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The method of external excitation for solving Laplace singular eigenvalue problems

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ABSTRACT

In this paper a new numerical technique for Laplace eigenvalue problems in the plane: $\nabla^2 w + k^2 w = 0$, $\mathbf{x} \in \Omega \subset \mathscr{R}^2$, $B[w] = 0, \mathbf{x} \in \partial\Omega$ is presented. We consider the case when the solution domain has boundary singularities like a reentrant corner, or an abrupt change in the boundary conditions. The method is based on mathematically modelling of physical response of a system to excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. We use the local Fourier–Bessel basis functions to describe the behaviour of the solution near the singular point. The results of the numerical experiments justifying the method are presented. In particular, the L-shaped domain and the cracked beam eigenvalue problems are considered.

1. Introduction

In this paper we continue the development of the meshless technique first presented in [1–4] for solution of eigenvalue problems of the type

$$\nabla^2 w + k^2 w = 0, \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2, \; B[\mathbf{x}, w] = 0, \; \mathbf{x} \in \partial\Omega. \tag{1}$$

Here the boundary operator $B[\ldots]$ specifies the boundary conditions and is considered to be of the two types: the Dirichlet $B[w] = w$ and Neumann $B[w] = \partial w/\partial n$ conditions. The eigenvalue problem is to find such real k for which there exist non-null functions w verifying (1). Here our study is focused on the case when the solution domain Ω has boundary points like a reentrant corner, or points with an abrupt change in the boundary conditions (see Fig. 1). This problem is important as a component in the design of many engineering devices, for example, in determining the propagation characteristics of hollow metallic waveguides.

Problem (1) is a classical problem of mathematical physics [5]. However, apart from a few analytically solvable cases with simple, regular domains, there is no general solution of this problem. Therefore, a large number of numerical methods have been developed for many practical problems.

The boundary methods, in particular, the method of fundamental solutions (MFS) [6] are convenient in application to problem (1). A general approach is as follows. First, using the MFS approximation, one gets a homogeneous linear system $\mathscr{A}(k)\mathbf{q} = \mathbf{0}$ with matrix elements depending on the wave number k. To obtain the non-trivial solution the determinant of this matrix must be zero:

$$\det[\mathscr{A}(k)] = 0. \tag{2}$$

To get the eigenvalues this equation must be investigated analytically or numerically (see [7,8] for more detailed description and the references).

However, this method faces great difficulties when applied to the problems with boundary singularities. To overcome them, the Trefftz method (TM) [9] and the method of particular solutions (MPS) [10–12] were developed. These techniques use various particular solutions of the eigenvalue equation which describe the local behaviour of the eigenfunction near the singular points. It can be shown that the convenient sets of particular solutions near a corner of angle π/α are the functions:

$$\varphi_n(\rho, \theta) = J_{n\alpha}(k\rho)\sin(n\alpha\theta), \quad n = 1, 2, \ldots, \infty, \tag{3}$$

$$\varphi_n(\rho, \theta) = J_{n\alpha}(k\rho)\cos(n\alpha\theta), \quad n = 0, 1, \ldots, \infty, \tag{4}$$

$$\varphi_n(\rho, \theta) = J_{(n-1/2)\alpha}(k\rho)\cos((n - 1/2)\alpha\theta), \quad n = 1, 2, \ldots, \infty \tag{5}$$

for the three cases of the boundary conditions shown in Fig. 2. Here (ρ, θ) is the local polar coordinate system with the origin at the singular point. The advantage of these functions (Fourier–Bessel functions) is that not only do they satisfy the governing equation, they also satisfy the boundary conditions along the adjacent line segments. For more details see the original papers.

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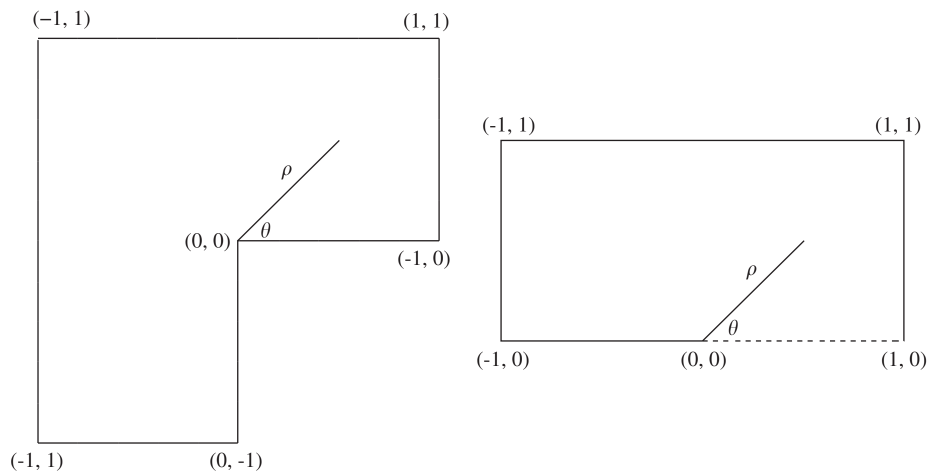

Fig. 1. The domains with boundary singularities. In the right part of the figure the solid line corresponds to the Dirichlet boundary condition and the dashed line denotes Neumann's condition. Here (ρ, θ) is the local polar coordinate system with the origin at the singular point.

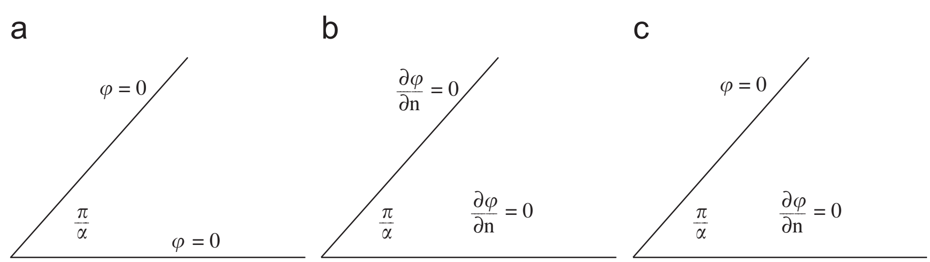

Fig. 2. A wedge with interior angle π/α and different boundary conditions along the adjacent line segments: (a) Dirichlet; (b) Neumann; and (c) mixed.

In the paper presented we use these Fourier–Bessel functions in the framework of the approach of [1–4]. This technique is based on the following quite trivial statement. Let $w_e(\mathbf{x})$ be a smooth enough function defined in the solution domain below named as the *exciting field*. If the *response field* w_r is a solution of the boundary value problem (BVP)

$$\nabla^2 w_r + k^2 w_r = -\nabla^2 w_e - k^2 w_e, \tag{6}$$

$$B[\mathbf{x}, w_r] = -B[\mathbf{x}, w_e], \tag{7}$$

then the sum $w(\mathbf{x}, k) = w_r + w_e$ satisfies the initial problem (1). Let $F(k)$ be some norm of the solution w. This function of k has extremums at the eigenvalues and, under some conditions described below, can be used for their determining. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the matrix of the linear algebraic system which approximates the BVP. From this point of view the presented approach is similar to the one described in [9], where the degeneracy is measured by the infinitesimal values of the minimal eigenvalue of the stiffness matrix of the problem.

Generally, no conditions are imposed on $w_e(\mathbf{x})$. As a result, one gets the sequence of the inhomogeneous PDEs (6) and (7) with a non-null right-hand side which can be solved by an appropriate volume method. For example, the FD method and Kanza's method [13,14] were used in [15,16]. However, when the exciting field is chosen in such a way that the right-hand side of (6) is equal to zero:

$$\nabla^2 w_e + k^2 w_e = 0, \tag{8}$$

then the response field w_r also satisfies the homogeneous Helmholtz equation

$$\nabla^2 w_r + k^2 w_r = 0, \tag{9}$$

which can be solved by a boundary method. Note that we can take *any* solution of (8) as the *exciting field*, e.g. we can take it in the form of a travelling field or as a field of a point source. On the other hand, w_r depends on this choice because it should satisfy the boundary condition (7).

The 2D Helmholtz equation has the known fundamental solutions $\Phi(\mathbf{x} - \zeta) = H_0^{(1)}(k|\mathbf{x} - \zeta|)$, where $H_0^{(1)}$ is the Hankel function. This admits of applying very effective meshless numerical techniques to solve (6) and (7): the MFS, the boundary knot method [17], the boundary integral method [18]. When applied to the non-singular problems these techniques provide a high accuracy of solutions. To handle with the eigenvalue problems with boundary singularities, we combine the global approximation of the solution by the Hankel function and the local approximation by the Fourier–Bessel functions near the singular points.

The outline of this paper is as follows: to explain the technique of regularization we begin by describing the non-singular case in Section 2. In Section 3, we present the extension of the algorithm to problems with boundary singularities. Finally, in Section 4, we give the conclusion.

2. Non-singular problems

2.1. One-dimensional case

To illustrate the method presented in the simplest case, let us consider the wave equation in homogeneous medium $\partial_{tt}^2 u = \partial_{xx}^2 u$ with the Dirichlet conditions at the endpoints of the interval $[0, 1]$, i.e., $u(0, t) = u(1, t) = 0$. Considering the time dependence $u(x, t) = \mathrm{e}^{-ikt} w(x)$, we get the one-dimensional analog of the

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Helmholtz equation:

$$\frac{d^2w}{dx^2}+k^2w=0, \tag{10}$$

with the boundary conditions

$$w(0)=w(1)=0. \tag{11}$$

The problem admits of an analytic solution $k_n=n\pi$.

According to the method presented in the paper, we take the *response field* w_r as a solution of the BVP:

$$\frac{d^2w_r}{dx^2}+k^2w_r=-\frac{d^2w_e}{dx^2}-k^2w_e, \tag{12}$$

$$w_r(0)=-w_e(0),\quad w_r(1)=-w_e(1), \tag{13}$$

then the sum $w=w_e+w_r$ satisfies the initial BVP (10) and (11). The right-hand side of (12) can be considered as an external exciting source. If we take $w_e(x)=e^{ikx}$, then the right-hand side of (12) is equal to zero. This excitation corresponds to the travelling wave which propagates along the x-axis from the source placed in $-\infty$. Let us introduce the norm of the solution as

$$F(k)=\sqrt{\sum_{n=1}^{N_t}|w(x_n)|^2\Big/N_t}, \tag{14}$$

where the points x_n are randomly distributed in [0, 1]. Note that the method is not very sensitive to the number of points x_n. But what is more important, they should be placed in an irregular way, i.e. where the eigenmode being investigated has non-zero values. E.g. the mode $w_n=\sin(n\pi x)$ is equal to zero when $x=1/n$. We also use the dimensionless form of this function: $F_d(k)=F(k)/F(1)$. The function $F(k)$ characterizes the value of the response of the system to the excitation with the wave number k. Varying k, we get the response curve and calculate the eigenvalues as positions of maxima.

However, this initial form of the method is unfit for our goal. Indeed, looking for the response field in the form

$$w_r=A_r\exp(ikx)+B_r\exp(-ikx),$$

we get the linear system for A_r, B_r

$$A_r+B_r=-w_e(0)=-1,$$
$$A_r\exp(ik)+B_r\exp(-ik)=-w_e(1)=-\exp(ik). \tag{15}$$

For $k\neq n\pi$ the system has the unique solution $A_r=-1, B_r=0$. Thus, $w=w_e+w_r\equiv 0$ and $F(k)=0$ with the precision error. As a result, the function $F(k)$ is not smooth and does not admit of finding the maxima with high precision. In Fig. 3 we place the corresponding graph to illustrate this situation. The function $F(k)$ is computed using $N_t=7$ points x_n randomly distributed in [0,1].

The two regularizing procedures which give a smooth response curve were proposed in [1–3]. Applying the first one, we substitute BVP (12) and (13) as follows:

$$\frac{d^2w_r}{dx^2}+(k^2+i\varepsilon k)w_r=0,$$
$$w_r(0)=-w_e(0),\quad w_r(1)=-w_e(1), \tag{16}$$

where $\varepsilon>0$ is a small value. Here we take into account that for $w_e(x)=e^{ikx}$ the right-hand side of (12) is equal to zero. This means that we shift the spectra of differential operator from the real axis. Resulting BVP has a unique non-zero solution for all real k.

As a result, we get the following system instead of (15):

$$A_r+B_r=-1,$$

$$A_re^{ik_\varepsilon}+B_re^{-ik_\varepsilon}=-e^{ik},\quad k_\varepsilon=\sqrt{k^2+i\varepsilon k} \tag{17}$$

and $w=w_e+w_r\neq 0$. The dimensionless response curves $F_d(k)$ depicted in Fig. 4 correspond to $\varepsilon=10^{-15}$ (left) and $\varepsilon=10^{-10}$ (right). The value $\varepsilon=10^{-15}$ is too small to regularize the solution. The response curve $F_d(k)$ has separate maximums at the positions of eigenvalues but is not smooth. The value $\varepsilon=10^{-10}$ provides a smooth curve.

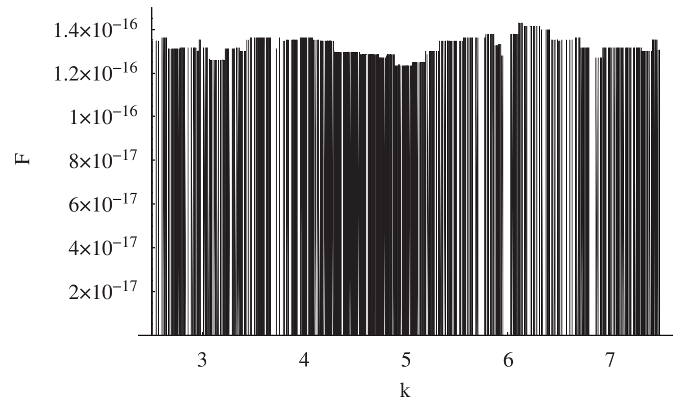

Fig. 3. The response curve without regularization.

Another regularizing procedure can be described in the following way. Let us introduce the constant shift Δk between the wave numbers of the exciting source and the studied mode, i.e., we take the exciting field $w_e(x)=w_e(x,k+\Delta k)=\exp[i(k+\Delta k)x]$ and get the linear system

$$A_r+B_r=-1,\quad A_re^{ik}+B_re^{-ik}=-e^{i(k+\Delta k)}, \tag{18}$$

which provides $w=w_e+w_r\neq 0$. The solution exists for all k except the eigenvalues k_n when the system becomes degenerate. However, due to iterative procedure of solution and rounding errors we never solve the system with the *exact* k_n. We observe degeneration of the system as a considerable growth of the solution in a neighbourhood of the eigenvalues.

The response curves corresponding to $\Delta k=10^{-15}$ and 10^{-10} are absolutely similar to the curves depicted in Fig. 4 for $\varepsilon=10^{-15}$ and 10^{-10}. The value $\Delta k=10^{-15}$ is too small to regularize the solution. But the value $\Delta k=10^{-10}$ yields a smooth curve.

When comparing these two procedures, it should be noted that they provide approximately the same precision in the calculations of eigenvalues. However, dealing with a real PDE and using the ε-procedure, we have to perform the calculations with complex variables. The use of the k-procedure provides calculations with real variables only.

Having got a smooth response curve, we apply the following simple algorithm. First, we localize these maxima of $F(k)$ on the intervals $[a_i,b_i]$. Next, we solve the univariate optimization problem inside each one. In particular, we apply Brent's method based on a combination of parabolic interpolation and bisection of the function near to the extremum (see [20]).

2.2. Two-dimensional case

The same technique can be applied in the 2D case. Solving Eq. (9), we look for the solution in the form of the linear combination

$$w_r(\mathbf{x})=\sum_{n=1}^{N}q_nH_0^{(1)}(k|\mathbf{x}-\zeta_n|). \tag{19}$$

Here q_n are free parameters of the problem and the source points ζ_n are placed outside the solution domain. The free parameters are obtained from the boundary condition (7)

$$\sum_{n=1}^{N}q_nB[\mathbf{x},H_0^{(1)}(k|\mathbf{x}-\zeta_n|)]=-B[\mathbf{x},w_e]. \tag{20}$$

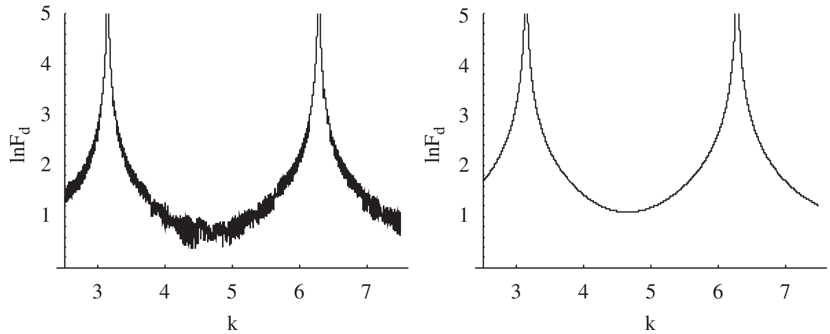

Fig. 4. The response curve for the one-dimensional eigenvalue problem in [0, 1]; ε-procedure with $\varepsilon = 10^{-15}$—left and $\varepsilon = 10^{-10}$—right.

We take the exciting field in the form of the travelling wave

$$w_e(\mathbf{x}, k) = \exp[ik(x\cos\upsilon + y\sin\upsilon)], \tag{21}$$

which satisfies (8) for any angle of incidence υ.

To find the unknowns q_n we solve the collocation problem

$$\sum_{n=1}^{N} q_n B[\mathbf{x}_i, H_0^{(1)}(k|\mathbf{x}_i - \zeta_n|)] = -B[\mathbf{x}_i, w_e(\mathbf{x}_i)], \quad \mathbf{x}_i \in \partial\Omega, \quad i = 1, \ldots, N_c. \tag{22}$$

The collocation points $\mathbf{x}_i$ are uniformly distributed on the boundary. The number of the collocation points N_c is taken twice as large as the number of unknowns N and the resulting overdetermined linear system is solved by the procedure of the least squares. Then, having the solution $w_r(\mathbf{x})$ and consequently $w(\mathbf{x}) = w_r(\mathbf{x}) + w_e(\mathbf{x})$, we introduce the norm $F(k)$ like (14). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima.

To get a smooth response curve we utilize the regularizing procedures described above. According to the ε-procedure one we substitute (9) by the equation

$$\nabla^2 w_r + (k^2 + i\varepsilon k) w_r = 0, \tag{23}$$

where $\varepsilon > 0$ is a small value. So, we replace k with $k_\varepsilon = \sqrt{k^2 + i\varepsilon k}$ in the left-hand side of (22).

Applying the k-procedure, we take the exciting field w_e instead of (21) in the form

$$w_e(\mathbf{x}, k + \Delta k) = \exp[i(k + \Delta k)(x\cos\upsilon + y\sin\upsilon)]. \tag{24}$$

The data placed in Table 1 correspond to the following problem: the solution domain Ω is the disk with the radius $R_c = 1$ and the Dirichlet condition $w = 0$ is taken on the boundary $\partial\Omega = \{x, y | x^2 + y^2 = R_c^2\}$. The problem is solved using three different solver of the Helmholtz equation in the framework of the technique presented. They are:

(1) The MFS described above.
(2) The boundary knot method (BKM) [17]. According to this technique the approximate solution is looked for in the form:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n J_0(k|\mathbf{x} - \zeta_n|),$$

cf. (19). Here the basis functions $J_0(k|\mathbf{x} - \zeta|)$ are non-singular and the source points ζ_n can be placed on the boundary $\partial\Omega$ or even inside the solution domain Ω. The rest part of the algorithm is the same as the one considered above for the MFS.
(3) The null-field boundary integral equation method [18,19]. It is based on the integral formulation

$$\int_{\partial\Omega} T(\mathbf{s}, \mathbf{x}) w_r(\mathbf{s})\, dl(\mathbf{s}) - \int_{\partial\Omega} U(\mathbf{s}, \mathbf{x}) \frac{\partial w_r}{\partial n}(\mathbf{s})\, dl(\mathbf{s}) = \begin{cases} 0, & \mathbf{x} \in R^2 \backslash \Omega, \\ 2\pi w_r(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{25}$$

Table 1
The relative errors in solution of eigenvalue problem for the circle with the radius $R = 1$

#	$k_i^{(ex)}$	MFS	BKM	BIM
1	2.404825560	1.1×10^{-9}	2.7×10^{-9}	1.2×10^{-9}
2	3.831705970	2.5×10^{-12}	3.7×10^{-9}	1.4×10^{-11}
3	5.135622307	1.0×10^{-9}	4.9×10^{-9}	1.0×10^{-9}
4	5.520078106	8.4×10^{-10}	3.2×10^{-9}	8.7×10^{-9}
5	6.380161905	1.4×10^{-9}	1.3×10^{-9}	1.4×10^{-9}
6	7.015586677	1.1×10^{-9}	4.9×10^{-9}	1.1×10^{-9}
7	7.588342447	1.6×10^{-9}	2.1×10^{-9}	1.6×10^{-9}
8	8.417244162	2.6×10^{-9}	9.4×10^{-10}	2.6×10^{-9}
9	8.653727935	2.5×10^{-9}	5.3×10^{-9}	2.5×10^{-9}
10	8.771483842	3.0×10^{-9}	6.5×10^{-9}	2.9×10^{-9}
15	11.086370039	1.8×10^{-9}	3.2×10^{-9}	1.8×10^{-9}
20	13.015200730	6.0×10^{-10}	1.8×10^{-9}	6.1×10^{-9}

Dirichlet condition.

where $U(\mathbf{s}, \mathbf{x})$ and $T(\mathbf{s}, \mathbf{x})$ are well-known Green's function and its normal derivative, respectively:

$$U(\mathbf{s}, \mathbf{x}) = -\frac{1}{2} i\pi H_0^{(1)}(kr), \quad T(\mathbf{s}, \mathbf{x}) = -\frac{1}{2} i\pi k H_1^{(1)}(kr) \frac{y_i n_i}{r}, \tag{26}$$

where $H_n^{(1)}$ is the nth-order Hankel function of the first kind, $r = |\mathbf{x} - \mathbf{s}|$, $y_i = s_i - x_i$, and n_i is the ith component of the outer normal vector at the boundary point $\mathbf{s}$. For the Dirichlet problem we find the unknown normal derivative $\partial w/\partial n$ as a solution of the integral equation

$$\int_{\partial\Omega} U(\mathbf{s}, \mathbf{x}) \frac{\partial w_r}{\partial n}(\mathbf{s})\, dl(\mathbf{s}) = \int_{\partial\Omega} T(\mathbf{s}, \mathbf{x}) w_r(\mathbf{s})\, dl(\mathbf{s}) = -\int_{\partial\Omega} T(\mathbf{s}, \mathbf{x}) w_e(\mathbf{s})\, dl(\mathbf{s}), \quad \mathbf{x} \in \Gamma, \tag{27}$$

where the auxiliary contour Γ contains the solution domain Ω and does not intersect its boundary $\partial\Omega$. So, the integrals are not singular. Having got the boundary value of the normal derivative $\partial w_r/\partial n(\mathbf{s})$, we obtain the solution $w_r(\mathbf{x})$ in the interior of Ω

$$w_r(\mathbf{x}) = \frac{1}{2\pi} \int_{\partial\Omega} T(\mathbf{s}, \mathbf{x}) w_r(\mathbf{s})\, dl(\mathbf{s}) - \frac{1}{2\pi} \int_{\partial\Omega} U(\mathbf{s}, \mathbf{x}) \frac{\partial w_r}{\partial n}(\mathbf{s})\, dl(\mathbf{s}). \tag{28}$$

Note that when writing (27), we use the top part of (25) because here $\mathbf{x} \in R^2 \backslash \Omega$. And to write (28) we use the bottom part of (25) for $\mathbf{x}$ which are inside Ω. Solving (27) we replace $w_r(\mathbf{s})$ by its known boundary value $w_r(\mathbf{s}) = -w_e(\mathbf{s})$ from (7). In the case of Neumann's boundary condition we solve the equation

$$\int_{\partial\Omega} T(\mathbf{s}, \mathbf{x}) w_r(\mathbf{s})\, dl(\mathbf{s}) = \int_{\partial\Omega} U(\mathbf{s}, \mathbf{x}) \frac{\partial w_r}{\partial n}(\mathbf{s})\, dl(\mathbf{s}) = -\int_{\partial\Omega} U(\mathbf{s}, \mathbf{x}) \frac{\partial w_e}{\partial n}(\mathbf{s})\, dl(\mathbf{s}), \quad \mathbf{x} \in \Gamma, \tag{29}$$

where $\partial w_r/\partial n(\mathbf{s})$ in the right-hand side is replaced by its known boundary value $\partial w_r/\partial n(\mathbf{s}) = -\partial w_e/\partial n(\mathbf{s})$. Having the boundary value $w_r(\mathbf{s})$, we use the same formula (28) to get the solution $w_r(\mathbf{x})$ in the interior of Ω. Then, having the solution $w_r(\mathbf{x})$ and consequently $w(\mathbf{x}) = w_r(\mathbf{x}) + w_e(\mathbf{x})$, we introduce the norm $F(k)$ like (14). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima.

All the integrals in (27)–(29) are approximated by the finite sums using an appropriate quadrature rule. The description of this technique with more details can be found in [18,19] and in the literature presented here.

The calculations presented in Table 1 correspond to the following parameters: MFS utilizes $N = 40$ sources placed on the circle with the radius $R_a = 5$; the BKM is applied with the same number of the sources placed on the boundary $\partial\Omega$; using BIM, we take $N = 40$ unknown values of the normal derivative $\partial w/\partial n$. Other examples showed a high precision in calculations of the eigenvalues for the non-singular boundary conditions are presented in papers listed above.

3. Singular problems

To extend the technique described in the previous section onto the case of the boundary singularities we use the MFS as a solver of BVP for w_r and look for the response field in the form

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=1}^{M} p_j \varphi_j(\rho, \theta), \tag{30}$$

where (ρ, θ) is the local polar coordinate system with the origin at the singular point and the functions $\varphi_j(\rho, \theta)$ correspond to the kind of the singularity. Only the singular functions (3)–(5) are considered in this paper.

We find the unknowns q_n, p_j as a solution of the collocation problem

$$B[w_r(\mathbf{x}_i)] = \sum_{n=1}^{N} q_n B[H_0^{(1)}(k|\mathbf{x}_i - \zeta_n|)] + \sum_{j=1}^{M} p_j B[\varphi_j(\rho_i, \theta_i)]$$
$$= -B[w_e(\mathbf{x}_i)], \quad \mathbf{x}_i \in \partial\Omega. \tag{31}$$

The collocation points $\mathbf{x}_i$ are uniformly distributed on the boundary. The number of the collocation points is taken twice as large as the number of unknowns $N + M$ and the resulting linear system is solved by the procedure of the least squares. Then, having the solution $w_r(\mathbf{x})$ and consequently $w(\mathbf{x}) = w_r(\mathbf{x}) + w_e(\mathbf{x})$, we introduce the norm $F(k)$ like (14). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima.

Example 1. Let us consider the eigenvalue problem for L-shaped domain with the Dirichlet boundary conditions. The response field is looked for in the form:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=1}^{M} p_j J_{2j/3}(k\rho)\sin(2j\theta/3), \tag{32}$$

with the singular functions satisfying the boundary conditions $\varphi_j(\rho, 0) = \varphi_j(\rho, 3\pi/2) = 0$. The data, placed in Table 2, are obtained using the k-procedure with $\Delta k = 10^{-6}$. The exciting field (21) is taken with the angle $\upsilon = \pi/4$. The MFS source points are placed on the circle with the radius $R_s = 3$. The data in the last column of the table are taken from [12]. To compare these data with our result we place the squares k_i^2 in the table. It looks like the data corresponding to $N = 120, M = 20$ give the eigenvalues with the 10 true digits.

Example 2. To solve the Neumann problem we use the expansion:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=0}^{M} p_j J_{2j/3}(k\rho)\cos(2j\theta/3), \tag{33}$$

with the singular functions satisfying the boundary conditions $\partial\varphi_j/\partial n(\rho, 0) = \partial\varphi_j/\partial n(\rho, 3\pi/2) = 0$. Thus, only the last term is modified. In Table 3 we test a convergence of the eigenvalues. The results are compared with the results of Shu and Chew [21] obtained by the global method of generalized differential quadrature (GDQ). Note that the L-shaped domain considered in [21] is smaller than the one depicted in Fig. 1. The similarity coefficient 0.635 is taken into account in the data presented in the table.

Example 3. The eigenvalue problem for a cracked beam (Fig. 1, right) is the example of problems with an abrupt change in the boundary conditions. The problem is considered in detail in [9]. We look for the MFS solution in the form:

$$w_r(\mathbf{x}) = \sum_{n=1}^{N} q_n H_0^{(1)}(k|\mathbf{x} - \zeta_n|) + \sum_{j=1}^{M} p_j J_{j-1/2}(k\rho)\cos((j - 1/2)\theta), \tag{34}$$

with the singular functions corresponding to the boundary conditions $\partial\varphi_j/\partial n(\rho, 0) = \varphi_j(\rho, \pi) = 0$. Some results of the calculations are presented in Table 4. Using the package Mathematica, the first two eigenvalues were calculated in [9] with 13 significant digits. They are shown in the last column of the table. One can see that the method presented gives the eigenvalues of the problem with 10 true digits.

4. Conclusion

In this paper, a numerical technique is proposed to solve eigenvalue problems in domains with boundary singularities. This is a mathematical model of the physical measurements when the resonant frequencies of a system are determined by the amplitude

Table 2
The L-shaped domain

i	$N/M = 40/15$	$N/M = 60/20$	$N/M = 80/20$	$N/M = 100/20$	$N/M = 120/20$	B&T
1	9.639726	9.639723840	9.639723841	9.639723842	9.639723841	9.6397238
2	15.197253	15.19725193	15.197251927	15.197251926	15.197251926	15.197252
3	19.739209	19.73920880	19.739208803	19.739208802	19.739208802	19.739209
4	29.521481	29.52148112	29.521481113	29.521481114	29.521481113	29.521481
5	31.912648	31.91263592	31.912635960	31.912635951	31.912635961	31.912636
6	41.474483	41.47450984	41.474509891	41.474509893	41.474509894	41.474510
7	44.948476	44.94848776	44.948487781	44.948487782	4.948487781	–
8	49.348022	49.34802200	49.348022007	49.348022006	49.348022006	–
9	56.709605	56.70960993	56.709609889	56.709609885	56.709609884	–
10	65.376529	65.37653575	65.376535711	65.376535708	65.376535709	–

Dirichlet condition. Convergence with the growth of the number of free parameters. The value of k^2 is shown.

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Table 3
The L-shaped domain

i	$N/M=40/10$	$N/M=40/15$	$N/M=60/10$	$N/M=80/10$	$N/M=100/10$	Shu and Chen
1	1.475623	1.475623	1.475622	1.475621813	1.475621793	1.47455
2	3.534032	3.534032	3.534031	3.534031369	3.534031369	3.53409
3	9.869604	9.869604	9.869604	9.869604401	9.869604401	9.86964
4	11.389478	11.389478	11.389479	11.389479388	11.389479386	11.38951
5	12.572380	12.572383	12.572388	12.572386842	12.572386864	12.56893
6	19.739209	19.739209	19.739209	19.739208802	19.739208802	19.73940
7	21.424745	21.424742	21.424734	21.424732805	21.424734381	21.41668
8	23.344379	23.344378	23.344372	23.344372031	23.344372008	23.34424
9	28.490787	28.490807	28.490812	28.490823385	28.490814353	28.48205
10	35.878650	35.878648	35.878631	35.878631521	35.878631920	

Neumann condition. Convergence with the growth of the number of free parameters. The value of k^2 is shown.

Table 4
The eigenvalues of the cracked beam

i	$N/M=50/15$	$N/M=60/10$	$N/M=75/10$	$N/M=80/10$	[9]
1	2.0116970	2.01169717	2.011697117	2.011697117	2.011697117212
2	3.2931526	3.29315261	3.293152635	3.293152635	3.293152635104
3	4.0798642	4.07986416	4.079864129	4.079864128	–
4	4.8863146	4.88631468	4.886314665	4.886314665	–
5	5.2893785	5.28937862	5.289378620	5.289378620	–
6	6.1326889	6.13268898	6.132689008	6.132689010	–
7	6.4719151	6.47191515	6.471915149	6.471915149	–
8	6.8246202	6.82462027	6.824620261	6.824620261	–
9	7.3939713	7.39397129	7.393971287	7.393971287	–
10	7.9781250	7.97812500	7.978125002	7.978125001	–

of response to some excitation. Varying the wave number k, we get the eigenvalues as positions of maxima of some the norm function $F(k)$. The growth of the amplitude of response near the eigenvalue is a sequence of the degeneracy of the collocation matrix (31). From this point of view the approach presented is similar to the one described in [9], where the degeneracy is measured by the infinitesimal values of the minimal eigenvalue of the stiffness matrix of the problem. This technique is convenient for determining some first eigenvalues of the system which are often of the most interest from the point of view of engineering applications.

The version of the technique [1–4] presented in the paper is based on the use of the local singular functions together with the global basis function. The analysis of L-shaped domain and the cracked beam eigenvalue problems has shown that the method provides a very high precision in determining eigenvalues with a moderate number of free parameters. The method is presented in the framework of the MFS but any boundary or volume method can be used as the Helmholtz solver.

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