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ORIGINAL PAPER

On solvability of a boundary integral equation of the first kind for Dirichlet boundary value problems in plane elasticity

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Abstract The solution of a Dirichlet boundary value problem of plane isotropic elasticity by the boundary integral equation (BIE) of the first kind obtained from the Somigliana identity is considered. The logarithmic function appearing in the integral kernel leads to the possibility of this operator being non-invertible, the solution of the BIE either being non-unique or not existing. Such a situation occurs if the size of the boundary coincides with the so-called critical (or degenerate) scale for a certain form of the fundamental solution used. Techniques for the evaluation of these critical scales and for the removal of the non-uniqueness appearing in the problems with critical scales solved by the BIE of the first kind are proposed and analysed, and some recommendations for BEM code programmers based on the analysis presented are given.

Keywords Boundary integral equation of the first kind · Critical scale · Symmetric Galerkin boundary element method · Plane elasticity

1 Introduction

The direct Boundary Element Method (BEM), when applied to a Dirichlet Boundary Value Problem (DBVP) of plane isotropic elasticity (with prescribed boundary displacements), usually leads to the solution of the Boundary Integral Equation (BIE) of the first kind obtained from the Somigliana displacement identity. The integral kernel of the operator of this BIE includes the logarithmic function. It is well known that the logarithmic function present in the integral kernel may cause the corresponding operator to be non-invertible, the solution of the BIE either being non-unique or not existing. Although, the occurrence of the phenomenon is strictly size dependent and arises only under very specific conditions, it requires accounting for. The corresponding boundary sizes are usually referred to as critical (or degenerate) scales. For each domain with a bounded boundary there exist one or two such scales, see Vodička and Mantič [15].

The phenomenon of critical scales can be avoided when solving DBVPs using second-kind BIEs, e.g. by applying the hypersingular BIE (the Somigliana traction identity) solved by the direct BEM or the strongly-singular double-layer potential BIE solved by the indirect BEM, see Vodička and Mantič [16] and Linkov [10], respectively, for other references and discussions of these approaches in the potential theory and elasticity. However, the Galerkin discretizations of these BIEs do not lead to symmetric linear systems, which is the case of the present first-kind BIE. Moreover, error estimates can be obtained more easily for the present symmetric first-kind BIE, see Hsiao and Wendland [9], than in the second-kind BIEs, in particular for non-smooth boundaries.

The above phenomenon appears not only in elasticity problems, but also in other two-dimensional BVPs which lead to BIEs with logarithmic kernels, e.g. those governed by the Laplace or biharmonic equations, see [3,4,6]. For the plane elasticity theory, the proof of the critical scale existence was given by Constanda [5], introducing a 3×3 constant matrix to identify the critical scales of a boundary. Recently, the theory has been further developed by the authors [15]

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reducing the dimension of this matrix by one and determining its scaling properties, which made it possible to show the existence of two single or one double critical scale for any bounded boundary. Additionally an upper bound for the size of the boundary which ensures the positivity and invertibility of the single-layer operator has been determined. At the same time, closely related results concerning the possible lack of positivity of the single-layer operator have been obtained by Steinbach [13], where, in addition to positivity dealt with in [15], a stronger property of the ellipticity of the single-layer operator has been ensured for an appropriate scaling of the fundamental solution.

Several approaches to the elimination of the non-trivial null-space in the resulting linear system caused by the critical scale of the boundary have been presented in the past. Simple re-scaling of the domain or an equivalent modification of the fundamental solution represent the most often used approaches in BEM. Nevertheless, there are also some other theoretically well-based approaches for removing the non-uniqueness from the solution of the single-layer potential BIE, which may be solved by BEM, proposed in [2,8].

Two techniques, based on the Fredholm theory of integral operators [2], for the removal of the non-uniqueness appearing in the BVPs solved by the BIE of the first kind with critical scales are discussed in the present work. For the sake of brevity, the present paper relies highly upon the previous results of the authors, published especially in [15] and [18], some previous theoretical results being shown, tested and explained through numerical solutions by BEM. Moreover, propositions only mentioned in [15] are described here in detail and proven.

First, starting from an analysis of the positiveness of the integral operator of the first kind [15], this operator is augmented as proposed in [6,7], yielding an invertible BIE system. Following [6], this augmenting leads to a definition of the operator $\boldsymbol{B}_\Gamma$ defined on $\mathbb{R}^2$, which represents a generalization of the Robin constant in the potential theory to plane elasticity and permits an easy search for critical scales of a boundary, see [15]. Moreover, some important properties of this operator obtained from its tensor character are shown here, together with a simple proof of the tensor character itself. Note that the 2×2 matrix of the operator $\boldsymbol{B}_\Gamma$ is in fact the left-upper corner submatrix of the 3×3 matrix studied in [5] and is also represented by the matrix of Lagrange multipliers introduced in [13].

Second, the nontrivial null-space of the original BIE operator is eliminated by adding an operator with a suitable degenerate kernel.

The Symmetric Galerkin Boundary Element Method (SGBEM) [1] has been used for the numerical analysis of DBVPs. A procedure for the evaluation of critical scales is implemented and tested using two boundary shapes. The values of the critical scales are computed by using SGBEM for the evaluation of the numerical approximations of the corresponding operators $\boldsymbol{B}_\Gamma$. Examples of the matrices of $\boldsymbol{B}_\Gamma$, associated to these boundary shapes, are presented, for the first time to the authors' knowledge. At the same time, the numerical results obtained by both aforementioned techniques for the non-uniqueness removal in the solution of the DBVPs are presented and discussed. The differences in the solution of the BIEs in their original or modified form are discussed in order to give, thinking in particular of the programmers of BEM codes, ideas on how to treat the results obtained.

2 BIE

The traction $\boldsymbol{t} = (t_1, t_2)$ solution of a DBVP with the displacement boundary condition $\boldsymbol{u} = (u_1, u_2) = \boldsymbol{g}$ prescribed on the bounded Lipschitz boundary Γ of a domain $\Omega \subset \mathbb{R}^2$ can be computed by solving the BIE obtained from the Somigliana displacement identity:

$$\int_\Gamma U_{ij}(x,y)t_j(y)\mathrm{d}\Gamma(y) = \frac{1}{2}g_i(x) + \fint_\Gamma T_{ij}(x,y)g_j(y)\mathrm{d}\Gamma(y) =: \tilde{g}_i(x), \qquad (1)$$

with $i = 1, 2$. The equation can be written in a matrix operator notation as

$$\boldsymbol{U}_\Gamma \boldsymbol{t} = \tilde{\boldsymbol{g}}. \qquad (2)$$

The integral kernel

$$U_{ij}(x,y) = \Lambda\left(\kappa\delta_{ij}\ln\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right) \qquad (3)$$

with

$$\kappa = 3-4\nu, \quad \Lambda = \frac{1}{8\pi G(1-\nu)} \qquad (4)$$

represents the symmetric tensor of a fundamental solution of the Navier equation for the plane strain case, with G being the elastic shear modulus, ν the Poisson ratio, and T_{ij} the tensor of the pertinent tractions, defined using the unit normal vector outward with respect to Ω. Note that ν for practical materials lies in the interval $\langle 0; \frac{1}{2})$. In the standard numerical solution the case of ν very close to $\frac{1}{2}$ originates difficulties as the operator $\boldsymbol{U}_\Gamma$ loses its ordinary positivity property for $\nu = \frac{1}{2}$. DBVPs with ν very close to $\frac{1}{2}$ should be studied separately [12] and will not be considered in what follows.

The same BIE (1) is valid for both interior and exterior DBVPs, where for the exterior DBVPs we assume the radiation condition:

$$u_i(x) = U_{ij}(x,0)b_j + \mathrm{O}(\|x\|^{-1}),$$
$$b_j = \int_\Gamma t_j(y)\mathrm{d}\Gamma(y), \quad \|x\| \to \infty. \tag{5}$$

It should be noted that this solution may be different from that known as a generalized regular solution, see [2], which assumes zero integral of the traction along the boundary.

3 Prediction and analysis of critical scales

3.1 $\boldsymbol{B}_\Gamma$ as a generalization of the Robin constant

In this section the results obtained in [15] will be resumed and completed, see also [5–7,13]. The invertibility of the left-hand side operator in (2) depends on the size and shape of the boundary and on the fundamental solution considered. With the present U_{ij}, if Γ is contained in the interior of a disk with radius $R = \exp\left(\frac{1}{2\kappa}\right)$, then $\boldsymbol{U}_\Gamma$ is positive, hence invertible. There exist one or two positive constants ρ, such that $\boldsymbol{U}_{\rho\Gamma}$ is not invertible, where $\rho\Gamma = \left\{\rho x \in \mathbb{R}^2 | \, x \in \Gamma\right\}$ naturally violates the previous condition. Nevertheless, augmenting the operator $\boldsymbol{U}_\Gamma$ may lead to an invertible operator [6,7]. The system

$$\boldsymbol{U}_\Gamma \boldsymbol{t} - \sum_{k=1}^{2} \omega_k \boldsymbol{\mu}^k = \tilde{\boldsymbol{g}}, \quad \int_\Gamma \mu_i^k t_i \mathrm{d}\Gamma = \int_\Gamma t_k \mathrm{d}\Gamma = \xi_k,$$
$$\boldsymbol{\mu}^1(x) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \boldsymbol{\mu}^2(x) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \tag{6}$$

has for any $\tilde{\boldsymbol{g}}$ and for each $\boldsymbol{\xi}$ a unique solution $(\boldsymbol{t}, \boldsymbol{\omega})$. The functions $\boldsymbol{\mu}^k$ form a basis for the space of rigid body translations in plane.

Thus, we can introduce a linear matrix operator $\boldsymbol{B}_\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ [6,15] which for a given $\boldsymbol{\xi}$ and the fixed $\tilde{\boldsymbol{g}} = \boldsymbol{0}$ finds the corresponding $\boldsymbol{\omega}$, i.e.

$$\boldsymbol{B}_\Gamma \boldsymbol{\xi} = \boldsymbol{\omega} \quad \Leftrightarrow \quad \boldsymbol{U}_\Gamma \boldsymbol{t} = \omega_1 \boldsymbol{\mu}^1 + \omega_2 \boldsymbol{\mu}^2 \wedge \int_\Gamma \boldsymbol{t} \mathrm{d}\Gamma = \boldsymbol{\xi}. \tag{7}$$

The operator $\boldsymbol{B}_\Gamma$ inherits the properties of invertibility, symmetry and positiveness of the operator $\boldsymbol{U}_\Gamma$, and the reciprocity also holds. Therefore, the task of finding the critical scales of the operator $\boldsymbol{U}_{\rho\Gamma}$ is reduced to the investigation of these properties for the operator $\boldsymbol{B}_{\rho\Gamma}$.

Considering $\boldsymbol{\xi}^1 = \boldsymbol{\mu}^1$ and $\boldsymbol{\xi}^2 = \boldsymbol{\mu}^2$ in the definition (7) of the 2×2 constant symmetric matrix $\boldsymbol{B}_\Gamma$, it can be shown, in view of the invertibility of the system (6), that there exists a corresponding unique 2×2 matrix of the so-called natural densities $\boldsymbol{\mathcal{T}}(x) = \left(\boldsymbol{\tau}^1(x), \boldsymbol{\tau}^2(x)\right)$ defined on Γ such that

$$\boldsymbol{U}_\Gamma \boldsymbol{\mathcal{T}} = \boldsymbol{B}_\Gamma. \tag{8}$$

In this sense the matrix $\boldsymbol{B}_\Gamma$ represents a generalization of the Robin constant in the potential theory to plane elasticity. Relation (8) also generalizes in some way a theorem by Constanda [5] for Γ given by a single curve (Theorem 2 therein).

The key point in an analysis of the critical scales is to know how the operators change with a domain scaling. The scaling properties of the operator $\boldsymbol{B}_\Gamma$ have been analysed in [15], where also a relation between the real eigenvalues σ of the symmetric operator $\boldsymbol{B}_\Gamma$ and the critical scales $\rho = \rho_c$ of the operator $\boldsymbol{U}_{\rho\Gamma}$ was deduced:

$$\boldsymbol{B}_{\rho\Gamma} = \boldsymbol{B}_\Gamma - \Lambda\kappa \ln \rho \boldsymbol{I} \quad \Rightarrow \quad \rho_c = \exp\left(\frac{\sigma}{\Lambda\kappa}\right), \tag{9}$$

$\boldsymbol{I}$ being the unit operator. If there is only one critical scale, ρ_c, the null-space dimension of $\boldsymbol{U}_{\rho_c\Gamma}$ is two. Otherwise for each particular ρ_c this dimension is one.

3.2 $\boldsymbol{B}_\Gamma$ as a tensor

Another important property of the operator $\boldsymbol{B}_\Gamma$ is its tensor character, $\boldsymbol{B}_\Gamma$ representing a second order tensor. The proof of its transformation property between two orthogonal cartesian coordinate systems in $\mathbb{R}^2$ is rather straightforward.

Let $\mathbf{Q}$ be the orthogonal matrix of the transformation from the original coordinate system (x_1, x_2) to the transformed one (x_1', x_2'). Rewriting (7) for the new coordinate system,

$$\boldsymbol{B}'_\Gamma \boldsymbol{\xi}' = \boldsymbol{\omega}' \Leftrightarrow \boldsymbol{U}'_\Gamma \boldsymbol{t}' = \omega_1' \boldsymbol{\mu}'^1 + \omega_2' \boldsymbol{\mu}'^2 \wedge \int_\Gamma \boldsymbol{t}' \mathrm{d}\Gamma = \boldsymbol{\xi}', \tag{10}$$

introduces the transformed vector $\boldsymbol{t}'$, operator $\boldsymbol{B}'_\Gamma$ and rigid body translation $\boldsymbol{\mu}'$. The tensor character of tractions and displacements makes valid the following relations, transforming the quantities to the original coordinate system:

$$\boldsymbol{t}' = \mathbf{Q}\boldsymbol{t}, \quad \boldsymbol{U}'_\Gamma \boldsymbol{t}' = \mathbf{Q}\left(\boldsymbol{U}_\Gamma \boldsymbol{t}\right). \tag{11}$$

Substituting these relations into (10), we have

$$\mathbf{Q}\boldsymbol{U}_\Gamma \boldsymbol{t} = \omega_1' \boldsymbol{\mu}'^1 + \omega_2' \boldsymbol{\mu}'^2 \wedge \mathbf{Q} \int_\Gamma \boldsymbol{t} \mathrm{d}\Gamma = \boldsymbol{\xi}', \tag{12}$$

and realizing that the rigid body translations are the same in both coordinate systems, i.e. $\boldsymbol{\mu}^i(x) = \boldsymbol{\mu}'^i(x')$ renders

$$\boldsymbol{U}_\Gamma \boldsymbol{t} = \mathbf{Q}^T \left(\omega_1' \boldsymbol{\mu}^1 + \omega_2' \boldsymbol{\mu}^2\right) \wedge \int_\Gamma \boldsymbol{t} \mathrm{d}\Gamma = \mathbf{Q}^T \boldsymbol{\xi}'. \tag{13}$$

Then, definition of $\boldsymbol{B}_\Gamma$ in (7) can be used to find the transformation rule for $\boldsymbol{B}'_\Gamma$:

$$\boldsymbol{B}_\Gamma \mathbf{Q}^T \boldsymbol{\xi}' = \mathbf{Q}^T \boldsymbol{\omega}' \Rightarrow \boldsymbol{B}'_\Gamma = \mathbf{Q}\boldsymbol{B}_\Gamma \mathbf{Q}^T. \tag{14}$$

The tensor character of $\boldsymbol{B}_\Gamma$ expressed by $(14)_2$, together with its natural invariance with respect to a translation tranformation of Γ, helps to understand its form for a boundary Γ with a non-trivial symmetry transformation group.

Consider, first, a boundary with a reflection symmetry. Let the x_1-axis of the coordinate system be the symmetry axis of Γ. Then, the reflection transformation given by the matrix $\mathbf{Q}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ transforms Γ to itself. Therefore for the matrix of the operator $\boldsymbol{B}_\Gamma$ we have

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} = \boldsymbol{B}_\Gamma = \boldsymbol{B}'_\Gamma = \mathbf{Q}_1 \boldsymbol{B}_\Gamma \mathbf{Q}_1^T \;\Rightarrow\; b_{12} = 0, \tag{15}$$

this matrix being thus diagonal. It should be noted that this diagonal form is maintained also in the case where only the outer contour of a bounded domain with holes has a reflection symmetry. The reason for this is that $\boldsymbol{B}_\Gamma$ does not depend on the number, shape and position of holes in a bounded domain, as has been shown in [15].

Second, if the symmetry transformation group includes a rotation by an angle α different from $k\pi$ (k being an integer), then the matrix of the operator $\boldsymbol{B}_\Gamma$ is up to a multiplicative constant a unit matrix. In this case there exists only one double critical scale as mentioned above. Actually, in this case we can rotate the coordinate system, without a change of $\boldsymbol{B}_\Gamma$ by the angle α, the matrix of the corresponding transformation being $\mathbf{Q}_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$. Analogously to (15) the following relation can be obtained:

$$\boldsymbol{B}_\Gamma = \boldsymbol{B}'_\Gamma = \mathbf{Q}_\alpha \boldsymbol{B}_\Gamma \mathbf{Q}_\alpha^T,$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{16}$$

The solution of this matrix equation leads to the two following conditions, which render the desired form of $\boldsymbol{B}_\Gamma$:

$$\left.\begin{aligned} (b_{11} - b_{22})\sin^2\alpha &= 2b_{12}\sin\alpha\cos\alpha \\ -(b_{11} - b_{22})\sin\alpha\cos\alpha &= 2b_{12}\sin^2\alpha \end{aligned}\right\} \overset{\alpha\neq k\pi}{\Rightarrow} \\ \Rightarrow b_{12} = 0 \wedge b_{11} = b_{22}. \tag{17}$$

This simple character of $\boldsymbol{B}_\Gamma$ is obtained for a bounded domain with holes also in the case where only the outer contour has the above symmetry property.

Notice, that a natural consequence of the above studied properties of $\boldsymbol{B}_\Gamma$ is that its real eigenvalues σ, and subsequently also the critical scales ρ_c associated to Γ through (9), are invariant with respect to a translation and an orthogonal transformation (reflection or rotation) of Γ.

4 Solutions of the modified BIEs

When solving DBVPs by (1) it is necessary to eliminate unwanted solutions if Γ has a critical size. Two distinct approaches based on the theory of Fredholm operators (see [2]) will be discussed.

The first approach uses the augmented system (6), which has a unique solution for any given data. Unfortunately, if $\boldsymbol{\omega}$ is not equal zero, the found traction field $\boldsymbol{t}$ is not the solution of the original DBVP.

The idea of the second approach to modify (1) is to prescribe an equation in the form of the Somigliana identity at a point x_0 lying in the exterior to the outermost contour of Γ and to impose a vanishing displacement at such a point.

Let us discuss the interior DBVP first. As the integral of the tractions $\boldsymbol{t}$ along the whole boundary always vanishes, the solution of (6) with $\boldsymbol{\xi} = 0$ gives the solution of the interior DBVP, which is also a solution of (2), having vanishing $\boldsymbol{\omega}$, independently of the size of Γ.

If the solution of (2) is not unique due to a critical scale of Γ, there exists a non-trivial solution of (6) with $\tilde{\boldsymbol{g}} = 0$ and $\boldsymbol{\omega} = 0$. Nevertheless, the solution of (2) exists so we can apply the modification technique proposed in [18]. Let us introduce functions $\boldsymbol{v}$ and $\boldsymbol{w}$ via relations $v_l(x) = U_{kl}(x_0, x)$ and $w_l(x) = T_{kl}(x_0, x)$, where k, is a suitably chosen index for the fixed point x_0 lying out of the domain and its holes. Note that the functions $\boldsymbol{v}$ and $\boldsymbol{w}$ are not restricted to the present particular choice, as they can be chosen arbitrarily satisfying some conditions [2,14,18].

The modified system then reads:

$$(\boldsymbol{U}_\Gamma \boldsymbol{t})(x) + \boldsymbol{v}(x) \int_\Gamma \boldsymbol{v}(y)\boldsymbol{t}(y)\mathrm{d}\Gamma(y) \\ = \tilde{\boldsymbol{g}}(x) + \boldsymbol{v}(x) \int_\Gamma \boldsymbol{w}(y)\boldsymbol{g}(y)\mathrm{d}\Gamma(y). \tag{18}$$

It should be noted that in the case of a two-dimensional null-space, the above modification has to be done twice with distinct suitable choices of the couple (x_0, k).

The situation with exterior DBVPs is a bit more complicated. The solution of (2) may not exist, because $\tilde{\boldsymbol{g}}$ does not necessary lie in the range of the operator $\boldsymbol{U}_\Gamma$ if it is not invertible. For example, in the case of $\boldsymbol{g} = \boldsymbol{\mu}^k$ also $\tilde{\boldsymbol{g}} = \boldsymbol{\mu}^k$ and this function may be out of the range of $\boldsymbol{U}_\Gamma$, see [15].

If the solutions $\boldsymbol{t}$ of (2) and also of (18) exist and we denote the solution of (6) by $\hat{\boldsymbol{t}}$ then we have

$$\boldsymbol{U}_\Gamma \boldsymbol{t} = \tilde{\boldsymbol{g}}, \quad \int_\Gamma t_k \mathrm{d}\Gamma = b_k, \text{ (unknown } b_k), \tag{19}$$

$$\boldsymbol{U}_\Gamma \hat{\boldsymbol{t}} - \sum_{k=1}^{2} \omega_k \boldsymbol{\mu}^k = \tilde{\boldsymbol{g}}, \quad \int_\Gamma \hat{t}_k \mathrm{d}\Gamma = \xi_k, \tag{20}$$

which renders

$$\boldsymbol{U}_{\Gamma}(\hat{\boldsymbol{t}}-\boldsymbol{t})=\sum_{k=1}^{2}\omega_k\boldsymbol{\mu}^k,$$
$$\int_{\Gamma}(\hat{t}_k-t_k)\mathrm{d}\Gamma=\xi_k-b_k \Rightarrow \boldsymbol{B}_{\Gamma}(\boldsymbol{\xi}-\boldsymbol{b})=\boldsymbol{\omega}. \tag{21}$$

Then, we can identify the obtained solutions by defining $\boldsymbol{\xi}=\boldsymbol{b}+\boldsymbol{B}_{\Gamma}^{-1}\boldsymbol{\omega}$ if $\boldsymbol{B}_{\Gamma}$ is invertible. Otherwise, we cannot do it uniquely as the solution of (2) is not unique.

In the case, when the solution of (2) does not exist, no $\boldsymbol{\xi}$ makes $\boldsymbol{\omega}$ to vanish in (6). In this case the obtained solution $\hat{\boldsymbol{t}}$ of (6) corresponds to the following BIE:

$$\boldsymbol{U}_{\Gamma}\hat{\boldsymbol{t}}=\tilde{\boldsymbol{g}}+\sum_{k=1}^{2}\omega_k\boldsymbol{\mu}^k, \tag{22}$$

where the right-hand side can be written as:

$$\begin{aligned}\tilde{g}_i(x)&+\sum_{k=1}^{2}\omega_k\mu_i^k(x)\\ &=\frac{1}{2}g_i(x)+\int\!\!\!\!\!\!-_{\Gamma} T_{ij}(x,y)g_j(y)\mathrm{d}\Gamma(y)\\ &\quad+\sum_{k=1}^{2}\omega_k\left[\frac{1}{2}\mu_i^k(x)+\int\!\!\!\!\!\!-_{\Gamma} T_{ij}(x,y)\mu_j^k(y)\mathrm{d}\Gamma(y)\right]\\ &=\frac{1}{2}\left[g_i(x)+\sum_{k=1}^{2}\omega_k\mu_i^k(x)\right]\\ &\quad+\int\!\!\!\!\!\!-_{\Gamma} T_{ij}(x,y)\left[g_i(y)+\sum_{k=1}^{2}\omega_k\mu_i^k(y)\right]\mathrm{d}\Gamma(y)\end{aligned} \tag{23}$$

due to the properties of the operator with the kernel $T_{ij}(x,y)$. Thus the function $\hat{\boldsymbol{t}}$ is the traction solution of the DBVP with the boundary condition $\hat{\boldsymbol{u}}|_{\Gamma}=\boldsymbol{g}+\sum_{k=1}^{2}\omega_k\boldsymbol{\mu}^k$. The function $\hat{\boldsymbol{u}}$ satisfies the radiation condition (5), which was not satisfied by the solution of the original DBVP with the critical scale. However, shifting all the solution by $-\sum_{k=1}^{2}\omega_k\boldsymbol{\mu}^k$, we come to the conclusion that prescribing $\boldsymbol{u}|_{\Gamma}=\boldsymbol{g}$ leads to the solution with the radiation condition (for $\|x\|\to\infty$):

$$u_i(x)=U_{ij}(x,0)\xi_j-\sum_{k=1}^{2}\omega_k\boldsymbol{\mu}^k(x)+\mathrm{O}\big(\|x\|^{-1}\big). \tag{24}$$

In simple terms it means that, on one hand, neither (2) nor its modification (18) can be generally used for solving an exterior DBVP for a domain scaled to its critical size, because its solution satisfying the condition (5) may not exist. On the other hand, the system (6) can be used for solving the exterior DBVP also for the critical scale, although the solution obtained may not satisfy condition (5) (if $\boldsymbol{\omega}\neq 0$), only condition (24). In fact, this solution can be considered a generalized regular solution of the exterior DBVP, see [2], if we put $\boldsymbol{\xi}=0$.

5 Notes on the numerical solution

The discretized versions in SGBEM of (1) and (18) convert them to systems of linear equations of the form:

$$\mathbf{U}_{\Gamma}\mathbf{t}=\tilde{\mathbf{g}} \quad \text{and} \quad \left[\mathbf{U}_{\Gamma}+\mathbf{V}\mathbf{V}^T\right]\mathbf{t}=\tilde{\mathbf{g}}+\mathbf{V}\mathbf{W}^T\mathbf{g}, \tag{25}$$

respectively, where $\mathbf{U}_{\Gamma}$ is a square $N\times N$ matrix of the system (N is the number of unknowns), and the other matrices are row or column matrices of nodal unknowns $\mathbf{t}$, given data $\mathbf{g}$ and $\tilde{\mathbf{g}}$ and modification data $\mathbf{V}$ and $\mathbf{W}$.

The discretization of the system (6) leads to a block matrix system:

$$\begin{pmatrix}-\mathbf{U}_{\Gamma} & \mathbf{M}^T\\ \mathbf{M} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{t}}\\ \boldsymbol{\omega}\end{pmatrix}=\begin{pmatrix}-\tilde{\mathbf{g}}\\ \boldsymbol{\xi}\end{pmatrix}. \tag{26}$$

The matrix $\mathbf{M}$ appears due to integrating the traction approximation over the boundary Γ: $\mathbf{M}\hat{\mathbf{t}}=\int_{\Gamma}\mathbf{N}\hat{\mathbf{t}}\mathrm{d}\Gamma$, where the matrix $\mathbf{N}$ contains all the nodal shape functions pertinent to unknown nodal values $\hat{\mathbf{t}}$.

The critical sizes are found using the discretized version of (7). We have two possibilities. In the first, for two particular choices of $\boldsymbol{\omega}$ the solution of the BIE $(25)_1$ and the pertinent $\boldsymbol{\xi}$ are found, namely

$$\mathbf{U}_{\Gamma_0}\mathbf{t}^1=\boldsymbol{\mu}^1,\quad \mathbf{U}_{\Gamma_0}\mathbf{t}^2=\boldsymbol{\mu}^2,$$
$$\int_{\Gamma_0}\mathbf{N}\mathbf{t}^1\mathrm{d}\Gamma_0=\boldsymbol{\xi}^1,\quad \int_{\Gamma_0}\mathbf{N}\mathbf{t}^2\mathrm{d}\Gamma_0=\boldsymbol{\xi}^2, \tag{27}$$

where Γ_0 is the outermost contour of the boundary Γ for interior DBVPs and $\Gamma_0=\Gamma$ for exterior DBVPs, and where the boundary conditions are prescribed as $\boldsymbol{u}^1|_{\Gamma_0}=\boldsymbol{\mu}^1$ and $\boldsymbol{u}^2|_{\Gamma_0}=\boldsymbol{\mu}^2$, respectively. Thus, we have for the approximation $\mathbf{B}_{\Gamma}=\mathbf{B}_{\Gamma_0}$ of the operator $\boldsymbol{B}_{\Gamma}$

$$\mathbf{B}_{\Gamma}\left(\boldsymbol{\xi}^1\ \boldsymbol{\xi}^2\right)=\left(\boldsymbol{\omega}^1\ \boldsymbol{\omega}^2\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \Rightarrow\ \mathbf{B}_{\Gamma}=\left(\boldsymbol{\xi}^1\ \boldsymbol{\xi}^2\right)^{-1}. \tag{28}$$

Notice that initially we have to solve the problem for the boundary Γ, which can be placed into a small disc with radius R (see Sect. 3) guaranteeing in this way the invertibility of the matrix $\mathbf{B}_{\Gamma}$.

The other possibility uses directly (26) defining $\mathbf{g}=\tilde{\mathbf{g}}=\mathbf{0}$ and solving it twice with the following two choices of $\boldsymbol{\xi}$: $\boldsymbol{\xi}^1$, $\boldsymbol{\xi}^2$

$$\left(\boldsymbol{\xi}^1\ \boldsymbol{\xi}^2\right)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \mathbf{B}_{\Gamma}=\left(\boldsymbol{\omega}^1\ \boldsymbol{\omega}^2\right). \tag{29}$$

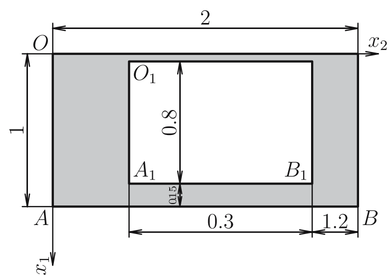

Fig. 1 The bounded domain of an interior DBVP

This second procedure has the advantage that we do not need to invert any matrix. Thus, it can also be used if the scale of the domain is critical.

6 Numerical examples

The author's SGBEM code has been used to test numerically some of the properties discussed above. Three examples of this section include one interior and two exterior problems in a plane strain state. All calculations have been performed without considering physical units, defining the elastic constants as $G = 1$ and $\nu = 0.25$. The tests presented include the determination of the matrix $\boldsymbol{B}_\Gamma$ and the evaluation of tractions, especially in the case where the domain is scaled to one of its critical sizes.

6.1 An interior DBVP

Consider a domain shown in Fig. 1. A DBVP with the displacement boundary conditions derived from the following Airy stress function will be considered: $F(x_1, x_2) = \sin\left(\frac{x_1}{h}\right) \exp\left(-\frac{x_2}{h}\right)$, $h = 1$, together with an additional assumption for fixing the displacements: $u_1(0, 0) = 0$, $u_2(0, 0) = 0$ and $u_2(h, 0) = 0$. The size of the domain is not critical, thus the DBVP has a unique solution.

To check initially the SGBEM implementation we have tested that the solutions obtained by $(25)_1$ and (26) with $\boldsymbol{\xi} = \mathbf{0}$ coincide with the analytic solution. The approximation is obtained using a uniform boundary element mesh with elements of the length 0.05. Figure 2 shows that the numerical results obtained by both approaches are the same—marks 'BIE' and 'Aug', respectively, representing the results from $(25)_1$ and (26) with $\boldsymbol{\xi} = \mathbf{0}$, and almost coincident with the analytical solution 'Anal'.

In what follows the matrix $\mathbf{B}_\Gamma$ will be computed by (28) and its properties will be tested. Finally, it will be applied to determine the critical scaling factors ρ_c, for which this matrix is singular, one of them being used to scale the domain to its critical size and to analyze the corresponding DBVP.

Let us consider a discretization with the same number of elements as above. With this discretization, we obtain the following approximation of the operator $\boldsymbol{B}_\Gamma$:

$$\mathbf{B}_\Gamma = \begin{pmatrix} 0.0328128 & 2.07696 \times 10^{-14} \\ 2.07696 \times 10^{-14} & 0.0469869 \end{pmatrix}, \quad (30)$$

which, due to (9), gives the following critical scales: $\rho_1 = 1.362407$ and $\rho_2 = 1.557126$. We can observe that the matrix is approximately diagonal, which corresponds to the fact that the outer contour has two symmetry axes parallel to the coordinate axes. The tensor character of $\boldsymbol{B}_\Gamma$ should cause the change of the diagonal form when rotating Γ. Applying a rotation angle $\alpha = \frac{\pi}{4}$ (it is clear that the absolute value of the off-diagonal terms reaches its maximum for this particular α) should transform this matrix to

$$\mathbf{B}'_\Gamma = \begin{pmatrix} 0.0398999 & 0.00708707 \\ 0.00708707 & 0.0398999 \end{pmatrix}, \quad (31)$$

and the actual calculation using (28) for the rotated boundary Γ_α has given the same result:

$$\mathbf{B}_{\Gamma_\alpha} = \begin{pmatrix} 0.0398999 & 0.00708707 \\ 0.00708707 & 0.0398999 \end{pmatrix}. \quad (32)$$

Now, let us change the scale by ρ_1 and let us also change the parameter h of $F(x_1, x_2)$ to ρ_1. The matrix in $(25)_1$ is singular with one zero eigenvalue, which can be easily verified numerically. The results are again plotted along the same faces as above, see Fig. 3. One more data set has been added to the results, as the plots also contain the results of $(25)_2$ denoted by the mark 'Mod'.

We can see that the unmodified system contains an eigen-solution which causes the solution at the outer contour to be different from the analytical one. Nevertheless, at the hole boundary the eigen-solution is zero, and thus there is no significant difference between the solution of $(25)_1$ and the other numerical solutions or the analytical one. This is in accordance with the fact that the holes do not influence the critical scales of a bounded domain. The solutions of the modified system $(25)_2$ and of the augmented system (26) with $\boldsymbol{\xi} = \mathbf{0}$ are the same and coincide with the analytical solution.

6.2 Exterior DBVPs

The next two examples deal with the exterior DBVPs for a circular boundary, see Fig. 4.

The critical scale for the circle, with the present fundamental solution and material properties, equals $\rho_c = \exp(0.25) = 1.284025$. Due to the symmetry of Γ there is only one such scale and the matrix $\mathbf{B}_\Gamma$ is a multiple of the unit matrix. For the approximation of $\mathbf{B}_\Gamma$ a boundary element mesh consisting of 80 straight linear elements has been used, the approximated value of the critical scale obtained by (9) being $\rho_c = 1.284682$.

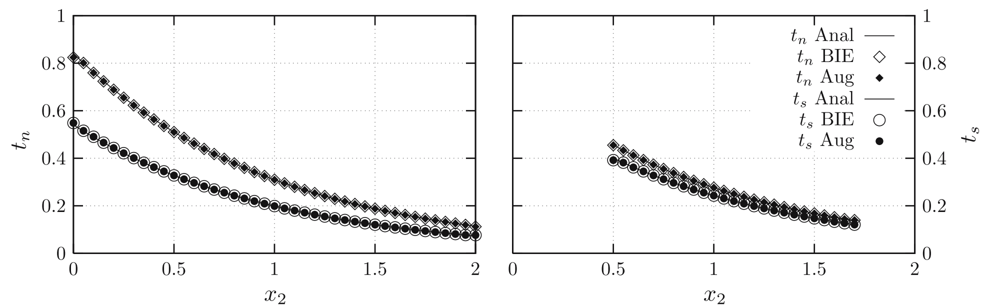

Fig. 2 The tractions for a non-critical scale at the faces AB (*left*) and A_1B_1 (*right*)

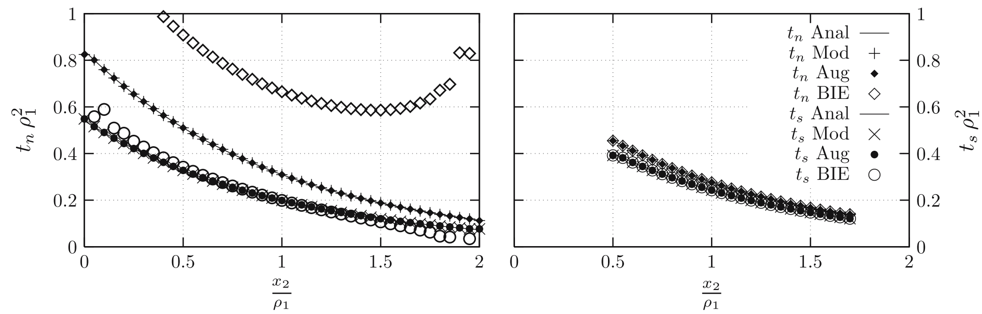

Fig. 3 The tractions for the critical scale ρ_1 at the faces AB (*left*) and A_1B_1 (*right*)

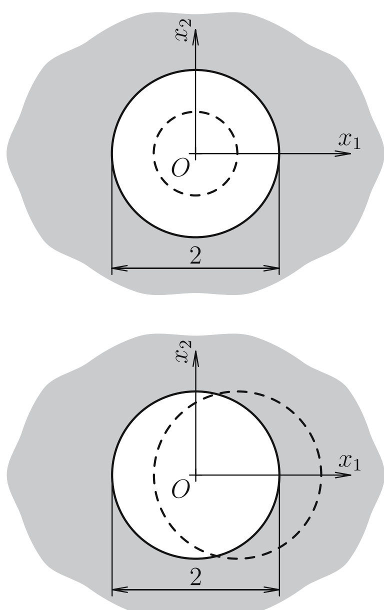

Fig. 4 The unbounded domain with circular boundary: uniform radial displacement (*top*), rigid body displacement (*bottom*)

Consider, first, the Dirichlet boundary condition in the form of a uniform radial displacement Fig. 4 (top). It is expected that the solution for the normal tractions will be constant along the boundary and the solution for the tangential tractions will vanish. Nevertheless, with a non-zero integral of the tractions along the boundary the behavior of the traction solution is different.

The solution of $(25)_2$ is found for the domain scaled to the critical size. Inasmuch as the dimension of the kernel of the operator in $(25)_1$ is two, and $\mathbf{B}_{\rho_c \Gamma}$ is vanishing, we need two couples (x_0, k) to modify the BIE according to (18). The first choice (the mark 'M0') for these two couples is $x_0 = (1.8, 1.8)$ for both $k = 1$ and $k = 2$, which finds the solution with vanishing displacements at x_0 and a non-vanishing unknown integral of tractions. When we consider another choice (the mark 'M5') for the couple (x_0, k), namely $x_0 = (1.8 \times 10^5, 1.8)$ for both $k = 1$ and $k = 2$, the displacements vanish far from the boundary, and the integral of the traction should be rather close to zero, although also not exactly so the integral does not vanish completely. The results of these two modifications are plotted on Fig. 5 (Φ is the angle measured from the positive x_1-axis clockwise).

Moreover, the plot contains the solution of (26), denoted by the mark 'Aug', with $\boldsymbol{\xi}_0 = \mathbf{0}$, for which the pertinent $\boldsymbol{\omega}$

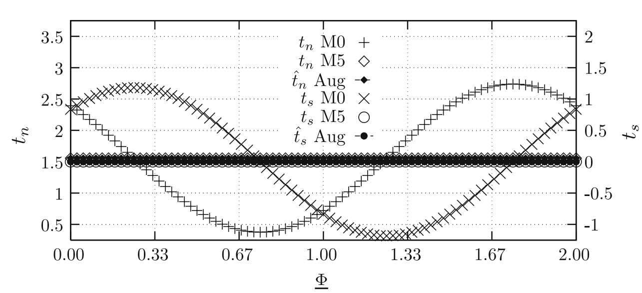

Fig. 5 The tractions for the critical scale with uniform radial displacement

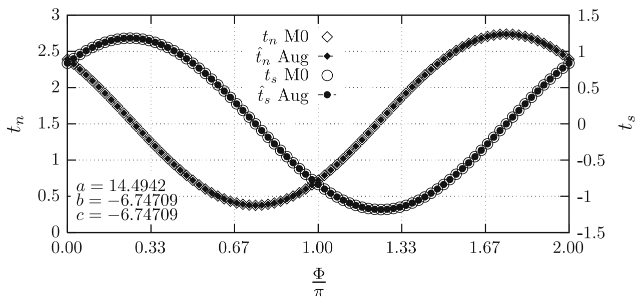

Fig. 6 Identifying the tractions calculated by $(25)_2$ and by (26) for the critical scale with uniform radial displacement

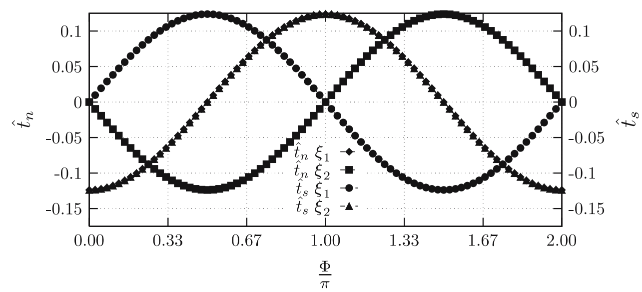

Fig. 7 The solutions of (26) for the critical scale with the unit displacement along the x_1-axis

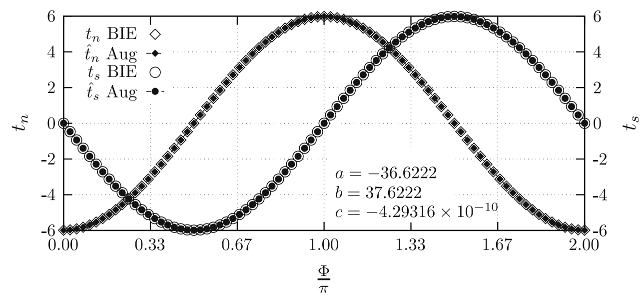

Fig. 8 Identifying the tractions calculated by $(25)_1$ and by (26) for the unit radius with the unit displacement along the x_1-axis

also vanishes. This occurs due to the fact that (2) is solvable and that the dimension of its null space is two; thus, for any $\boldsymbol{\xi}$ the pertinent $\boldsymbol{\omega}$ is zero. The results of 'Aug' and 'M5' are rather close to each other, because the traction integral for the case 'M5' is rather close to zero. In contrast, this is not true for the case 'M0', where these integrals are far from zero. They could be evaluated, but the purpose was to identify this solution with some solution of (26). Therefore, the system (26) has been solved two times with $\boldsymbol{\xi}_1 = (1, 0)^T$ and $\boldsymbol{\xi}_2 = (0, 1)^T$ and the results have been compared with the case 'M0', see Fig. 6.

We can see a perfect fit between both results, where the 'Aug' solution contains a combination of the solutions obtained with various $\boldsymbol{\xi}$. If we denote $\hat{\mathbf{t}}_i$ the solution of (26) with $\boldsymbol{\xi} = \boldsymbol{\xi}_i$, $i = 0, 1, 2$, the data $\hat{\mathbf{t}}$ corresponding to the curve 'Aug' are $\hat{\mathbf{t}} = a\hat{\mathbf{t}}_0 + b\hat{\mathbf{t}}_1 + c\hat{\mathbf{t}}_2$ with a, b and c given in the picture. The constants a, b and c could not be found from (21), as the matrix $\boldsymbol{B}_\Gamma$ is singular, and only a numerical regression fit could be used.

The situation will be totally different in the third example, see Fig. 4 (bottom). The prescribed displacements are equal to a rigid body translation of the boundary. When the size of the domain is scaled to the critical size, the solution of (1) does not exist, and therefore the equation $(25)_1$ does not have a solution and the solution of $(25)_2$ does not have a clear sense. Thus, we can only test that the solution of (26) has for any $\boldsymbol{\xi}$ a corresponding non-zero $\boldsymbol{\omega}$: taking $\boldsymbol{\xi} = \boldsymbol{\xi}_i$ $(i = 0, 1, 2)$ shows that in all these cases we have the same $\boldsymbol{\omega} = (-1, 0)^T$. This means that according to (23) all the obtained solutions are the solutions of the exterior DBVP with the vanishing displacements prescribed along the boundary. Thus for zero $\boldsymbol{\xi}_0$ the solution is also zero, and for the other two $\boldsymbol{\xi}$ they are shown in Fig. 7. The distributions fit with the expected sine and cosine behavior.

The solution of $(25)_1$ exists when the scale of the domain is not critical, for example with the original domain shown in Fig. 4 (bottom). Here, we can use the relation (21) to identify the obtained solution with the solution of (26). We will continue in the same way as in the case above, but we can also directly find the coefficients a, b, c. Using (21) we obtain

$$\boldsymbol{\xi}_1 = \boldsymbol{b} + \boldsymbol{B}_\Gamma^{-1}\boldsymbol{\omega}_1, \tag{33}$$

where the matrix of $\boldsymbol{B}_\Gamma$ can be found by (29) and $\boldsymbol{\omega}_1$ by solving (26):

$$\boldsymbol{B}_\Gamma = \begin{pmatrix} 0.0265801 & -3.44956 \times 10^{-14} \\ -3.44956 \times 10^{-14} & 0.0265801 \end{pmatrix},$$
$$\boldsymbol{\omega}_1 = \begin{pmatrix} -0.973420 \\ -2.78416 \times 10^{-13} \end{pmatrix} \Rightarrow \boldsymbol{b} = \begin{pmatrix} 36.6222 \\ \approx 0 \end{pmatrix}, \tag{34}$$

thus the parameters should be $a = 1 - b$, $b = 37.6222$, $c = 0$.

The tractions calculated by (25) (the 'BIE' mark) together with those obtained by (26), denoted by the mark 'Aug', are shown in Fig. 8, where these data, as in the example above, are plotted in the form $\hat{\mathbf{t}} = a\hat{\mathbf{t}}_0 + b\hat{\mathbf{t}}_1 + c\hat{\mathbf{t}}_2$.

The values of constants a, b and c shown in the picture are obtained by numerical regression, but nevertheless they fit nicely with the calculation in (34).

Finally it should be noted that in both of these last cases the split of the solution $\hat{\mathbf{t}}$ into the form $\hat{\mathbf{t}} = a\hat{\mathbf{t}}_0 + b\hat{\mathbf{t}}_1 + c\hat{\mathbf{t}}_2$ is in accordance with the possibility of solving an exterior DBVP on a multiple-connected domain with a non-equilibrated loading as a superposition of a particular solution for each hole with non-equilibrated load and the generalized regular solution. Related considerations in the context of the complex variable BIEs can be found in [10,11].

7 Conclusions

The DBVPs of linear plane elasticity have been studied. The phenomenon of the critical scale can influence the solvability of these problems, when a boundary element scheme which uses the integral operator of the first kind with a logarithmic kernel is implemented. Procedures which make it possible to determine whether this phenomenon takes place or not and to treat it have been presented.

The first part of the work has dealt with the reduction of the problem to the linear matrix operator $\boldsymbol{B}_\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$, which facilitates the search for the critical scales for a particular choice of the integral kernel including the logarithmic function. Then, two techniques have been described which help to understand the nature of the whole problem and allow a BEM code programmer to deal with the cases where the original BIE (2) does not have a unique solution.

In what follows some recommendations for BEM code programmers based on the analysis presented are given:

- The modification of the BIE with non-unique solutions described by (18) and $(25)_2$, motivated by an approach developed in [17,18] for a different kind of BVP solved by SGBEM, is always applicable for interior problems. Nevertheless, it has been shown that for the exterior problems it has only a limited applicability, as it requires that the original BIE has a solution. It is recommended that the point x_0 used in the BIE modification be taken farther from the boundary in order to obtain the solution with a small contour integral of the tractions and close to the generalized regular solution.
- The augmenting operator technique, based on the theory from [6,15] and described by (6) and (26), is applicable to all kinds of DBVPs, both interior and exterior. However, the solution of the BIE obtained can be a solution of a problem different from that originally considered if an inappropriate radiation condition is applied. In the present paper, this limitation is related to the fact that a special type of radiation condition has been considered, such that the BIE has the same form for both interior and exterior DBVPs. Nevertheless, if the generalized regular solution is required, the choice $\boldsymbol{\xi} = \boldsymbol{0}$ produces the correct result.
- Probably the easiest technique to implement in a BEM code applied for the DBVP solution is to scale the size of the boundary Γ to ensure the invertibility of the operator $\boldsymbol{U}_\Gamma$. This can be specified rather precisely, which can be useful in BEM applications, in particular due to the fact that the condition number of the corresponding discretized linear system increases with diminishing boundary scale when considering the boundary scales smaller than the critical scales ρ_c [15]. Let us denote $R_{\min}$ the minimal radius of a circle which contains Γ, and $l_{\max}$ the diameter of Γ, i.e. $l_{\max} = \max_{x,y\in\Gamma} |x - y|$, which can be found rather easily. Then it can be shown that the relation $\sqrt{3}R_{\min} \le l_{\max} \le 2R_{\min}$ holds, where, for example, the former equality occurs for an equilateral triangle and the latter one for a circle. Therefore, any boundary can be placed within a circle of the radius $\frac{l_{\max}}{\sqrt{3}}$ and due to the result mentioned in the first paragraph of Sect. 3 it is sufficient to have $l_{\max} \le \sqrt{3}\exp\left(\frac{1}{2\kappa}\right)$. Hence, the unique solution of BIE (2) always exists for $l_{\max} \le \sqrt{3}\exp\left(\frac{1}{2\kappa}\right) \le \sqrt{3}\exp\frac{1}{6} \approx 2.046$, taking $\nu \in \left[0; \frac{1}{2}\right)$. As a conclusion, the diameter of the boundary equal to 2, or less than and close to 2, will work well for any practical application of BEM.
- Recall that modifying the kernel U_{ij} in (3) by a constant changes the values of the critical scales ρ_c, as analysed in [13,15]. In fact, in view of the above analysis, there is a possibility to set this constant according to the current size of the boundary, characterized by $l_{\max}$, in such a way that the critical scales are avoided and a reasonable matrix conditioning is obtained, defining the modified kernel as follows:

$$U_{ij}^{\mathrm{mod}} = U_{ij} + \left(-\frac{\Lambda}{2} + \Lambda\kappa \ln \frac{l_{\max}}{\sqrt{3}}\right)\delta_{ij}. \tag{35}$$

Finally, it should be stressed that for bounded domains with holes the critical scales do not depend on the holes. The boundary conditions at these holes can be either of Dirichlet or Neumann type, and the effect of the critical scales of the outermost contour can still take place, see [14] for an analysis of the SGBEM approach.

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