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Regularization of hypersingular integrals in 3-D fracture mechanics: Triangular BE, and piecewise-constant and piecewise-linear approximations

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ABSTRACT

In this article the hypersingular integrals that arise when boundary integral equation (BIE) methods are used to solve fracture mechanics problems are considered. An approach for hypersingular integral regularization is based on the theory of distribution and Green's theorems. This approach is applied for regularization of the hypersingular integrals over triangular boundary elements (BEs) for the case of piecewise-constant and piecewise-linear approximations. The hypersingular integrals are transformed into regular contour integrals that can be easily calculated analytically.

1. Introduction

Materials used in engineering often contain cracks and other structural defects. Therefore investigation of 3-D crack problems is very important in engineering practice. Since analytical solutions to these problems have been limited to the case of relatively simple geometry cracks in an infinite body with a simple load, numerical methods such as the boundary element method (BEM) have been developed. The boundary integral equation (BIE) is a very powerful tool for solution of mathematical problems in science and engineering [1,2]. BIE and BEMs are now established in many engineering disciplines as an alternative numerical technique for domain approaches, for example the finite-element method. The attraction of BEM can be largely attributed to the reduction in dimensionality of the problem. Another advantage of the BEM is large accuracy of results, especially for stress concentration problems. The solution at an internal point of analyzed domain is exactly expressed through the boundary values and no discretization of domain is required. This is the main reason why BEM is the most accurate computational method for solution of crack problems. A familiar complication of BIE and BEM methods is, however, that they must in general be formulated in terms of hypersingular integral operators [3–9].

It is known that the overall accuracy of BEM is largely dependent on the precision with which various integrals are evaluated. No doubt, the evaluation of hypersingular integrals requires much more sensitive treatment than that of regular integrals. Numerical methods developed for regular integral calculation cannot be used for their calculation. There are many methods for calculation of divergent integrals, for references see the review article by Tanaka et al. [10] and references there. We will not discuss here advantages and disadvantages of these methods; this has already been done in the above-mentioned review. We will consider here in more detail the method for divergent integral regularization, which is based on the theory of distributions and idea of finite-part integrals according to Hadamard [11].

We apply the approach based on the theory of distributions and finite-part integrals for problems of fracture mechanics firstly in Zozulya [9]. Then it was further developed for regularization of the hypersingular integrals in static and dynamic problems of fracture mechanics in [12,13,23], respectively. More applications of the developed regularization method can be found in review articles [14–16]. Further development of this approach and application of Green's theorems in the sense of the theory of distribution has bean done in Zozulya [17]. The equations presented in Zozulya [18] and Zozulya and Gonzalez-Chi [19] permit transformation of divergent hypersingular integrals into regular ones. The developed approach can be applied not only for hypersingular integral regularization but also for a wide class of divergent integral regularizations.

In the present paper, the above-mentioned approach for divergent integral regularization is further developed and applied for the case of 3-D elastostatic crack problems. We consider 2-D hypersingular integrals over arbitrary convex polygons for piecewise-constant approximation and over triangular boundary elements (BEs) for piecewise-linear approximation as Hadamard's finite-part integrals (*F.P.*). Regularized equations for 2-D hypersingular integral calculation have been presented here. It is

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important to mention that in the presented equations all calculations can be done analytically, no numerical integration is needed.

2. Boundary integral equations

Let us consider an infinite elastic medium occupying the whole space R^3 that contains arbitrarily oriented plane cracks. The cracks are described by corresponding oriented surfaces $\Omega^+ \cup \Omega^-$, where Ω^+ and Ω^- are opposite edges. The crack surfaces Ω^+ and Ω^- are locally parallel and their curvatures are relatively small. In $V := R^3 \Omega^+ \cup \Omega^-$ we consider the behavior of the medium governed by the linear Lame equations of elastostatics for the displacement field $u_i(\mathbf{x})$, i.e.

$$A_{ij} u_j(\mathbf{x}) = 0, \quad A_{ij} = \mu \delta_{ij} \partial_k \partial_k + (\lambda + \mu) \partial_i \partial_j, \quad \mathbf{x} \in V, \tag{2.1}$$

subject to the boundary conditions

$$p_i(\mathbf{x}) = p_i^+ \text{ for } x_i \in \Omega^+, \tag{2.2}$$

$$p_i(\mathbf{x}) = p_i^- \text{ for } x_i \in \Omega^-.$$

Because we consider infinite region, additional conditions must be satisfied in the form

$$u_j(\mathbf{x}) = O(r^{-1}), \quad \sigma_{ij}(\mathbf{x}) = O(r^{-2}) \quad \text{for} \quad r \to \infty, \tag{2.3}$$

where λ and μ are Lame constants, $\mu > 0$, $\lambda > -\mu$, δ_{ij} is the Kronecker's symbol, $\partial_i = \partial / \partial x_i$ denotes partial derivatives with respect to space and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance in the 3-D Euclidian space. Throughout this paper we use the Einstein summation convention.

We introduce Cartesian coordinates system, with x_1- and x_2-axes in the plane of the crack and x_3-axis perpendicular to this plane. Following Guz and Zozulya [14,20] we suppose the opposite crack edge surfaces are identified ($\Omega^+ = \Omega^- = \Omega$) and are distinguished only by the direction of the external normal vectors ($\mathbf{n}^+ = \mathbf{n}^- = \mathbf{n}$). Then deformation of the crack edges is defined by crack opening

$$\Delta u_i(\mathbf{x}) = u_i^+(\mathbf{x}) - u_i^-(\mathbf{x}) \forall x \in \Omega, \tag{2.4}$$

since we suppose that only small deformations occur.

In Guz and Zozulya [14,15,20] it was shown that in this case the BIE that relates load $p_i(\mathbf{y})$ on the crack faces Ω^+ and Ω^- and crack opening $\Delta u_i(\mathbf{x})$ may be written in the form

$$p_i(\mathbf{y}) = -\int_\Omega F_{ij}(\mathbf{x}, \mathbf{y}) \Delta u_j(\mathbf{x}) dS. \tag{2.5}$$

The kernels $F_{ij}(\mathbf{x},\mathbf{y})$ in BIE (2.5) may be presented in the form

$$F_{11} = \frac{\mu}{4\pi(1-\upsilon)} \left[\frac{(1-2\upsilon)}{r^3} + 3\upsilon \frac{(x_1 - y_1)^2}{r^5} \right], \quad F_{12} = \frac{\mu\upsilon}{4\pi(1-\upsilon)} \frac{(x_1 - y_1)(x_2 - y_2)}{r^5},$$

$$F_{22} = \frac{\mu}{4\pi(1-\upsilon)} \left[\frac{(1-2\upsilon)}{r^3} + 3\upsilon \frac{(x_2 - y_2)^2}{r^5} \right], \quad F_{33} = \frac{\mu}{4\pi(1-\upsilon)} \frac{1}{r^3}. \tag{2.6}$$

where μ and υ are the elastic modulus and Poison ratio, respectively.

Simple observation shows that the kernels in BIE (2.5) tend to infinity when $r \to 0$. More detailed analysis of the Eq. (2.5) and kernels (2.6) gives us the following result, with $\mathbf{x} \to \mathbf{y}$:

$$F_{ij}(\mathbf{x}, \mathbf{y}) \to r^{-3}. \tag{2.7}$$

Integrals with these kernels are divergent and therefore need special consideration. Usually such integrals are considered in the sense of finite part according to Hadamard [11].

Definition 1.1. Integrals with kernels $F_{ij}(\mathbf{x},\mathbf{y})$ are hypersingular and must be considered in the sense of the Hadamard finite part as

$$F.P. \int_{\partial V} u_i(\mathbf{x}) F_{ji}(\mathbf{x} - \mathbf{y}) dS = \lim_{\varepsilon \to 0} \times \left(\int_{\partial V \backslash \partial V(r < \varepsilon)} u_i(\mathbf{x}) W_{ji}(\mathbf{x} - \mathbf{y}) dS + 2u_j(\mathbf{x}) \frac{f_j(\mathbf{x})}{\partial V(r < \varepsilon)} \right). \tag{2.8}$$

Here functions $f_j(\mathbf{x})$ are chosen from the condition of the limit existence. Refer to Gel'fand and Shilov [21] for details.

The hypersingular character of kernels in (2.5) determines boundary properties of the corresponding potentials. Analyses of these formulae show that the boundary potentials with kernels $F_{ij}(\mathbf{x},\mathbf{y})$ contain hypersingular kernels. They continuously cross the boundary ∂V.

3. BEM equations

In order to transform the BIE into finite-dimensional BEM equations we have to split the crack surface Ω into finite elements, which are called boundary elements (BEs).

$$\Omega = \bigcup_{n=1}^{N} \Omega_n, \quad \Omega_n \cap \Omega_k = \varnothing \quad \text{if } n \neq k. \tag{3.1}$$

On each BE we shall choose Q nodes of interpolation and shape functions $\varphi_{nq}(\mathbf{x})$. Then the vectors of displacement discontinuity and traction on the BE Ω_n will be represented approximately in the form

$$\begin{aligned} \Delta u_i(\mathbf{x}) &\approx \sum_{q=1}^{Q} \Delta u_i^n(\mathbf{x}_q) \varphi_{nq}(\mathbf{x}), \quad \mathbf{x} \in \Omega_n, \\ p_i(\mathbf{x}) &\approx \sum_{q=1}^{Q} p_i^n(\mathbf{x}_q) \varphi_{nq}(\mathbf{x}), \quad \mathbf{x} \in \Omega_n, \end{aligned} \tag{3.2}$$

and on the hold crack surface Ω in the form

$$\begin{aligned} \Delta u_i(\mathbf{x}) &\approx \sum_{n=1}^{N} \sum_{q=1}^{Q} \Delta u_i^n(\mathbf{x}_q) \varphi_{nq}(\mathbf{x}), \quad \mathbf{x} \in \bigcup_{n=1}^{N} \Omega_n = \Omega, \\ p_i(\mathbf{x}) &\approx \sum_{n=1}^{N} \sum_{q=1}^{Q} p_i^n(\mathbf{x}_q) \varphi_{nq}(\mathbf{x}), \quad \mathbf{x} \in \bigcup_{n=1}^{N} \Omega_n = \Omega. \end{aligned} \tag{3.3}$$

Substituting expressions (4) in (1) gives us the BE equations that relate the vectors of displacements discontinuity and traction on the crack surface in the form

$$p_i^m(\mathbf{y}_r) = -\sum_{n=1}^{N} \sum_{q=1}^{Q} F_{ji}^n(\mathbf{y}_r, \mathbf{x}_q) \Delta u_j^n(\mathbf{x}_q), \tag{3.4}$$

where

$$F_{ji}^n(\mathbf{y}_r, \mathbf{x}_q) = \int_{\Omega_n} F_{ji}(\mathbf{y}_r, \mathbf{x}) \varphi_{nq}(\mathbf{x}) dS. \tag{3.5}$$

More detailed information on transition from the BIE to the BEM equations can be found in Balas et al. [1] and Banerjee [2].

4. Piecewise-constant approximation

The piecewise-constant approximation is the simplest one. Interpolation functions in this case do not depend on the FE form and dimension of the domain. They have the form

$$\varphi_q(\mathbf{x}) = \begin{cases} 1 & \forall \mathbf{x} \in S_n, \\ 0 & \forall \mathbf{x} \notin S_n. \end{cases} \tag{4.1}$$

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In order to simplify the situation we transform the global system of coordinates such that the origin is located at the nodal point, where $\mathbf{y}^0=0$, the coordinate axes x_1 and x_2 are located in the plane of the element, while axis x_3 is perpendicular to that plane. In this case $x_3=0$, $n_1=0$ $n_2=0$, $n_3=1$ and fundamental solutions have the form presented by Eq. (2.6).

Regular representations for integrals with kernels (2.6) can be fond in our previous publications [17–19]. For the piecewise-constant approximation they have the form

$$\begin{aligned}
J_3^{0,0} &= F.P.\int_{S_n}\frac{dS}{r^3} = -\int_{\partial S_n}\frac{r_n}{r^3}dl,\\
J_5^{2,0} &= F.P.\int_{S_n}\frac{x_1^2}{r^5}dS = \int_{\partial S_n}\left(\frac{r_n}{3r^3}-\frac{x_2^2 r_n}{r^5}-\frac{2x_1n_1}{3r^3}\right)dl,\\
J_5^{0,2} &= F.P.\int_{S_n}\frac{x_2^2}{r^5}dS = \int_{\partial S_n}\left(\frac{r_n}{3r^3}-\frac{x_1^2 r_n}{r^5}-\frac{2x_2n_2}{3r^3}\right)dl,\\
J_5^{1,1} &= F.P.\int_{S_n}\frac{x_1x_2}{r^5}dS = \int_{\partial S_n}\left(\frac{x_1x_2r_n}{r^5}-\frac{r_+}{3r^3}\right)dl,
\end{aligned} \tag{4.2}$$

where S_n is a polygonal boundary element, ∂S_n is its boundary, $r_n=x_\alpha n_\alpha$, $r_+=x_1n_2+x_2n_1$ and $r_-=x_2n_1-x_1n_2$.

Divergent integrals of the type (4.2) have been transformed into regular integrals and may be easily calculated. For example, the hypersingular integral $J_3^{0,0}$ for a circular area with point $\mathbf{y}$ located at the center of the circle leads to the following result:

$$J_3^{0,0} = -\int_{\partial S_n}\frac{r_n}{r^3}dl = -\frac{1}{r}\int_0^{2\pi}d\varphi = -\frac{2\pi}{r}. \tag{4.3}$$

Here polar coordinates are used, r and φ are the circle radius and polar angle, respectively.

In the application of divergent integrals in the BEM, it is necessary to calculate the above integrals over any triangular, rectangular or polygonal elements. In the case of convex polygon with K vertexes, Eq. (3.5), taking into account (4.2), has the form

$$F_{ji}^n(\mathbf{y}_r,\mathbf{x}_q) = \int_{\partial S_n}F_{ji}(\mathbf{y}_r,\mathbf{x})dl = \sum_{k=1}^{K}\int_{l_k}F_{ji}(\mathbf{y}_r,\mathbf{x})dl. \tag{4.4}$$

Here indexes r and q indicate the number of nodes.

Let us consider a polygon S_n with K vertexes as shown in Fig. 1. To calculate the divergent integrals of the type (4.2) the approach developed in [17–19] will be used. All the calculations will be done using the local rectangular coordinate system with its origin located at the point $\mathbf{y}^q$, x_1- and x_2-axis located in the plane of the polygon and x_3-axis perpendicular to this plane.

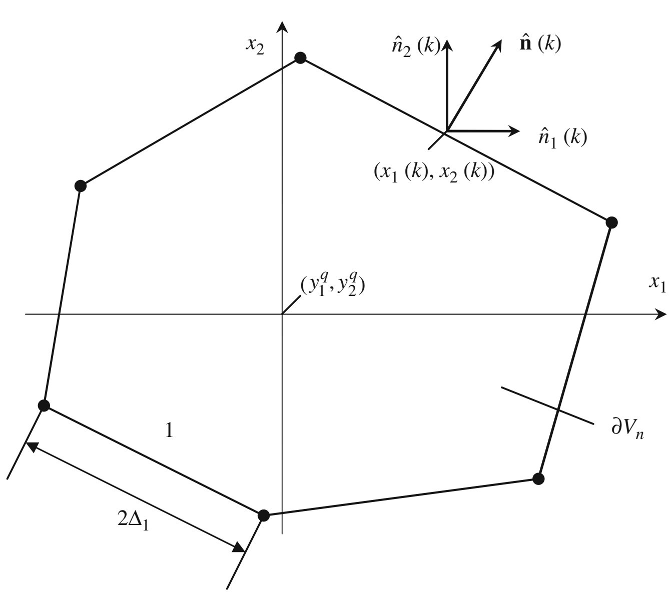

Fig. 1. Polygon with K vertexes.

Global coordinates of the vertexes are (x_1^k, x_2^k). The coordinates of an arbitrary point on the contour ∂S_n may be represented in the form

$$x_1(\xi)=x_1(k)+\Delta_k\hat{n}_2(k)\xi \text{ and } x_2(\xi)=x_2(k)+\Delta_k\hat{n}_1(k)\xi, \tag{4.5}$$

where $x_1(k)$ and $x_2(k)$ are the coordinates of the kth side of the contour, $\hat{\mathbf{n}}(\hat{n}_1,\hat{n}_2)$ is a unit vector normal to the contour, $\xi\in[-1,1]$ is a parameter of integration along the kth side and $2\Delta_k$ is the length of the kth side of the contour ∂S_n.

Coordinates $x_1(k)$ and $x_2(k)$, unit vector normal to the contour ∂S_n and its length can be calculated though the nodal points:

$$\begin{aligned}
&x_i(k)=\frac{x_i^{k+1}+x_i^k}{2},\ \hat{n}_1(k)=\frac{x_2^{k+1}-x_2^k}{2\Delta_k},\\
&\hat{n}_2(k)=\frac{x_1^{k+1}-x_1^k}{2\Delta_k},\ 2\Delta_k=\sqrt{(x_1^{k+1}-x_1^k)^2+(x_2^{k+1}-x_2^k)^2}.
\end{aligned} \tag{4.6}$$

These are some more useful notations that will be used below.

$$\begin{aligned}
&r(\xi)=\sqrt{\Delta_k^2\xi^2+2\xi\Delta_k r_+(k)+r^2(k)},\ r(k)=\sqrt{x_1^2(k)+x_2^2(k)},\ r_n(k)=x_\alpha(k)\hat{n}_\alpha(k),\\
&r_+(k)=x_1(k)\hat{n}_2(k)+x_2(k)\hat{n}_1(k),\ r_-(k)=x_2(k)\hat{n}_1(k)-x_1(k)\hat{n}_2(k),\\
&r_n(\xi)=r_n(k)+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)\xi,\ r_+(\xi)=r_+(k)+\xi\Delta_k,\ r_-(\xi)=r_-(k)+\xi\Delta_k(\hat{n}_2^2-\hat{n}_1^2).
\end{aligned} \tag{4.7}$$

Using these notations the integrals under consideration may be represented in a convenient form for calculations. Substituting (4.5)–(4.7) into (4.2) we obtain formulae for calculating the corresponding integrals over each side l_k of the S_n polygon in the form

$$\begin{aligned}
J_3^{0,0}(k) &= -\Delta_k\int_{-1}^{1}\frac{r_n(k)+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)\xi}{r^3(\xi)}d\xi,\\
J_5^{2,0}(k) &= \frac{1}{3}\int_{-1}^{1}\frac{r_n(k)+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)\xi}{r^3(\xi)}\Delta_k\,d\xi\\
&\quad-\frac{2}{3}\int_{-1}^{1}\frac{x_2(k)\hat{n}_2(k)+\hat{n}_1(k)\hat{n}_2(k)\Delta_k\xi}{r^3(\xi)}\Delta_k\,d\xi\\
&\quad-\int_{-1}^{1}\frac{1}{r^5(\xi)}(x_1(k)r_n(k)+2\hat{n}_2(k)x_1(k)(r_n(k)+\hat{n}_1(k)x_1(k))\Delta_k\xi\\
&\quad+\hat{n}_2(k)^2(r_n(k)+4\hat{n}_1(k)x_1(k))\Delta_k^2\xi^2+2\hat{n}_1(k)\hat{n}_2^3(k)\Delta_k^3\xi^3)\Delta_k\,d\xi,\\
J_5^{0,2}(k) &= \frac{1}{3}\int_{-1}^{1}\frac{r_n(k)+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)\xi}{r^3(\xi)}\Delta_k\,d\xi\\
&\quad-\frac{2}{3}\int_{-1}^{1}\frac{x_1(k)\hat{n}_1(k)+\hat{n}_1(k)\hat{n}_2(k)\Delta_k\xi}{r^3(\xi)}\Delta_k\,d\xi\\
&\quad-\int_{-1}^{1}\frac{1}{r^5(\xi)}(x_2(k)r_n(k)+2\hat{n}_1(k)x_2(k)(r_n(k)+\hat{n}_1(k)x_2(k))\Delta_k\xi\\
&\quad+\hat{n}_1(k)^2(r_n(k)+4\hat{n}_2(k)x_2(k))\Delta_k^2\xi^2+2\hat{n}_1^3(k)\hat{n}_2(k)\Delta_k^3\xi^3)\Delta_k\,d\xi,\\
J_5^{1,1}(k) &= -\frac{1}{3}\int_{-1}^{1}\frac{r_+(k)+\xi\Delta_k}{r^3(\xi)}\Delta_k\,d\xi\\
&\quad+\int_{-1}^{1}\frac{1}{r^5(\xi)}(x_1(k)x_2(k)r_n(k)+(r_n^2(k)+2\hat{n}_1(k)x_1(k)\hat{n}_2(k)x_2(k))\Delta_k\xi\\
&\quad+3\hat{n}_1(k)\hat{n}_2(k)r_n(k)\Delta_k^2\xi^2+2\hat{n}_1^2(k)\hat{n}_2^2(k)\Delta_k^3\xi^3)\Delta_k\,d\xi.
\end{aligned} \tag{4.8}$$

These formulae may be represented in the convenient form

$$\begin{aligned}
J_3^{0,0} &= -\sum_{k=1}^{K}(r_n(k)I_{3,0}+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)I_{3,1}),\\
J_5^{2,0} &= \sum_{k=1}^{K}\left(\frac{1}{3}(r_n(k)I_{3,0}+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)I_{3,1})\right.\\
&\quad-\frac{2}{3}(x_2(k)\hat{n}_2(k)I_{3,0}+\hat{n}_1(k)\hat{n}_2(k)\Delta_k I_{3,1})
\end{aligned}$$

$$
\begin{aligned}
&-x_1^2(k)r_n(k)I_{5,0}+2\hat{n}_2(k)x_1(k)(r_n(k)+\hat{n}_1(k)x_1(k))I_{5,1}\\
&+\hat{n}_2^2(k)(r_n(k)+4\hat{n}_1(k)x_1(k))I_{5,2}+2\hat{n}_1(k)\hat{n}_2^3(k)I_{5,3}),\\
J_5^{0,2}=&\sum_{k=1}^{K}\left(\frac{1}{3}(r_n(k)I_{3,0}+2\Delta_1\hat{n}_1(k)\hat{n}_2(k)I_{3,1})\right.\\
&-\frac{2}{3}(x_1(k)\hat{n}_1(k)I_{3,0}+\hat{n}_1(k)\hat{n}_2(k)\Delta_k I_{3,1})\\
&-x_2^2(k)r_n(k)I_{5,0}+2\hat{n}_1(k)x_2(k)(r_n(k)+\hat{n}_1(k)x_2(k))I_{5,1}\\
&+\hat{n}_1(k)^2(r_n(k)+4\hat{n}_2(k)x_2(k))I_{5,2}+2\hat{n}_1^3(k)\hat{n}_2(k)I_{5,3}),\\
J_5^{1,1}=&\sum_{k=1}^{Q}(x_1(k)x_2(k)r_n(k)I_{5,0}+(r_n^2(k)+2\hat{n}_1(k)x_1(k)\hat{n}_2(k)x_2(k))I_{5,1}\\
&\left.+3\hat{n}_1(k)\hat{n}_2(k)r_n(k)I_{5,2}+2\hat{n}_1^2(k)\hat{n}_2^2(k)I_{5,3}-\frac{1}{3}(r_+(k)I_{3,0}+I_{3,1})\right).
\end{aligned}
\tag{4.9}
$$

Here we use the following notation for the corresponding integrals:

$$I_{p,l}=(\Delta_k)^{l+1}\int_{-1}^{1}\frac{\xi^l}{r^p(\xi)}d\xi. \tag{4.10}$$

These integrals may be calculated analytically.

$$
\begin{aligned}
I_{3,0}&=\Delta_k\int_{-1}^{1}\frac{1}{r(\xi)^3}d\xi=\frac{\Delta_k\xi+r_+(k)}{(r^2(k)-r_+^2(k))r(\xi)}\Big|_{-1}^{1},\\
I_{3,1}&=(\Delta_k)^2\int_{-1}^{1}\frac{\xi}{r(\xi)^3}d\xi=-\frac{r_+(k)\Delta_k\xi+r^2(k)}{(r^2(k)-r_+^2(k))r(\xi)}\Big|_{-1}^{1},\\
I_{5,0}&=\Delta_k\int_{-1}^{1}\frac{1}{r(\xi)^5}d\xi=\frac{I_{3,0}}{r^2(k)-r_+^2(k)}+\frac{\Delta_k\xi+r_+(k)}{2(r^2(k)-r_+^2(k))r(\xi)^3}\Big|_{-1}^{1},\\
I_{5,1}&=(\Delta_k)^2\int_{-1}^{1}\frac{\xi}{r(\xi)^5}d\xi=r_+(k)I_{5,0}-\frac{1}{3r(\xi)^3}\Big|_{-1}^{1},\\
I_{5,2}&=(\Delta_k)^3\int_{-1}^{1}\frac{\xi^2}{r(\xi)^5}d\xi=\frac{(\Delta_k\xi+r_+(k))^3}{3(r^2(k)-r_+^2(k))r(\xi)^3}\Big|_{-1}^{1}\\
&-2r_+(k)I_{5,1}-r^2(k)I_{5,0},\\
I_{5,3}&=(\Delta_k)^4\int_{-1}^{1}\frac{\xi^3}{r(\xi)^5}d\xi=\\
&-\frac{2r^2(k)r(\xi)^2+r^2(k)\Delta_k^2\xi^2+3\Delta_k^2\xi^2r_+^2(k)(r_+(k)\Delta_k\xi+r^2(k))}{3\Delta_k r(\xi)^3(r^2(k)-r_+^2(k))}\Big|_{-1}^{1}.
\end{aligned}
\tag{4.11}
$$

Now divergent integrals with hypersingular kernels (2.6) may be represented though regular contour integrals in the form

$$
\begin{aligned}
F_{11}^n(\mathbf{y}_r,\mathbf{x}_q)&=\frac{\mu}{4\pi(1-\upsilon)}[(1-2\upsilon)J_3^{0,0}+3\upsilon J_5^{2,0}],\\
F_{22}^n(\mathbf{y}_r,\mathbf{x}_q)&=\frac{\mu}{4\pi(1-\upsilon)}[(1-2\upsilon)J_3^{0,0}+3\upsilon J_5^{0,2}],\\
F_{33}^n(\mathbf{y}_r,\mathbf{x}_q)&=-\frac{\mu}{4\pi(1-\upsilon)}J_3^{0,0},\quad F_{12}^n(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}J_5^{1,1}.
\end{aligned}
\tag{4.12}
$$

It is important to mention that all calculations here can be done analytically, no numerical integration is needed.

5. Piecewise-linear approximation

Let us consider the triangular BE that is shown in Fig. 2. In order to simplify the situation we transform the global system of coordinates such that the origins of global and local systems of coordinates coincide. Coordinate axes x_1 and x_2 are located in the plane of the element, while axis x_3 is perpendicular to that plane. In this case x_3=0 and n_1=0, n_2=0, n_3=1. The axes of local coordinates ξ_1 and ξ_2 coincide with the sides of the triangular BE that join at the nodal point 3 (see Fig. 2).

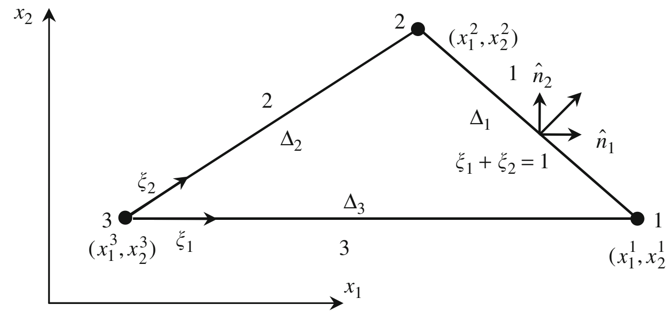

Fig. 2. Triangular BE.

The triangular BE is defined by its angular nodes and its shape functions are

$$
\begin{aligned}
&\varphi_1(\xi_1,\xi_2)=\xi_1,\quad \varphi_2(\xi_1,\xi_2)=\xi_2,\\
&\varphi_3(\xi_1,\xi_2)=(1-\xi_1-\xi_2)\quad \xi_1\in[0,1],\quad \xi_2\in[0,1].
\end{aligned}
\tag{5.1}
$$

Then the global coordinates can be expressed as functions of local ones in the form

$$x_i(\xi_1,\xi_2)=\sum_{q=1}^{3}x_i^q\varphi_q(\xi_1,\xi_2),\quad y_i(\xi_1,\xi_2)=\sum_{q=1}^{3}y_i^q\varphi_q(\xi_1,\xi_2), \tag{5.2a}$$

or

$$x_1=x_1^3+\Delta x_1^3\,\xi_1-\Delta x_1^2\,\xi_2,\quad x_2=x_2^3+\Delta x_2^3\,\xi_1-\Delta x_2^2\,\xi_2, \tag{5.2b}$$

where $\Delta x_i^3=(x_i^1-x_i^3)$ and $\Delta x_i^2=-(x_i^2-x_i^3)$

Derivatives of the shape functions are

$$
\begin{aligned}
&\frac{\partial\varphi_1(\boldsymbol{\xi})}{\partial\xi_1}=1,\quad \frac{\partial\varphi_1(\boldsymbol{\xi})}{\partial\xi_2}=0,\quad \frac{\partial\varphi_2(\boldsymbol{\xi})}{\partial\xi_1}=0,\quad \frac{\partial\varphi_2(\boldsymbol{\xi})}{\partial\xi_2}=1,\\
&\frac{\partial\varphi_3(\boldsymbol{\xi})}{\partial\xi_1}=-1,\quad \frac{\partial\varphi_3(\boldsymbol{\xi})}{\partial\xi_2}=-1,
\end{aligned}
\tag{5.3}
$$

Taking into account that the coordinates ξ_1 and ξ_2 are oblique, normal derivative has to be calculated using the formula

$$\partial_n=\hat{n}_1\frac{\partial}{\partial x_1}+\hat{n}_2\frac{\partial}{\partial x_2}=\left(\frac{\hat{n}_1\Delta_2}{\Delta}\hat{n}_\alpha(2)+\frac{\hat{n}_2\Delta_3}{\Delta}\hat{n}_\alpha(3)\right)\frac{\partial}{\partial\xi_\alpha}, \tag{5.4}$$

where $\Delta=\Delta x_1^3\,\Delta x_2^2-\Delta x_1^2\,\Delta x_2^3$.

Normal derivatives of the shape functions are

$$
\begin{aligned}
\partial_n\varphi_1(k)&=\frac{\hat{n}_1(k)\hat{n}_1(2)\Delta_2}{\Delta}+\frac{\hat{n}_2(k)\hat{n}_1(3)\Delta_3}{\Delta},\\
\partial_n\varphi_2(k)&=\frac{\hat{n}_1(k)\hat{n}_2(2)\Delta_2}{\Delta}+\frac{\hat{n}_2(k)\hat{n}_2(3)\Delta_3}{\Delta},\\
\partial_n\varphi_3(k)&=-\frac{\hat{n}_1(k)(\hat{n}_1(2)+\hat{n}_2(2))\Delta_2+\hat{n}_2(k)(\hat{n}_1(3)+\hat{n}_2(3))\Delta_3}{\Delta}.
\end{aligned}
\tag{5.5}
$$

The coordinates of nodal points are: point 1, (x_1^1,x_2^1); point 2, (x_1^2,x_2^2) and point 3, (x_1^3,x_2^3). Lengths of the triangle sides and radius are

$$\Delta_k=\sqrt{(x_1^{k+1}-x_1^k)^2+(x_2^{k+1}-x_2^k)^2},\quad r(\boldsymbol{\xi},\mathbf{y}^q)=\sqrt{(x_1-y_1^q)^2+(x_2-y_2^q)^2}. \tag{5.6}$$

Regular representations for these integrals can be fond in our previous publications [17,18]. They have the form

$$
\begin{aligned}
J_{q,3}^{0,0}&=F.P.\int_{S_n}\frac{\varphi_q(\boldsymbol{\xi})}{r^3}dS=-\int_{\partial S_n}\left(\varphi_q(\boldsymbol{\xi})\frac{r_n}{r^3}+\frac{1}{r}\partial_n\varphi_q(\boldsymbol{\xi})\right)dl,\\
J_{q,5}^{2,0}&=F.P.\int_{S_n}\varphi_q(\boldsymbol{\xi})\frac{x_1^2}{r^5}dS=\int_{\partial S_n}\left(\varphi_q(\boldsymbol{\xi})\left(\frac{r_n}{3r^3}-\frac{x_2^2r_n}{r^5}-\frac{2x_1n_1}{3r^3}\right)\right.\\
&\left.-\left(\frac{1}{r}+\frac{x_1^2}{r^3}\right)\partial_n\varphi_q(\boldsymbol{\xi})\right)dl,
\end{aligned}
$$

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$$J^{0,2}_{q,5} = F.P.\int_{S_n} \varphi_q(\xi)\frac{x_2^2}{r^5}dS = \int_{\partial S_n}\left(\varphi_q(\xi)\left(\frac{r_n}{3r^3} - \frac{x_1^2 r_n}{r^5} - \frac{2x_2 n_2}{3r^3}\right) - \left(\frac{1}{r} + \frac{x_2^2}{r^3}\right)\partial_n\varphi_q(\xi)\right)dl,$$

$$J^{1,1}_{q,5} = F.P.\int_{S_n}\varphi_q(\xi)\frac{x_1x_2}{r^5}dS = \int_{\partial S_n}\left(\varphi_q(\xi)\left(\frac{x_1x_2r_n}{r^5} - \frac{r_+}{3r^3}\right) - \frac{x_1x_2}{r^3}\partial_n\varphi_q(\xi)\right)dl. \quad (5.7)$$

The integrals under consideration may be represented in a convenient form for calculation:

$$F^n_{ji}(\mathbf{y}_r,\mathbf{x}_q) = \int_{S_n} F_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dS = \sum_{k=1}^{3}\int_{l_k} F_{ji}(\mathbf{y}_r,\mathbf{x})\varphi_q(\mathbf{x})dl. \quad (5.8)$$

Analysis of this equation and representations (5.5) show that we have to calculate the sum of integrals of the following type:

$$J^{l,m}_{q,p}(k) = \int_{l_k}\varphi_q(\xi)\frac{x_1^l(\xi)x_2^m(\xi)}{r^p(\xi)}dl(\xi). \quad (5.9)$$

Details of the calculations are presented in Appendix A. Final results side by side of the calculations are presented below.

Side 1–2: in this case the sums of integrals (5.9) are

$$J^{0,0}_{1,3}(1) = r_n(I_{3,0}+I_{3,1})/2 + \Delta_1\hat n_1\hat n_2(I_{3,1}+I_{3,2}) + \partial_n\varphi_1(1)I_{1,0},$$

$$J^{0,0}_{2,3}(1) = r_nI_{3,1}/2 + \Delta_1\hat n_1\hat n_2I_{3,2} + \partial_n\varphi_2(1)I_{1,0},$$

$$J^{0,0}_{3,3}(1) = \partial_n\varphi_3(1)I_{1,0},$$

$$\begin{aligned}
J^{2,0}_{1,5}(1) = {} & \frac{1}{3}(r_n(I_{3,0}-I_{3,1}) + 2\Delta_1\hat n_1\hat n_2(I_{3,1}-I_{3,2})) \\
& -(x_2^1)^2r_n(I_{5,0}-I_{5,1}) - 2\Delta_1\hat n_1r_nx_2^1(I_{5,1}-I_{5,2}) \\
& -2(x_2^1)^2\Delta_1\hat n_1\hat n_2(I_{5,1}-I_{5,2}) - r_n(\Delta_1\hat n_1)^2(I_{5,2}-I_{5,3}) \\
& -4\Delta_1^2\hat n_1^2\hat n_2^2(I_{5,2}-I_{5,3}) - 2\Delta_1^3\hat n_1^3\hat n_2(I_{5,3}-I_{5,4}) \\
& -\frac{2}{3}(x_1^1(I_{3,0}-I_{3,1}) + \Delta_1\hat n_2(I_{3,1}-I_{3,2}))\hat n_1 - \partial_n\varphi_2(1)((x_1^1)^2r_nI_{3,0} \\
& +2\Delta_1\hat n_2x_1^1I_{3,1} + (x_1^1)^2\Delta_1^2\hat n_2^2I_{3,2} + I_{1,0}),
\end{aligned}$$

$$\begin{aligned}
J^{2,0}_{2,5}(1) = {} & \frac{1}{3}(r_nI_{3,1} + 2\Delta_1\hat n_1\hat n_2I_{3,2}) - (x_2^1)^2r_nI_{5,1} \\
& -2\Delta_1\hat n_1r_nx_2^1I_{5,2} - 2(x_2^1)^2\Delta_1\hat n_1\hat n_2I_{5,2} - r_n(\Delta_1\hat n_1)^2I_{5,3} \\
& -4\Delta_1^2\hat n_1^2\hat n_2^2I_{5,3} - 2\Delta_1^3\hat n_1^3\hat n_2I_{5,4} - \frac{2}{3}(x_1^1I_{3,1} + \Delta_1\hat n_2I_{3,2})\hat n_1 \\
& -\partial_n\varphi_2(1)((x_1^1)^2r_nI_{3,0} + 2\Delta_1\hat n_2x_1^1I_{3,1} + (x_1^1)^2\Delta_1^2\hat n_2^2I_{3,2} + I_{1,0}),
\end{aligned}$$

$$J^{2,0}_{3,5}(1) = -\partial_n\varphi_3(1)((x_1^1)^2r_nI_{3,0} + 2\Delta_1\hat n_2x_1^1I_{3,1} + (x_1^1)^2\Delta_1^2\hat n_2^2I_{3,2} + I_{1,0}),$$

$$\begin{aligned}
J^{0,2}_{1,5}(1) = {} & \frac{1}{3}(r_n(I_{3,0}-I_{3,1}) + 2\Delta_1\hat n_1\hat n_2(I_{3,1}-I_{3,2})) \\
& -(x_1^1)^2r_n(I_{5,0}-I_{5,1}) - 2\Delta_1\hat n_2r_nx_1^1(I_{5,1}-I_{5,2}) \\
& -2(x_1^1)^2\Delta_1\hat n_1\hat n_2(I_{5,1}-I_{5,2}) - r_n(\Delta_1\hat n_2)^2(I_{5,2}-I_{5,3}) \\
& -4\Delta_1^2\hat n_1^2\hat n_2^2(I_{5,2}-I_{5,3}) - 2\Delta_1^3\hat n_1\hat n_2^3(I_{5,3}-I_{5,4}) \\
& -\frac{2}{3}\left(x_2^1(I_{3,0}-I_{3,1}) + \Delta_1\hat n_1(I_{3,1}-I_{3,2})\right)\hat n_2 - \partial_n\varphi_2(1)((x_2^1)^2r_nI_{3,0} \\
& +2\Delta_1\hat n_1x_2^1I_{3,1} + (x_2^1)^2\Delta_1^2\hat n_1^2I_{3,2} + I_{1,0}),
\end{aligned}$$

$$\begin{aligned}
J^{0,2}_{2,5}(1) = {} & \frac{1}{3}(r_nI_{3,1} + 2\Delta_1\hat n_1\hat n_2I_{3,2}) - (x_1^1)^2r_nI_{5,1} \\
& -2\Delta_1\hat n_2r_nx_1^1I_{5,2} - 2(x_1^1)^2\Delta_1\hat n_1\hat n_2I_{5,2} - r_n(\Delta_1\hat n_2)^2I_{5,3} \\
& -4\Delta_1^2\hat n_1^2\hat n_2^2I_{5,3} - 2\Delta_1^3\hat n_1\hat n_2^3I_{5,4} - \frac{2}{3}(x_2^1I_{3,1} + \Delta_1\hat n_1I_{3,2})\hat n_2 \\
& -\partial_n\varphi_2(1)((x_2^1)^2r_nI_{3,0} + 2\Delta_1\hat n_1x_2^1I_{3,1} + (x_2^1)^2\Delta_1^2\hat n_1^2I_{3,2} + I_{1,0}),
\end{aligned}$$

$$J^{0,2}_{3,5}(1) = -\partial_n\varphi_2(1)((x_2^1)^2r_nI_{3,0} + 2\Delta_1\hat n_1x_2^1I_{3,1} + (x_2^1)^2\Delta_1^2\hat n_1^2I_{3,2} + I_{1,0}),$$

$$\begin{aligned}
J^{1,1}_{1,5}(1) = {} & x_1^1x_2^1r_n(I_{5,0}-I_{5,1}) + x_2^1\Delta_1\hat n_2r_n(I_{5,1}-I_{5,2}) \\
& + x_1^1\Delta_1\hat n_1r_n(I_{5,1}-I_{5,2}) + 2\Delta_1\hat n_1\hat n_2x_1^1x_2^1(I_{5,1}-I_{5,2}) \\
& + \Delta_1^2\hat n_1\hat n_2r_n(I_{5,1}-I_{5,3}) + 2\Delta_1^2\hat n_1\hat n_2^2x_2^1(I_{5,2}-I_{5,3}) \\
& + 2\Delta_1^2\hat n_1^2\hat n_2x_1^1(I_{5,2}-I_{5,3}) + 2\Delta_1^4\hat n_1^2\hat n_2(I_{5,3}-I_{5,4}) \\
& -\frac{1}{3}(r_+(I_{3,0}-I_{3,1}) + \xi_2\Delta_k(I_{3,1}-I_{3,2})) - \partial_n\varphi_2(1)(x_1^1x_2^1r_nI_{3,0} \\
& + \Delta_1(\hat n_2x_1^1 + \hat n_1x_2^1)I_{3,1} + x_1^1x_2^1\Delta_1^2\hat n_1\hat n_2I_{3,2} + I_{1,0}),
\end{aligned}$$

$$\begin{aligned}
J^{1,1}_{2,5}(1) = {} & x_1^1x_2^1r_nI_{5,1} + x_2^1\Delta_1\hat n_2r_nI_{5,2} + x_1^1\Delta_1\hat n_1r_nI_{5,2} \\
& + 2\Delta_1\hat n_1\hat n_2x_1^1x_2^1I_{5,2} + \Delta_1^2\hat n_1\hat n_2r_nI_{5,3} + 2\Delta_1^2\hat n_1\hat n_2^2x_2^1I_{5,3} \\
& + 2\Delta_1^2\hat n_1^2\hat n_2x_1^1I_{5,3} + 2\Delta_1^4\hat n_1^2\hat n_2I_{5,4} - \frac{1}{3}(r_+I_{3,1} + \xi_2\Delta_kI_{3,2}) \\
& -\partial_n\varphi_2(1)(x_1^1x_2^1r_nI_{3,0} + \Delta_1(\hat n_2x_1^1 + \hat n_1x_2^1)I_{3,1} + x_1^1x_2^1\Delta_1^2\hat n_1\hat n_2I_{3,2} + I_{1,0}),
\end{aligned}$$

$$\begin{aligned}
J^{1,1}_{3,5}(1) = {} & -\partial_n\varphi_3(1)(x_1^1x_2^1r_nI_{3,0} + \Delta_1(\hat n_2x_1^1 + \hat n_1x_2^1)I_{3,1} \\
& + x_1^1x_2^1\Delta_1^2\hat n_1\hat n_2I_{3,2} + I_{1,0}). \quad (5.10)
\end{aligned}$$

Side 2–3: in this case the sums of integrals (5.9) are

$$J^{0,0}_{q,3}(2) = 0,\quad J^{2,0}_{q,5}(2) = 0,\quad J^{0,2}_{q,5}(2) = 0,$$

$$J^{1,1}_{1,5}(2) = 0,\quad J^{1,1}_{2,5}(2) = 0,\quad J^{1,1}_{3,5}(2) = \frac{1}{3\Delta_2}. \quad (5.11)$$

Side 3–1: in this case the sums of integrals (5.9) are

$$J^{0,0}_{q,3}(3) = 0,\quad J^{2,0}_{q,5}(3) = 0,\quad J^{0,2}_{q,5}(3) = 0,$$

$$J^{1,1}_{1,5}(3) = 0,\quad J^{1,1}_{2,5}(3) = 0,\quad J^{1,1}_{3,5}(3) = \frac{1}{3\Delta_2}. \quad (5.12)$$

We have taken into account that integrations in (5.10)–(5.13) have to be done in the way as shown by arrows in Fig. 3. Singular integrals of the type $\int_0^1 d\xi/\xi$ on sides 2–3 and 3–1 can be neglected because in the final relations they appear with opposite signs during integration over adjacent elements, as follows from Fig. 3.

All integrals of the type $I_{p,l}$ here can be calculated by the following formulae:

$$I_{3,0} = \Delta_k\int_{-1}^{1}\frac{1}{r(\xi)^3}d\xi = \frac{\Delta_k\xi + r_+(k)}{(r^2(k) - r_+^2(k))r(\xi)}\Big|_0^1,$$

$$I_{3,1} = (\Delta_k)^2\int_{-1}^{1}\frac{\xi}{r(\xi)^3}d\xi = -\frac{r_+(k)\Delta_k\xi + r^2(k)}{(r^2(k) - r_+^2(k))r(\xi)}\Big|_0^1,$$

$$I_{3,3} = (\Delta_k)^4\int_{-1}^{1}\frac{\xi^3}{r(\xi)^3}d\xi = -2\Delta_k\xi r_+(k)I_{3,1} - 3r_+(k)I_{3,2} + \left(\frac{2r^2(k)r(\xi)}{r^2(k) - r_+^2(k)} - \frac{\Delta_k^2\xi^2}{r(\xi)}\right)\Big|_{-1}^1,$$

$$I_{5,0} = \Delta_k\int_{-1}^{1}\frac{1}{r(\xi)^5}d\xi = \frac{I_{3,0}}{r^2(k) - r_+^2(k)} + \frac{\Delta_k\xi + r_+(k)}{2(r^2(k) - r_+^2(k))r(\xi)^3}\Big|_0^1,$$

$$I_{5,1} = (\Delta_k)^2\int_{-1}^{1}\frac{\xi}{r(\xi)^5}d\xi = r_+(k)I_{5,0} - \frac{1}{3r(\xi)^3}\Big|_0^1,$$

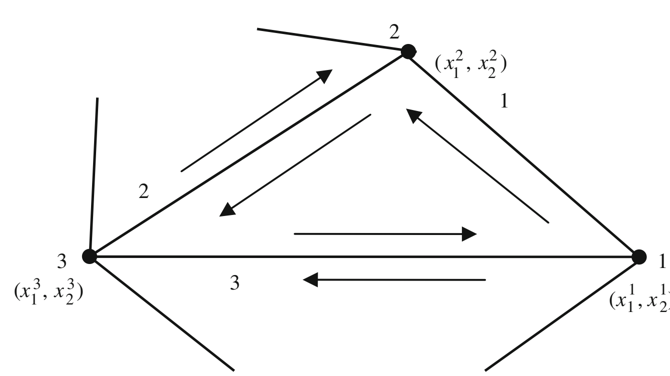

Fig. 3. Path of integration.

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$$I_{5,2}=(\varDelta_k)^3\int_{-1}^{1}\frac{\xi^2}{r(\xi)^5}d\xi=\frac{(\varDelta_k\xi+r_+(k))^3}{3(r^2(k)-r_+^2(k))r(\xi)^3}\Big|_0^1$$
$$-2r_+(k)I_{5,1}-r^2(k)I_{5,0},$$
$$I_{5,3}=(\varDelta_k)^4\int_{-1}^{1}\frac{\xi^3}{r(\xi)^5}d\xi=\frac{2r^2(k)r(\xi)^2+r^2(k)\varDelta_k^2\xi^2+3\varDelta_k^2\xi^2r_+^2(k)}{3\varDelta_k r(\xi)^2}I_{3,1}\Big|_0^1,$$
$$I_{5,4}=(\varDelta_k)^5\int_{-1}^{1}\frac{\xi^4}{r(\xi)^5}d\xi=\left(\frac{1}{\varDelta_k^3}\ln|r_+(k)+\varDelta_k\xi+r(\xi)|\right.$$
$$+\frac{1}{3\varDelta_k^3r(\xi)^3(r^2(k)-r_+^2(k))^2}(5r^6(\xi)r_+(k)-3r^4(\xi)\varDelta_k\xi$$
$$-4r^4(\xi)\varDelta_k^3\xi^3+7r_+^2(k)r(\xi)\varDelta_k\xi(3r^4(\xi)+\varDelta_k^2\xi^2)$$
$$-3r_+^2(k)r^2(\xi)(r^2(\xi)-6\varDelta_k^2\xi^2)-4r_+^4(k)r(\xi)\varDelta_k\xi(3r^2(\xi)$$
$$\left.+2\varDelta_k^2\xi^2)-12r_+^2(k)\varDelta_k^2\xi^2)\right)\Big|_{-1}^1. \quad (5.13)$$

Finally sums of the integrals in (5.9) have the form

$$J_{q,p}^{l,m}=\sum_{k=1}^{3}J_{q,p}^{l,m}(k). \quad (5.14)$$

All integrals of the type $J_{q,p}^{l,m}(k)$ here have already bean calculated above side by side for $k=1,2,3$ and represented by Eqs. (5.10)–(5.12).

Substituting everything obtained for each side results in (5.6) and taking into account (5.14) finally we have

$$F_{11}^n(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{3}J_{q,3}^{0,0}(k)+3\upsilon\sum_{k=1}^{3}J_{q,5}^{2,0}(k)\right],$$
$$F_{22}^n(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu}{4\pi(1-\upsilon)}\left[(1-2\upsilon)\sum_{k=1}^{3}J_{q,3}^{0,0}(k)+3\upsilon\sum_{k=1}^{3}J_{q,5}^{0,2}(k)\right],$$
$$F_{33}^n(\mathbf{y}_r,\mathbf{x}_q)=-\frac{\mu}{4\pi(1-\upsilon)}\sum_{k=1}^{3}J_{q,3}^{0,0}(k),\quad F_{12}^n(\mathbf{y}_r,\mathbf{x}_q)=\frac{\mu\upsilon}{4\pi(1-\upsilon)}\sum_{k=1}^{3}J_{q,5}^{1,1}(k). \quad (5.15)$$

It is important to mention here that all calculations can be done analytically, no numerical integration is needed.

6. Numerical calculations

Let a 3-D elastic unbounded body have a penny-shaped crack that is located in the plane $R^2=\{\mathbf{x}:x_3=0\}$ and let its surface have coordinates $\Omega=\{x_1^2+x_2^2\le R, x_3=0\}$ as shown in Fig. 4. Assume that the material has the following mechanical properties: elastic modulus $E=200$ GPa, Poisson ratio $\nu=0.25$ and specific density $\rho=7800\,\text{kg/m}^3$.

First we consider that the crack is subjected to a uniform static stress at infinity. Analytical solution is available for this problem [22]. The problem is axisymmetrical in this case. Displacement discontinuity of the crack surfaces is defined by

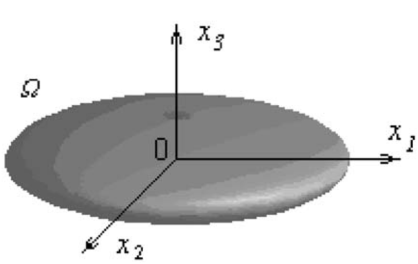

Fig. 4. Penny-shaped crack in 3-D elastic space.

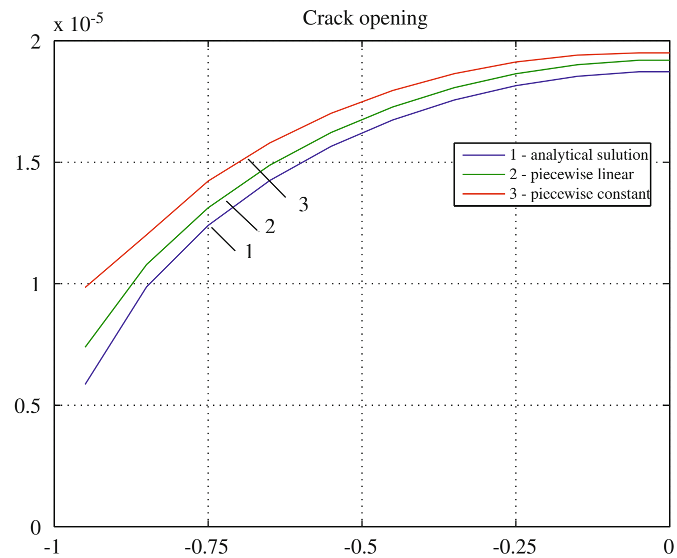

Fig. 5. Penny-shaped crack opening versus radius in 3-D elastic space for 250 BEs.

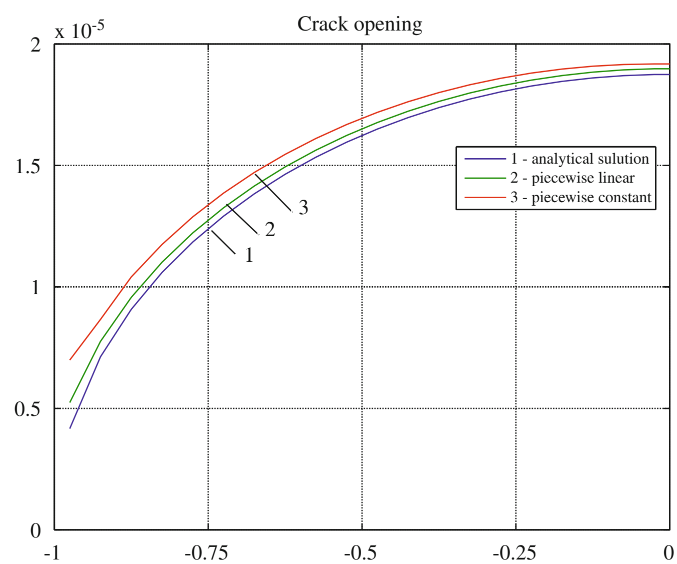

Fig. 6. Penny-shaped crack opening versus radius in 3-D elastic space for 548 BEs.

the equation

$$\Delta u_3(\mathbf{x})=\frac{2(1-\upsilon^2)pR}{\pi E}\sqrt{1-r^2/R^2}, \quad (6.1)$$

where R is the crack radius and r is the polar coordinate.

The BIE that relates the load on crack edges and their displacement discontinuity (2.5) in this case has the form

$$p_3(\mathbf{y})=-\int_{\Omega}F_{33}(\mathbf{x},\mathbf{y})\Delta u_3(\mathbf{x})dS. \quad (6.2)$$

The kernel $F_{33}(\mathbf{x},\mathbf{y})$ is defined by Eq. (2.6).

Results of analytical and numerical calculations for this case are shown in Figs. 5 and 6 for 250 BEs and 548 BEs, respectively. From the presented diagrams it follows that obtained numerical results coincide well with the analytical solution outside the small area near the crack tip.

Now let us consider a harmonic tension–compression wave that propagates in the direction perpendicular to the surface of the crack penny-shaped Ω. The incident wave is defined by the

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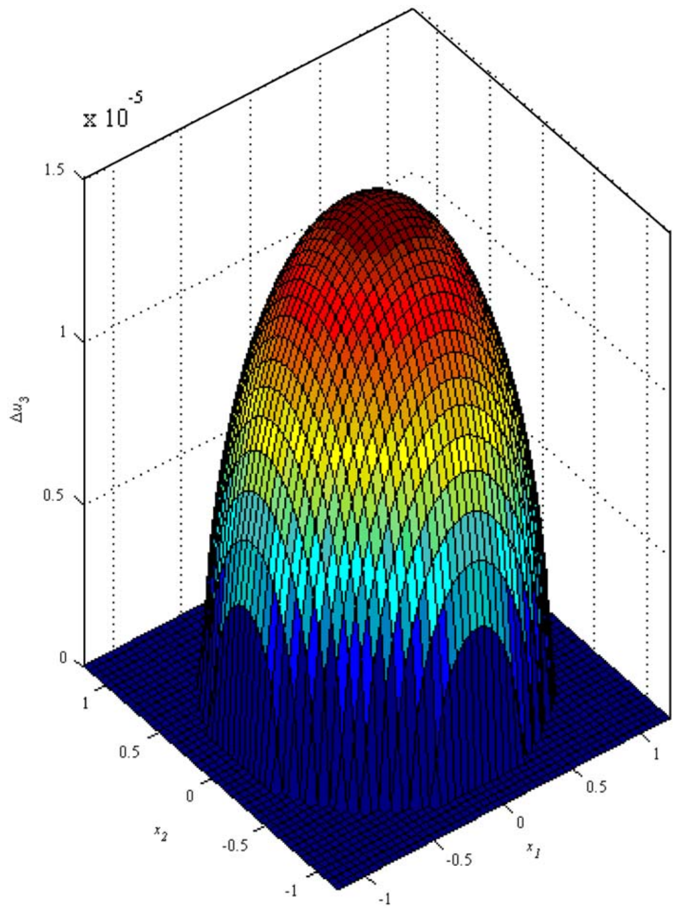

Fig. 7. Complex amplitude of the penny-shaped crack opening $\Delta u_3(\mathbf{x})$ for k_1=0.45.

potential function

$$\Phi(\mathbf{x},t)=\Phi_0(\mathbf{x})e^{i(k_1x_3-\omega t)},\quad k_1=\omega/c_1,\quad c_1=\sqrt{(\lambda+2\mu)/\rho},\tag{6.3}$$

where $\omega=2\pi/T$ is the frequency, T the period of vibration, Φ_0 the amplitude and c_1 the velocity of the dilatational wave.

We consider here the problem concerning reflected waves. The load on the crack edges caused by the incident wave has the form

$$p_3(\mathbf{x},t)=\mathrm{Re}\{p_3(\mathbf{x})e^{-i\omega t}\},\quad p_3(\mathbf{x})=-\mu k_2^2c_1^2/c_2^2\Phi_0(\mathbf{x}).\tag{6.4}$$

The BIE that relates complex amplitudes of the load on crack edges and their displacement discontinuities in this case has the form (6.2). The kernel $F_{33}(\mathbf{x},\mathbf{y})$ is a complex-valued function. Details of its calculation and regularization can be found in Zozulya [18]. Complex amplitude of the penny-shaped crack opening $\Delta u_3(\mathbf{x})$ for k_1=0.45 is presented in Fig. 7.

7. Conclusions

Based on the theory of distribution approach for divergent hypersingular integrals regularization is developed here and applied for the case of 3-D elastostatic crack problems. We consider 2-D hypersingular integrals over arbitrary convex polygons for piecewise-constant approximation and over triangular BE for piecewise-linear approximation and find regular formulae for their calculation. It is important to mention that in the presented equations all calculations can be done analytically, no numerical integration is needed.

Appendix A

Side 1–2: from Fig. 8 it follows that in this case $\xi_1=1-\xi_2$. The main parameters defined by (4.7), (5.1)–(5.4) are

$$x_1=x_1^1+\Delta_1\hat{n}_2\xi_2,\quad x_2=x_2^1+\Delta_1\hat{n}_1\xi_2,\quad dl=\Delta_1 d\xi_2,$$
$$r(\xi_2)=\sqrt{\Delta_1^2\xi_2^2+2\xi\Delta_1r_++r^2},\quad r=\sqrt{(x_1^1)^2+(x_2^1)^2},$$
$$r_n=x_\alpha^1\hat{n}_\alpha,\quad r_+=x_2^1\hat{n}_1+x_1^1\hat{n}_2,\quad r_-=x_2^1\hat{n}_1-x_1^1\hat{n}_2,$$
$$r_n(\xi_2)=r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2,\quad r_+(\xi_2)=r_++\xi_2\Delta_k,$$
$$r_-(\xi_2)=r_-(k)+\xi_2\Delta_k(\hat{n}_2^2-\hat{n}_1^2),$$
$$\varphi_1(\xi_1,\xi_2)=1-\xi_2,\quad \varphi_2(\xi_1,\xi_2)=\xi_2,\quad \varphi_3(\xi_1,\xi_2)=0$$
$$\partial_n\varphi_1(1)=\frac{\hat{n}_1(1)\hat{n}_1(2)\Delta_2}{\Delta}+\frac{\hat{n}_2(1)\hat{n}_1(3)\Delta_3}{\Delta},$$
$$\partial_n\varphi_2(1)=\frac{\hat{n}_1(1)\hat{n}_2(2)\Delta_2}{\Delta}+\frac{\hat{n}_2(1)\hat{n}_2(3)\Delta_3}{\Delta},$$
$$\partial_n\varphi_3(1)=-\frac{\hat{n}_1(1)(\hat{n}_1(2)+\hat{n}_2(2))\Delta_2+\hat{n}_2(1)(\hat{n}_1(3)+\hat{n}_2(3))\Delta_3}{\Delta}$$

Integrals of the type $J_{q,3}^{0,0}$:

$$J_{1,1}^{0,0}(1)=\int_0^1\left((1-\xi_2)\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{2r^3(\xi_2)}-\frac{\partial_n\varphi_1(1)}{r(\xi_2)}\right)\Delta_1\,d\xi_2,$$
$$J_{2,1}^{0,0}(1)=\int_0^1\left(\xi_2\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{2r^3(\xi_2)}-\frac{\partial_n\varphi_2(1)}{r(\xi_2)}\right)\Delta_1\,d\xi_2,$$
$$J_{3,1}^{0,0}(1)=\int_0^1\frac{\partial_n\varphi_3(1)}{r(\xi_2)}\Delta_1\,d\xi_2.$$

Integrals of the type $J_{q,5}^{2,0}$:

$$J_{1,5}^{2,0}(1)=\int_0^1(1-\xi_2)$$
$$\left(\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{3r^3(\xi_2)}-\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}\right.$$
$$\left.-\frac{2(x_1^1+\Delta_1\hat{n}_2\xi_2)\hat{n}_1}{3r^3(\xi_2)}\right)\Delta_1\,d\xi_2$$
$$-\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{r^3(\xi_2)}+\frac{1}{r(\xi_2)}\right)\partial_n\varphi_1(1)\,\Delta_1\,d\xi_2,$$
$$J_{2,5}^{2,0}(1)=\int_0^1\xi_2\left(\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{3r^3(\xi_2)}-\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}\right.$$
$$\left.-\frac{2(x_1^1+\Delta_1\hat{n}_2\xi_2)\hat{n}_1}{3r^3(\xi_2)}\right)\Delta_1\,d\xi_2$$
$$+\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{r^3(\xi_2)}+\frac{1}{r(\xi_2)}\right)\partial_n\varphi_2(1)\,\Delta_1\,d\xi_2,$$
$$J_{3,5}^{2,0}(1)=\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2}{r^3(\xi_2)}+\frac{1}{r(\xi_2)}\right)\partial_n\varphi_3(1)\,\Delta_1\,d\xi_2.$$

Integrals of the type $J_{q,5}^{0,2}$:

$$J_{1,5}^{0,2}(1)=\int_0^1(1-\xi_2)\left(\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{3r^3(\xi_2)}\right.$$

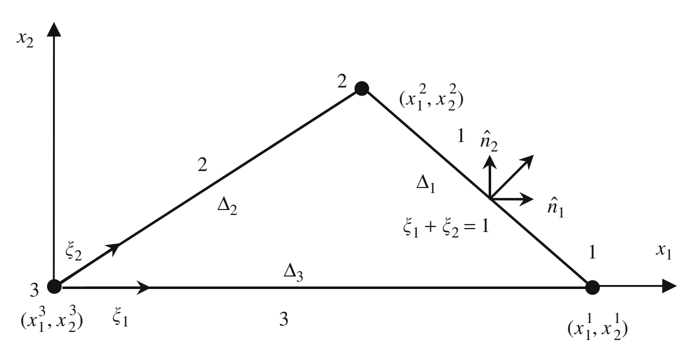

Fig. 8. Sides for integration.

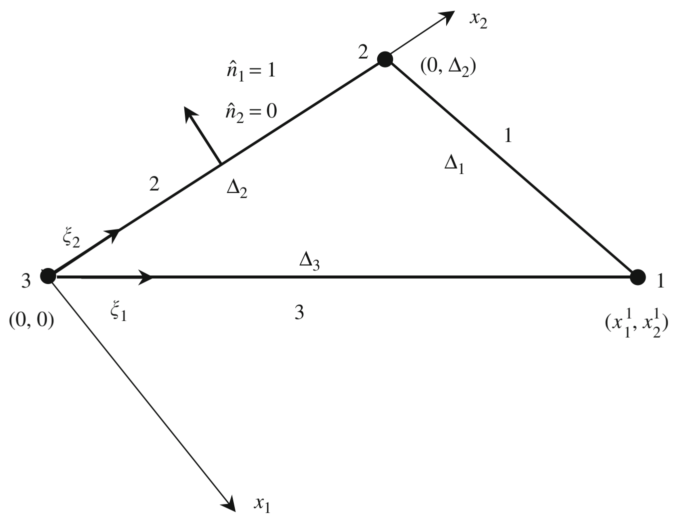

Fig. 9. Sides for integration.

$$-\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}$$
$$-\frac{2(x_2^1+\Delta_1\hat{n}_1\xi_2)\hat{n}_2}{3r^3(\xi_2)}\Bigg)\Delta_1\,d\xi_2$$
$$-\int_0^1\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{r^3(\xi_2)}+\frac{1}{r(\xi_2)}\right)\partial_n\varphi_1(1)\,\Delta_1\,d\xi_2,$$

$$J_{2,5}^{0,2}(1)=\int_0^1\xi_2\Bigg(\frac{r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2}{3r^3(\xi_2)}-\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)^2(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}$$
$$-\frac{2(x_2^1+\Delta_1\hat{n}_1\xi_2)\hat{n}_2}{3r^3(\xi_2)}\Bigg)\Delta_1\,d\xi_2$$
$$+\int_0^1\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{r^3(\xi_2)}+\frac{1}{r(\xi_2)}\right)\partial_n\varphi_2(1)\,\Delta_1\,d\xi_2,$$

$$J_{3,5}^{0,2}(1)=\int_0^1\left(\frac{(x_2^1+\Delta_1\hat{n}_1\xi_2)^2}{r^3(\xi_2)}+\frac{1}{r(\xi_2)}\right)\partial_n\varphi_3(1)\,\Delta_1\,d\xi_2.$$

Integrals of the type $J_{q,5}^{1,1}$:

$$J_{1,5}^{1,1}(1)=\int_0^1(1-\xi_2)\Bigg(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}$$
$$-\frac{r_++\xi_2\Delta_k}{3r^3(\xi_2)}\Bigg)\Delta_1\,d\xi_2$$
$$-\int_0^1\Bigg(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{r^3(\xi_2)}$$
$$+r(\xi_2)\Bigg)\partial_n\varphi_1(1)\,\Delta_1\,d\xi_2,$$

$$J_{2,5}^{1,1}(1)=\int_0^1\xi_2\Bigg(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)(r_n+2\Delta_1\hat{n}_1\hat{n}_2\xi_2)}{r^5(\xi_2)}$$
$$-\frac{r_++\xi_2\Delta_k}{3r^3(\xi_2)}\Bigg)\Delta_1\,d\xi_2$$
$$+\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{r^3(\xi_2)}+r(\xi_2)\right)$$
$$\partial_n\varphi_2(1)\,\Delta_1\,d\xi_2,$$

$$J_{3,5}^{1,1}(1)=\int_0^1\left(\frac{(x_1^1+\Delta_1\hat{n}_2\xi_2)(x_2^1+\Delta_1\hat{n}_1\xi_2)}{r^3(\xi_2)}+r(\xi_2)\right)\partial_n\varphi_3(1)\,\Delta_1\,d\xi_2.$$

Side 2–3. From Fig. 9 it follows that in this case $\xi_1=0$, $\hat{n}_1=1$ and $\hat{n}_2=0$.

The main parameters defined by (4.7) and (5.1)–(5.4) are

$$x_2=\Delta_2\xi_2,\quad x_1=0,\quad dl=\Delta_2\,d\xi_2,\quad r(\xi_2)=\Delta_2\xi_2,\quad r_n(\xi_2)=0,$$
$$r_n=0,\quad r_+=0,\quad r_-=0,\quad r_+(\xi_2)=\Delta_k\xi_2,\quad r_-(\xi_2)=-\Delta_k\hat{n}_1^2\xi_2,$$
$$\varphi_1(\xi_2)=0,\quad \varphi_2(\xi_2)=\xi_2,\quad \varphi_3(\xi_2)=1-\xi_2,\quad \partial_n\varphi_1(2)=\Delta_2/\Delta,$$
$$\partial_n\varphi_2(2)=0,\quad \partial_n\varphi_3(2)=-\Delta_2/\Delta.$$

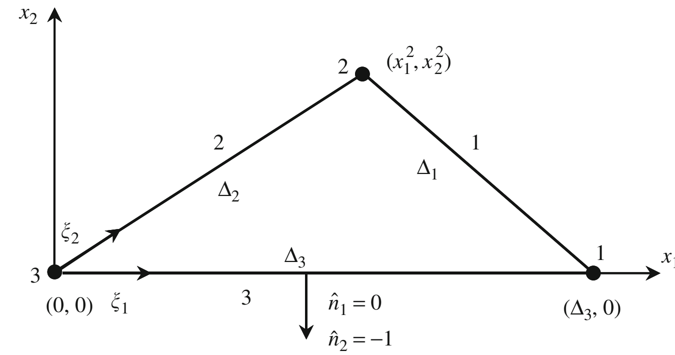

Fig. 10. Sides for integration.

Integrals of the type $J_{q,3}^{0,0}$:

$$J_{1,3}^{0,0}(2)=-\frac{\Delta_2}{\Delta}\int_0^1\frac{d\xi_2}{\xi_2},\quad J_{2,3}^{0,0}(2)=0,\quad J_{3,3}^{0,0}(2)=\frac{\Delta_2}{\Delta}\int_0^1\frac{d\xi_2}{\xi_2}.$$

Integrals of the type $J_{q,5}^{2,0}$:

$$J_{1,5}^{2,0}(2)=-2\frac{\Delta_2}{\Delta}\int_0^1\frac{d\xi_2}{\xi_2},\quad J_{2,5}^{2,0}(2)=0,\quad J_{3,5}^{2,0}(2)=2\frac{\Delta_2}{\Delta}\int_0^1\frac{d\xi_2}{\xi_2}.$$

Integrals of the type $J_{q,5}^{0,2}$:

$$J_{1,5}^{0,2}(2)=-\frac{\Delta_2}{\Delta}\int_0^1\frac{d\xi_2}{\xi_2},\quad J_{2,5}^{0,2}(2)=0,\quad J_{3,5}^{0,2}(2)=\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_2}{\xi_2}.$$

Integrals of the type $J_{q,5}^{1,1}$:

$$J_{1,5}^{1,1}(2)=0,\quad J_{2,5}^{1,1}(2)=-\frac{1}{3\Delta_2}\int_0^1\frac{1}{\xi_2}d\xi_2,$$
$$J_{3,5}^{1,1}(2)=-\int_0^1\frac{1}{3\Delta_2\xi_2^2}d\xi_2+\int_0^1\frac{1}{3\Delta_2\xi_2}d\xi_2.$$

Side 3–1: From Fig. 10 it follows that in this case $\xi_2=0$, $\hat{n}_1=0$ and $\hat{n}_2=-1$.

The main parameters defined by (4.7) and (5.1)–(5.4) are

$$x_1=\Delta_3\xi_1,\quad x_2=0,\quad dl=\Delta_3\,d\xi_1,\quad r(\xi_1)=\Delta_3\xi_1,\quad r_n(\xi_1)=0,$$
$$r_n=0, r_+=0,\quad r_-=0,\quad r_+(\xi_1)=\Delta_3\xi_1,\quad r_-(\xi_1)=\Delta_k\hat{n}_2^2\xi_1,$$
$$\varphi_1(\xi_1)=\xi_1,\quad \varphi_2(\xi_1)=0,\quad \varphi_3(\xi_1)=1-\xi_1,\quad \partial_n\varphi_1(3)=0,$$
$$\partial_n\varphi_2(3)=\Delta_3/\Delta,\quad \partial_n\varphi_3(3)=-\Delta_3/\Delta.$$

Integrals of the type $J_{q,3}^{0,0}$:

$$J_{1,3}^{0,0}(3)=0, J_{3,3}^{0,0}(3)=-\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_1}{\xi_1},\quad J_{3,3}^{0,0}(3)=\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_1}{\xi_1}.$$

Integrals of the type $J_{q,5}^{2,0}$:

$$J_{1,5}^{2,0}(3)=0,\quad J_{2,5}^{2,0}(3)=-2\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_1}{\xi_1},\quad J_{3,5}^{2,0}(3)=2\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_1}{\xi_1}.$$

Integrals of the type $J_{q,5}^{0,2}$:

$$J_{1,5}^{0,2}(3)=0,\quad J_{2,5}^{0,2}(3)=-\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_1}{\xi_1},\quad J_{3,5}^{0,2}(3)=\frac{\Delta_3}{\Delta}\int_0^1\frac{d\xi_1}{\xi_1}.$$

Integrals of the type $J_{q,5}^{1,1}$:

$$J_{1,5}^{1,1}(3)=-\frac{1}{3\Delta_3}\int_0^1\frac{1}{\xi_1}d\xi_1,\quad J_{2,5}^{1,1}(3)=0,$$
$$J_{3,5}^{1,1}(3)=-\int_0^1\frac{1}{3\Delta_3\xi_1^2}d\xi_1+\int_0^1\frac{1}{3\Delta_3\xi_1}d\xi_1.$$

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