# Isotropic Clamped-Free Thin Annular Circular Plate Subjected to a Concentrated Load

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*The problem of an isotropic annular plate clamped along one edge and free at the other and subjected to a concentrated load is solved by a series approximation. The continuity conditions of deflection, slope, shear and radial moments at the radius of load application are satisfied. Variations of deflection coefficient, radial moment coefficients and shear coefficients with radius and angle are presented.* [DOI: 10.1115/1.2165235]

## 1 Introduction

Annular plate problems occur in engineering application, for example in the design of structures where a load is supported by a circular overhang. Some of the early attempts to solve the annular plate problems include the work of Conway [1] who considered an annular plate with linearly varying thickness subjected to a uniformly distributed load and a line load uniformly distributed along the edge of the hole. The nature of the problem solved by Conway [1] ensures that variations along the circumference varnish and that simplifies the governing equations greatly. Sherborne and Murthy [2] considered the elastic bending of an anisotropic annular plate of variable thickness. But like the work of Conway [1], the solution is only valid for symmetrical loading. Minguez and Vogwell [3] had solved the problem of an isotropic clamped-free annular plate subjected to a uniform pressure. Lord and Yousef [4] had attempted a similar problem by using numerical methods. Bird and Steele [5] presented an elegant treatment of a circular plate with arbitrary number of circular holes subjected to loading along boundaries.

Recently, Sharafutdinov [6] has considered the problem of an annular plate subjected to concentrated load along its edges, using the theory of functions of a complex variable. Sharafutdinov [6] obtained stress distribution along the contour of the circular aperture. Frequently, however, the applied load in neither uniform nor symmetrical but concentrated. Furthermore, a more common mode of concentrated load application is not along the edge of an annular plate but normal to it. A common engineering design is an annular plate loaded by a load-bearing member, transmitting a concentrated load.

## 2 The Annular Plate Subjected to a Concentrated Load

Consider an isotropic annular circular plate clamped at the outer edge and free at the inner edge, such as shown in Fig. 1. The plate is subjected to normal concentrated load, $P$ applied at point A at distance $b$ from the center O of the plate. Timoshenko [7] solved a similar problem for the circular plate clamped along its edge. In this paper, an approximate solution is obtained for the annular problem. The solutions are obtained by dividing the circular plate into the inner and outer parts. The separate solutions are required to satisfy continuity relationship along the radius of the load application. The radial moment, shear and deflections variations with radius and angle are presented. This information can be used in predicting the failure in this type of structure.

## 3 Governing Equations

The general theory of plate deformations is well documented [7,8]. Consider an isotropic annular circular plate that is loaded at point A at a distance $b$ from the center of the plate (Fig. 1). The differential equation describing the deformation of the plate may be written in polar coordinates as [8]

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)=P/Drr \quad (1)$$

Following an approach used by Timoshenko [7], we may divide the plate into two parts by the cylindrical section of radius $b$ as shown in the figure by the dashed line. We then apply a series solution of the form

$$w=\sum_{m=0}^{\infty} R_m\cos(m\theta) \quad (2)$$

to the homogeneous equation each of the portions of the plate. Once an equation for deflection is known, other structural quantities like shear and moment may be readily obtained for the deflection equation. This approach avoids the existence of degenerate solutions that may exist when different series representations are introduced for the different structural quantities, as may become necessary in employing a numerical method [9]. Chen, Wu, Chen, and Lee [9] showed that by using different series representations for moment and shear forces, and by examining the resulting matrices, degenerate cases may result when the boundary integral equation and boundary element methods are employed. Equation (2) is convergent provided that $R_m$ is convergent. Substituting Eq. (2) into Eq. (1) gives

$$m^2(m^2-2)R_m(r)+rR_m'(r)-r^2R_m''(r) \\ -2m^2\{R_m(r)-rR_m'(r)+r^2R_m''(r)\}+2r^3R_m'''(r)+r^4R_m''''(r)=0 \quad (3)$$

Now substituting

$$R_m=r^n \quad (4)$$

into Eq. (3) gives

$$r^n\{(-2+n)(-1+n)^2n-2m^2(-1+n)^2+(-2m^2+m^4+2n-n^2)\} \\ =0 \quad (5)$$

Solving Eq. (5) for $n$ gives

$$\{n\}=\{m,m+2,-m+2,-m\} \quad (6)$$

Hence, in general, the equation describing the inner plate is

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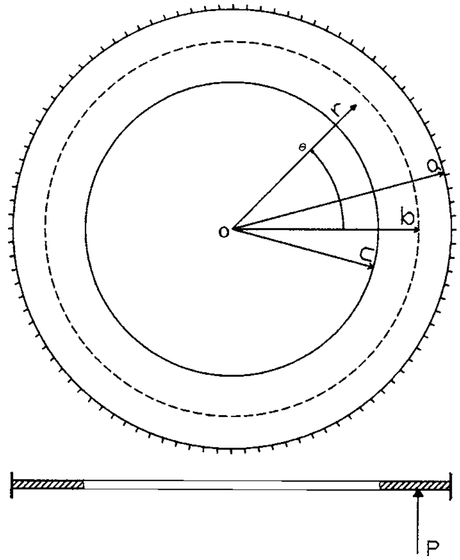


**Fig. 1 Annular circular plate subjected to a concentrated load**

$$R_m = A_m r^m + B_m r^{m+2} + C_m r^{-m+2} + D_m r^{-m} \tag{7}$$

Similar expressions to those given in Eq. (6) may be written for the functions $R_0'$, $R_1'$, and $R_m'$ corresponding to the inner portion of the plate. In the same sense, we employ the symbols $A_m'$, $B_m'$ and so on in the place of $A_m, B_m, \ldots$ for the constants of the outer part of the plate.

Specifically, the equation describing the outer plate becomes

$$R_m' = A_m' r^m + B_m' r^{m+2} + C_m' r^{-m+2} + D_m' r^{-m} \tag{8}$$

In this way, the basis for the inner and outer sections emerges naturally from the solution of the governing equations. This method of arriving at the basis has numerously been applied by Timoshenko [7], Gupta [10], and Carrier [11,12] in the solution of this type of problems. Recently, Chen, Wu, and Lee [13] have shown how different bases may be selected for the inner and outer sections of the plate of a circular plate. Using different bases for the inner and outer regions of a circular plate, as done for example by Chen et al. [13], averts the existence of singularity at the center of the plate, as $r$ tends to zero. Such a situation does not arise in an annular plate.

## 4 Boundary Conditions

For each of the terms in Eqs. (7) and (8), we have to determine four constants for the outer portion of the plate and four for the inner portion. Hence, a total of eight unknowns are involved in the solution, requiring eight independent equations. Four of these equations are obtained from the inner and outer boundary conditions at the edges of the plate. Four additional equations are obtained from the continuity conditions along the circle of radius $b$. The applied concentrated load is readily expanded as Fourier series,

$$P \approx P\left[\frac{1}{2} + \sum_{m=1}^{\infty} \frac{2\sin(m\pi/2)}{m\pi}\cos(m\theta)\right],\ m = 1,3,5,\ldots,\ 0 \leqslant \theta \leqslant \pi/2. \tag{9}$$

Thus, the boundary conditions involved are obtained as follows. Since the plate is clamped at the outer radius $r=a$, deflection and slope are zero on this boundary. Hence

$$R_m(r=a) = 0 \tag{10}$$

$$\frac{\partial R_m}{\partial r}(r=a) = 0 \tag{11}$$

Since the inner boundary of the plate is free, the requirements that the shear and moment be zero on this boundary lead, respectively, to

$$\left.\left(\frac{\partial^3 R_m}{\partial r^3} - \frac{1}{r^2}\frac{\partial R_m}{\partial r} + \frac{1}{r}\frac{\partial^2 R_m}{\partial r^2} - \frac{m^2}{r^2}\frac{\partial R_m}{\partial r}\right)\right|_{r=c} = 0 \tag{12}$$

$$\left.\left(\frac{\partial^2 R_m}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial R_m}{\partial r} - \frac{m^2}{r^2}R_m\right)\right)\right|_{r=c} = 0 \tag{13}$$

Along the circle $r=b$ where the concentrated load is placed, the following continuity equations are imposed [7],

$$R_m(r=b) = R_m'(r=b) \tag{14}$$

$$\frac{\partial R_m}{\partial r}(r=b) = \frac{\partial R_m'}{\partial r}(r=b) \tag{15}$$

$$\frac{\partial^2 R_m}{\partial r^2}(r=b) = \frac{\partial^2 R_m'}{\partial r^2}(r=b) \tag{16}$$

$$N_r(R_0)|_{r=b} - N_r(R_0')|_{r=b} = \frac{P}{4\pi b} \tag{17}$$

$$N_r(R_m)|_{r=b} - N_r(R_m')|_{r=b} = \frac{P\sin(m\pi/2)}{\pi^2 mb} m = 1,3,5\ldots \tag{18}$$

where

$$N_r(R) = -D\left\{\frac{\partial^3 R}{\partial r^3} + \frac{1}{r}\frac{\partial^2 R}{\partial r^2} - \frac{1}{r^2}\frac{\partial R}{\partial r} - \frac{m^2}{r^2}\frac{\partial R}{\partial r}\right) \tag{19}$$

Also, from Eq. (9),

$$R_m = 0, m = 2,4,8,\ldots \tag{20}$$

and $R_m$ converges as the coefficients of the Fourier series (9).

## 5 The Solution

Using the boundary conditions in the governing equations gives the desired solutions. The solution of the eight simultaneous equations is both tedious and prone to algebraic errors. For simplicity, the external radius of the annular circular plate is taken to be unity, since this does not affect the generalization of the results. Other dimensions are normalized with respect to the outer radius. Furthermore, a symbolic tool, Mathematica [14], has been used to carry out the necessary algebraic simplifications. Depending on the value of m, different solutions are obtained. For example, for $m=0$, Eqs. (6) yield the solutions,

$$R_0 = A_0 r^2 + B_0 r^2 \ln r + C_0 + D_0 \ln r \tag{21}$$

$$R_0' = A_0' r^2 + B_0' r^2 \ln r + C_0' + D_0' \ln r \tag{22}$$

Similarly, for $m=1$, Eq. (6) yields the solutions

$$R_1 = A_1/r + B_1 r + C_1 r \ln r + D_1 r^3 \tag{23}$$

$$R_1' = A_1'/r + B_1' r + C_1' r \ln r + D_1' r^3 \tag{24}$$

For $m>1$, the solution are written consistently as

$$R_m = A_m r^{m+2} + B_m r^{-m+2} + C_m r^{-m} + D_m r^m \tag{25}$$

$$R_m' = A_m' r^{m+2} + B_m' r^{-m+2} + C_m' r^{-m} + D_m' r^m \tag{26}$$

Using the boundary conditions expressed in Eqs. (10)–(19), the constants $A_n$, $B_n$, $C_n$, $D_n$, $A_n'$, $B_n'$, $C_n'$, $D_n'$, for $\{n=0,1,m\}$ are obtained. The expressions for the coefficients are given in the Appendix . Using Mathematica, the coefficients for $m=0$ and $m=1$ are found to reduce to the coefficients of the isotropic circular

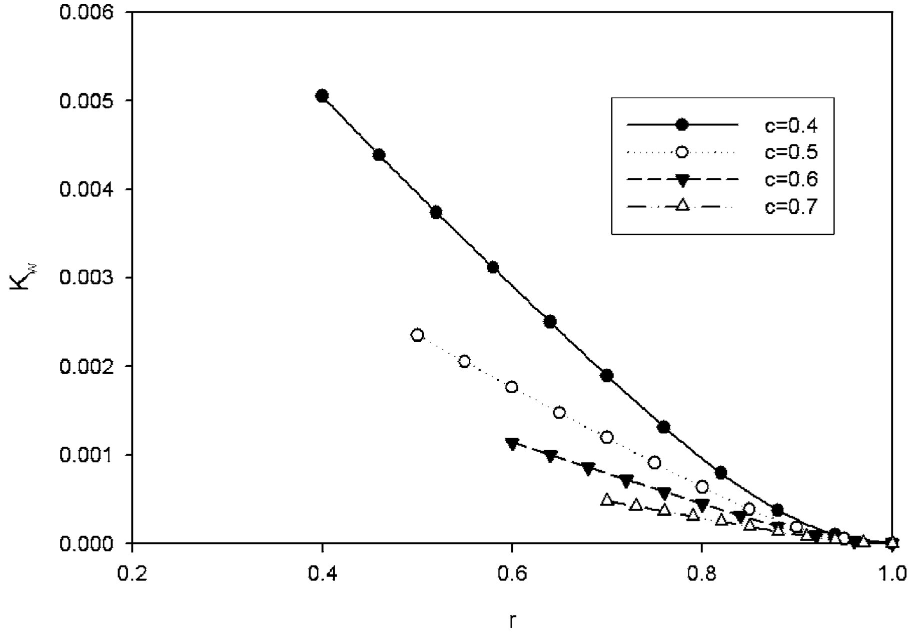


**Fig. 2 Variation of deflection coefficient versus $r[\theta=0, b=(a+c)/2]$**

plate in the limit as the inner radius becomes vanishingly small.

## 6 Results and Discussions

It is convenient to express the deflection, the radial moment, and shear, respectively, as

$$w=\frac{k_w P}{Drr} \tag{27}$$

$$M_r=\frac{k_{\mathrm{mr}} P}{Drr} \tag{28}$$

$$M_s=\frac{k_s P}{Drr} \tag{29}$$

Using a Poisson ratio of 0.3, the following results are obtained. Figure 2 shows the variation of deflection coefficient with radius for different annular radii, $c$, for a load placed at the center of the plate. The smaller the inner radius is, the less the deflection for the range of $c$, considered. Figure 3 shows the variation of radial moment with radius. Between the inner radius and the point of load application, moment is vanishingly small but increases suddenly from the point of load application to a maximum at the clamped outer radius. Figure 4 shows the variation of shear with radius. Shear is zero at the inner free boundary. From the point of load application, shear is suddenly finite and decreases in magnitude towards the outer clamped end. Figure 5 shows the variation of deflection with the circumferential angle. As expected, deflection is maximum along the meridian of load application and diminishes at the angle increases.

## 7 Conclusion

The problem of the point-loaded annular plate problem with one edge clamped and the other free has been solved using a series approximation. The approach had divided the plate into the inner and outer regions, based on the radius at which the concentrated load is located. The continuity of deflection, slope, shear, and radial moments at the radius of load application are satisfied. Variations of deflection coefficient, radial moment coefficients, and shear coefficients with radius and angle have been presented.

## Nomenclature

| | | |
|---|---|---|
| $a$ | = | outer radius of the annular plate |
| $c$ | = | inner radius of the annular plate |
| $b$ | = | distance of point load from the plate's center |
| $r,\theta$ | = | Cartesian coordinates axes |
| $w$ | = | deflection |
| $M_r$ | = | moment radial component |
| $Drr$ | = | uniform flexural rigidity of plate |
| $\nu$ | = | Poisson's ratio |
| $P$ | = | concentrated load |
| $K_w$ | = | deflection coefficient |
| $K_{\mathrm{mr}}$ | = | radial moment coefficient |
| $K_s$ | = | shear coefficient |
| $m$ | = | series index |

## Appendix

$$A_0=[1+b^2-(1+b^2-2c^2)v+2c^2(1+v2\log b)]/\Delta_0$$

$$B_0=1/(8\pi\Delta_0)$$

$$C_0=[v_1+b^2v_1-2c^2v_2(1+\log b)]/\Delta_0$$

$$D_0=2[b^2v_1-c^2v_2-2c^2v_2\log b]/\Delta_0$$

$$A_0'=[v_1-b^2v_1+2v_1\log b]/\Delta_0$$

$$B_0'=0$$

$$C_0'=[(b^2-1)(v_1-2c^2v_2)+2(-c^2v_2+b^2(v_1-c^2v_2))\log b]/\Delta_0$$

$$D_0'=2c^2v_2[-1+b^2-2\log b]/\Delta_0$$

$$\begin{aligned}A_1=[&-b^4(1-2v)-3c^4(2+v)+b^2c^2(-b^2(-3+v)\\&+c^2(9+5v)+v_2)+2b^2c^2(-c^2(3+v)+v_2)\log b-(-(-1\\&+2b^2)c^4(3+v)+b^4v_1)\log c]/\Delta_1\end{aligned}$$

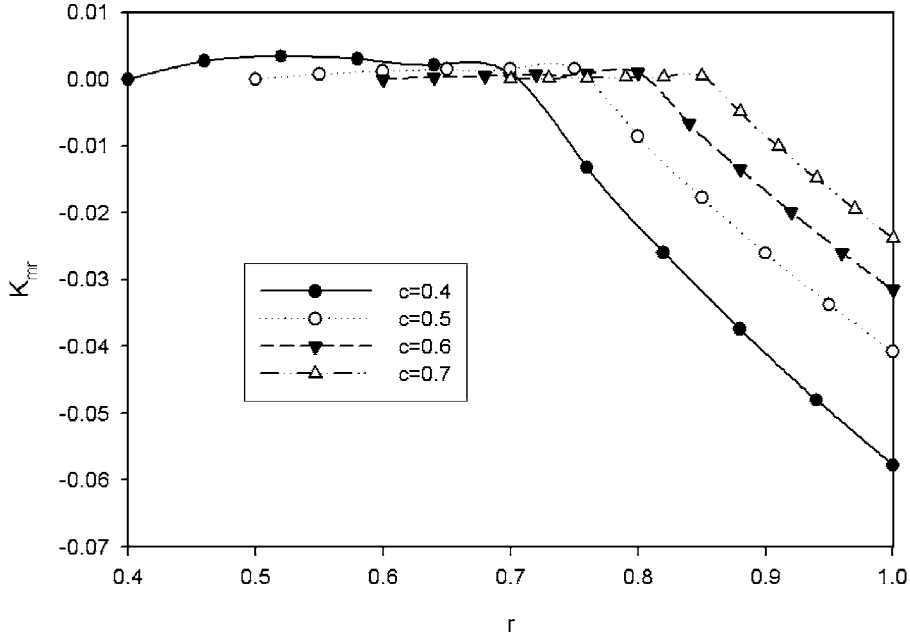


**Fig. 3 Variation of radial moment coefficient versus $r[\theta=0, b=(a+c)/2]$**

$$B_1 = [-2b^2(-1+c^2)(c^2(3+v)-v_1)\log b + (-1+b^2)(-9c^4-5c^4v+b^2(-1+c^2(-3+v))+3v) + X_5 - 2(c^4(3+v)-b^2v_1)\log c]/\Delta_1$$

$$C_1 = 2[-(b^2+3c^2)(1+b^2+(-1+3b^2)c^2) + (1+c^2)(b^4+c^2+b^2(3-5c^2))v + 2b^2(-c^4(3+v) + v_1)\log(c/b)]/\Delta_1$$

$$D_1 = [b^4v + c^4(3+2v) + b^2(1-3v+c^2v_2) - X_5 + 2b^2(v_1-c^2v_2)\log b + (c^4(3+v)+b^2(-2+b^2)v_1)\log(c)]/\Delta_1$$

$$A_1' = [c^2[2b^2(c^2(3+v)-v_2)\log b + (-1+b^2)\{c^2+(-3(2+v) + b^2(3+2v)) + b^2v_2 + (-1+b^2)c^2(3+v)\log c\}]]/\Delta_1$$

$$B_1' = [-[2b^2\log b(1-3v+c^4(9+5v)-2a^2X_5+2(c^4(3+v) - v_1)\log c) + (-1+b^2)(-9c^4-5c^4v+b^2(-1+c^2(-3+v)) + 3v) + X_5 - 2(c^4(3+v)-b^2v_1)\log c)]]/\Delta_1$$

$$C_1' = 2[(-1+b^2)(-c^4(3+v)+b^2(c^2(-3+v)+v_1)-X_5) + 2b^2(c^4(3+v)-v_1)\log b]/\Delta_1$$

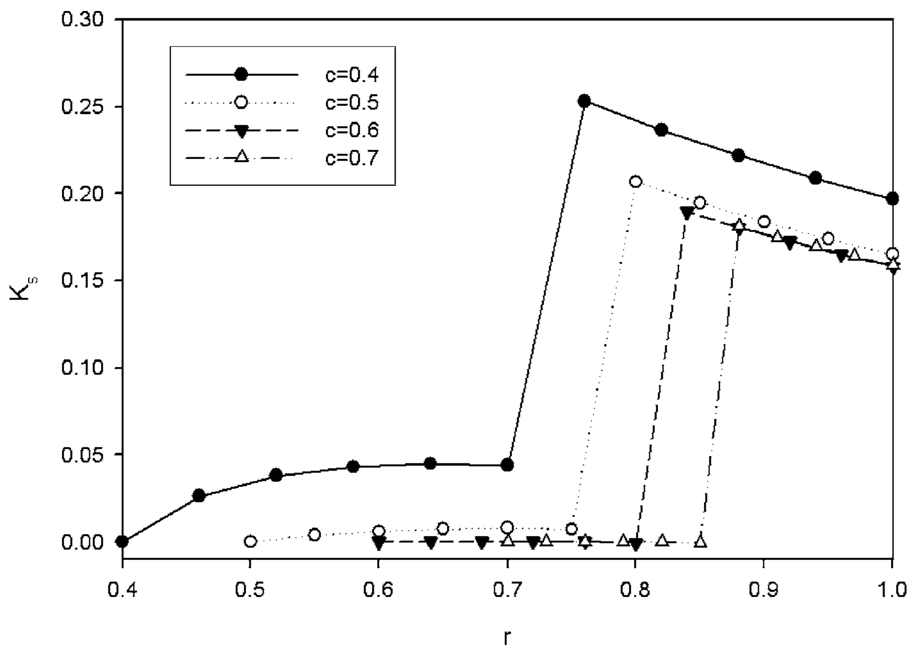


**Fig. 4 Variation of shear moment coefficient versus $r[\theta=0, b=(a+c)/2]$**

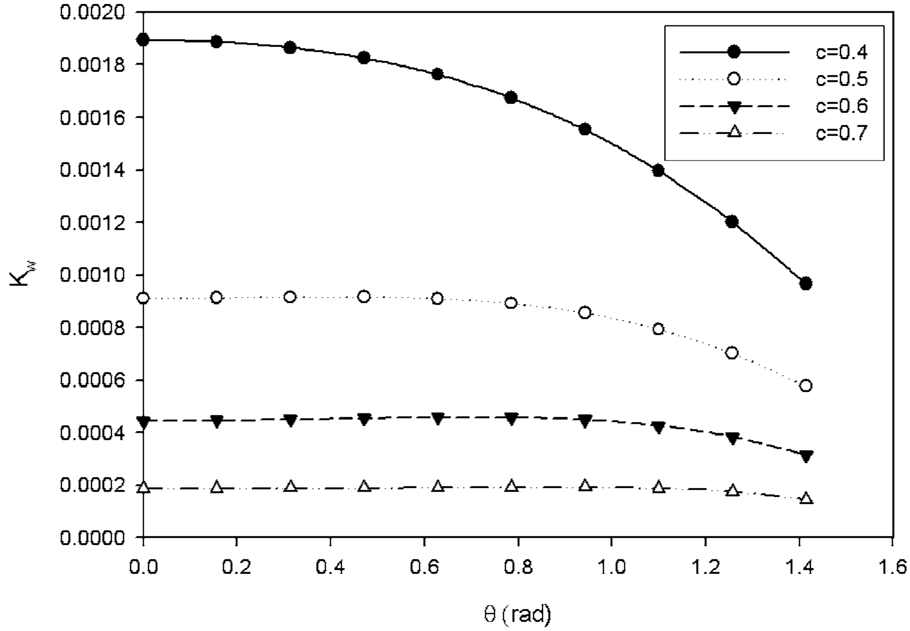


**Fig. 5 Variation of deflection coefficient versus $\theta[r=a+c)/2, b=(a+c)/2]$**

$$D_1' = [2b^2(v_1 - c^2v_2)\log b + (-1+b^2)(1-2v+b^2v+c^2(1+v)+(-1+b^2)v_1\log c)]/\Delta_1$$

$$A_m = [b^{-m}[c^{2+4m}(-mn_2+(-2+m^2)v) - b^{2m}c^2(b^2m-n_1)(-2v + mX_4) + c^{2m}(-b^2m(b^{2m}-n_1)v_1 - c^4(m^2(1+3v)-4v_2) + c^2n_1(m_1X_3+(b^{2m}-b^2m)X_4))]]/\Delta_2$$

$$B_m = [b^{-m}[c^{2+4m}(b^2m-m_1)(-mn_2+(-2+m^2)v) - b^{2m}c^2(2v - mX_4) + c^{2m}\{b^2mv_1 + c^2m_1X_3 + b^{2m}\{-c^4(-4(1+v)+m^2(1+3v)) + b^2mm_1v_1 + c^2(b^2m(2-mn_1+(-1+m)mv)\}\}]]/\Delta_2$$

$$C_m = [b^{-m}[c^{2+4m}m(m-b^2n_1)(-mn_2+(-2+m^2)v) - c^{4+2m}(-1 + b^{2m}n_1) + (-4(1+v)+m^2(1+3v)) - b^{2+2m}c^{2m}m^3v_1 - b^{2+2m}c^2m(-2v+mX_4) + c^{2+2m}mn_1(-b^2X_4 + b^{2m}(b^2(-2 + m(n_1+v-mv)) + mX_4))]]/\Delta_3$$

$$D_m = [b^{-m}[-b^2c^{2+4m}m + (-mn_2+(-2+m^2)v) - b^{2m}c^2m(m - b^2m_1)(-2v+mX_4) + c^{2m}\{-b^{2+2m}c^2m(2+m^2v-m(n_1+v)) + c^4m_1(m^2+3m^2v-4v_2) + b^{2m}c^4(m^2(1+3v)-4v_2) - c^2m^2m_1X_3 + b^2(-m^3v_1+c^2mm_1n_1)\}]]/\Delta_3$$

$$A_m' = [b^{-m}[-b^2c^{2+4m}m + (-mn_2+(-2+m^2)v) - b^{2m}c^2m(m - b^2m_1)(-2v+mX_4) + c^{2m}\{-b^{2+2m}c^2m(2+m^2v-m(n_1+v)) + c^4m_1(m^2+3m^2v-4v_2) + b^{2m}c^4(m^2(1+3v)-4v_2) - c^2m^2m_1X_3 + b^2(-m^3v_1+c^2mm_1n_1)\}]]/\Delta_3$$

$$B_m' = [b^{-m}c^{2m}[-c^{2+2m}(b^{2m}-b^2m+m_1)(-mn_2+(-2+m^2)v) + m(b^2+b^{2m}(-m+b^2m_1))v_1 + c^2(2+m^2v-m(n_1+v) + b^{2m}m_1(b^2m-n_1)X_3)]]/\Delta_2$$

$$C_m' = [b^{-m}c^{2+2m}[c^{2m}m(b^{2+2m}+m-b^2n_1)(-mn_2+(-2+m^2)v) + c^2\{4(1+v)-m^2(1+3v)+b^{2m}\{n_1(-4+m^2-4v+3m^2v) - b^2(m^3+3m^3v-4mv_2)\}\} - m(b^2+b^{2m}(-m+b^2m_1))]]/\Delta_3$$

$$D_m' = [b^{-m}c^2[b^{2m}(c^{2+2m}(-4+m^2-4v+3m^2v)+m^2(2v+m(m_2 - mv))) + c^{2m}m_1(c^2(-4+m^2-4v+3m^2v)-m^2X_3) + b^{2+2m}mm_1(-2v-c^{2m}X_3+mX_4) + b^2m(-2v+c^{2m}(-c^2(m^2 + 3m^2v-4v_2)+m_1n_1X_3)+mX_4)]]/\Delta_3$$

$$\Delta_0 = 32\pi(a^2v_1 - c^2v_2)D_{rr}/P$$

$$\Delta_1 = 16b\pi[1+4c^2-2v+c^4(3+2v)-(-c^4(3+v)+v_1)\ln c]D_{rr}/P$$

$$\Delta_2 = 8m^2\pi^2m_1[c^{2+4m}(-mn_2+(-2+m^2)v)+c^{2m}(-4c^2(-1+m^2) + m^2v_1 - c^4(m^2(1+3v)-4v_2)) + c^2(-2v+mX_4)]D_{rr}/P$$

$$\Delta_3 = m\Delta_1 n_1/m_1 m_1 = m-1, m_2 = m-2, n_1 = m+1, n_2 = m+2, v_1 = v-1, v_2 = v+1, X_1 = 2m-v-1$$

$$X_2 = 2m+v+1, X_3 = m(v-1)-2, X_4 = 2+m(v-1), X_5 = c^2(v-3)$$

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