of waveguides with multi connected cross sections and waveguides whose cross sections contain boundary singularities like a reentrant corner, or an abrupt change in the boundary conditions. In [32] it was applied to 3D problems so, it can be used for analyzing resonators too.

The method is easy to program and not expensive in the CPU time. Indeed, all the numerical examples are considered with the same placement of the source points and differ only by the positions of the collocation points on the contour of the waveguide.

This technique is convenient for determining some first eigenvalues of the system which are often of the most interest from the point of view of engineering applications.

The method is presented mainly in the framework of the MFS but any boundary or volume method can be used as the Helmholtz solver.

REFERENCES

1. Morse, P. M. and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.
2. Bulley, R. M., "Analysis of arbitrary shaped waveguide by polynomial approximation, *IEEE Trans. Microwave Theory Tech.*, Vol. 18, No. 12, 1022–1028, Dec. 1970.
3. Lin, W. K., L. W. Li, T. S. Yeo, and M. S. Leong, "Analysis of metallic waveguides of large class of cross sections using polynomial approximation and superquadric functions," *IEEE Trans. Microwave Theory Tech.*, Vol. 49, No. 6, 1136–1239, Jun. 2001.
4. Tomas, D. T., "Functional approximations for solving boundary value problems by computer," *IEEE Trans. Microwave Theory Tech.*, Vol. 17, No. 8, 447–454, Aug. 1969.
5. Swaminathan, M., E. Arvas, T. K. Sarkar, and A. R. Djordjevic, "Computation of cutoff wavenumbers of TE and TM modes in waveguides of arbitrary cross sections using a surface integral formulation," *IEEE Trans. Microwave Theory Tech.*, Vol. 38, No. 2, 154–159, Feb. 1990.
6. Guan, J. M. and C. C. Su, "Analysis of metallic waveguides with rectangular boundaries by using the finite-difference method

and the simultaneous iteration with the Chebyshev acceleration," *IEEE Trans. Microwave Theory Tech.*, Vol. 43, No. 2, 374–382, Feb. 1995.

7. Hernandez-Lopez, M. A. and M. Quintillan-Gonzalez, "A finite element method code to analyze waveguide dispersion," *J. of Electromagn. Waves and Appl.*, Vol. 21, No. 3, 397–408, 2007.
8. Yener, N., "Algebraic function approximation in eigenvalue problems of lossless metallic waveguides: Examples," *J. of Electromagnetic Waves and Applications*, Vol. 20, No. 6, 731–745, 2006.
9. Yener, N., "Advancement of algebraic function approximation in eigenvalue problems of lossless metallic waveguides to infinite dimensions, Part I: Properties of the operator in infinite dimensions," *J. of Electromagnetic Waves and Applications*, Vol. 20, No. 12, 1611–1628, 2006.
10. Khalaj-Amirhosseini, M., "Analysis of longitudinally inhomogeneous waveguides using Taylor's series expansion," *J. of Electromagnetic Waves and Applications*, Vol. 20, No. 8, 1093–1100, 2006.
11. Khalaj-Amirhosseini, M., "Analysis of longitudinally inhomogeneous waveguides using the Fourier series expansion," *J. of Electromagnetic Waves and Applications*, Vol. 20, No. 10, 1299–1310, 2006.
12. Xiao, J.-K., W.-S. Ji, S. Zhang, and Y. Li, "A field theoretical method for analyzing microwave cavity with arbitrary cross-section," *J. of Electromagnetic Waves and Applications*, Vol. 20, No. 4, 435–446, 2006.
13. Mei, Z. L. and F. Y. Xu, "A simple, fast and accurate method for calculating cutoff wavelengths for the dominant mode in elliptical waveguide," *J. of Electromagnetic Waves and Applications*, Vol. 21, No. 3, 367–374, 2007.
14. Yung, E. K. N. and W. Lin, "Theory of Cassinian waveguides," *J. of Electromagnetic Waves and Applications*, Vol. 21, No. 10, 1323–1331, 2007.
15. Shu, C. and Y. T. Chew, "Application of multi-domain GDQ method to analysis of waveguides with rectangular boundaries," *Progress In Electromagnetic Research*, PIER 21, 1–19, 1999.
16. Shu, C., W. X. Wu, and C. M. Wang, "Analysis of metallic waveguides using least square-based finite difference method," *CMC: Computers, Materials & Continua*, Vol. 2, 189–200, 2005.
17. Ooi, B. L. and G. Zhao, "Element-free method for analysis of

arbitrarily-shaped hollow conducting waveguides," *IEE Proceedings - Microwaves, Antennas and Propagation*, Vol. 152, No. 1, 31–34, Feb. 2005.

18. Zhao, G., B. L. Ooi, Y. J. Fan, Y. Q. Zhang, I. Ang, and Y. Gao, "Application of conformal meshless RBF coupled with coordinate transformation for arbitrary waveguide analysis," *Journal of Electromagnetic Waves and Applications*, Vol. 21, No. 1, 3–14, 2007.
19. Kondapalli, P. S. and D. J. Shippy, "Analysis of acoustic scattering in fluids and solids by the method of fundamental solutions," *Journal of the Acoustical Society of America*, Vol. 91, No. 4, 1844–1854, 1992.
20. Karageorghis, A., "The method of fundamental solutions for the calculation of the eigenvalues of the Helmholtz equation," *Applied Math. Letters*, Vol. 14, 837–842, 2001.
21. Chen, J. T., J. H. Lin, S. R. Kuo, and S. W. Chyuan, "Boundary element analysis for the Helmholtz eigenvalues problems with a multiply connected domain," *Proc. R. Soc. Lond. A*, Vol. 457, 2521–2546, 2001.
22. Chen, J. T., L. W. Liu, and H. K. Hong, "Spurious and true eigensolutions of Helmholtz BIEs and BEMs for a multiply connected problem," *Proc. R. Soc. Lond. A*, Vol. 459, 1897–1924, 2003.
23. Chen, J. T., M. H. Chang, K. H. Chen, and S. R. Lin, "The boundary collocation method with meshless concept for acoustic eigenanalysis of two-dimensional cavities using radial basis function," *Journal of Sound and Vibration*, Vol. 257, 667–711, 2002.
24. Chen, J. T., I. L. Chen, and Y. T. Lee, "Eigensolutions of multiply connected membranes using the method of fundamental solutions," *Engineering Analysis with Boundary Elements*, Vol. 29, 166–174, 2005.
25. Bucci, O. M., G. D'Elia, and M. Santojanni, "A fast multipole approach to 2D scattering evaluation based on a non redundant implementation of the method of auxiliary sources," *J. of Electromagnetic Waves and Applications*, Vol. 20, No. 13, 1715–1723, 2006.
26. Anastassiu, H. T., "Error estimation of the method of auxiliary sources (MAS) for scattering from an impedance circular cylinder," *Progress In Electromagnetics Research*, PIER 52, 109–128, 2005.
27. Liu, X., B.-Z. Wang, and S. Lai, "Element-free Galerkin

method in electromagnetic scattering field computation," *J. of Electromagnetic Waves and Applications*, Vol. 21, No. 14, 1915–1923, 2007.

28. Ooi, B. L. and G. Zhao, "Element-free method for the analysis of partially-filled dielectric waveguides," *J. of Electromagn. Waves and Appl.*, Vol. 21, No. 2, 189–198, 2007.
29. Li, Z. C., T. T. Lu, H. Y. Hu, H. S. Tsai, and A. H. D. Cheng, "The Trefftz method for solving eigenvalue problems," *Engineering Analysis with Boundary Elements*, Vol. 30, 292–308, 2006.
30. Reutskiy, S. Y., "The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains," *Engineering Analysis with Boundary Elements*, Vol. 30, 150–159, 2006.
31. Reutskiy, S. Y., "The method of fundamental solutions for eigenproblems with Laplace and biharmonic operators," *CMC: Computers, Materials Continua*, Vol. 2, 177–188, 2005.
32. Reutskiy, S. Y., "The Method of External Sources (MES) for eigenvalue problems with Helmholtz equation," *CMES: Computer Modeling in Engineering & Sciences*, Vol. 12, 27–39, 2006.
33. Reutskiy, S. Y., "The methods of external and internal excitation for problems of free vibrations of non-homogeneous membranes," *Engineering Analysis with Boundary Elements*, Vol. 31, 906–918, 2007.
34. Reutskiy, S. Y., "The Method of External Excitation for problems of free vibrations of non-homogeneous Timoshenko beams," *International Journal for Computational Methods in Engineering Science & Mechanics*, Vol. 8, 10–21, 2007.
35. Chen, W., "Symmetric boundary knot method," *Engineering Analysis with Boundary Elements*, Vol. 26, 489–494, 2002.
36. Chen, J. T., C. T. Chen, P. Y. Chen, and I. L. Chen, "A semi-analytical approach for radiation and scattering problems with circular boundaries," *Computer Methods in Applied Mechanics and Engineering*, Vol. 196, 2751–2764, 2007.
37. Alves, C. J. S. and P. R. S. Antunes, "The method of fundamental solutions applied to the calculation of eigenfrequencies and eigenmodes of 2D simply connected shapes," *CMC: Computers, Materials & Continua*, Vol. 2, 251–266, 2005.
38. Fox, L., P. Henrici, and C. Molor, "Approximations and bounds for eigenvalues of elliptic operators," *SIAM J. Numer. Anal.*, Vol. 4, 89–102, 1967.
39. Betcke, T. and L. N. Trefethen, "Reviving the method of

particular solutions," *SIAM Review*, Vol. 47, 469–491, 2005.

40. Trefethen, L. N. and T. Betcke, "Computed eigenmodes of planar regions," *AMS Contemporary Mathematics*, Vol. 412, 297–314, 2006.
41. Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipescipes in C++*, 2nd edition, Cambridge University Press, 2002.
42. Kupradze, V. D., "On approximate solution of problems in mathematical physics," *Russian Math. Surveys*, Vol. 22, 58–108, 1967.
43. Vekua, I. N., *New Methods for Solving Elliptic Equations*, North-Holland, Amsterdam, 1967.
44. Hafner, C., *The Generalized Multipole Technique for Computational Electromagnetics*, Artech House Books, Boston, 1990.