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Abstract
In this paper, the Cauchy problem for the Helmholtz equation is investigated. By Green's formulation, the problem can be transformed into a moment problem. Then we propose a numerical algorithm for obtaining an approximate solution to the Neumann data on the unspecified boundary. Error estimate and convergence analysis have also been given. Finally, we present numerical results for several examples and show the effectiveness of the proposed method.

1. Introduction

The Helmholtz equation arises in many physical applications, especially in wave propagation and vibration phenomena, such as the acoustic cavity problem, the scattering of a wave, vibration of the structure, electromagnetic scattering and so on (see [2, 3, 7, 8, 13]). The direct problems, i.e. Dirichlet, Neumann or mixed boundary value problems for the Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points of the concerned domain, which will lead to some inverse problems. The Cauchy problem for the Helmholtz equation is an inverse problem and is severely ill-posed. That means the solution does not depend continuously on the given Cauchy data and any small change in the given data may cause large change to the solution [9, 17]. Several numerical methods have been proposed to solve this problem, such as the alternating iterative boundary element method [13], the conjugate gradient boundary element method [14], and the method of fundamental solutions [11, 15, 19]. In paper [10], the boundary knot method was applied to solve the Cauchy problem of the inhomogeneous Helmholtz equation. In this paper, we propose a new numerical method for dealing with this problem in a special domain. The main idea is to transform the Cauchy problem into a moment problem whose numerical method has been studied extensively. The Neumann boundary value of the solution on the unspecified boundary can be obtained by solving a corresponding Hausdorff moment problem. Convergence analysis and numerical verification are also presented.

The paper is organized as follows. In section 2, we formulate the problem and transform the Cauchy problem into a moment problem according to the idea in [4]. In section 3, we propose a numerical algorithm for solving the moment problem and give error estimate and convergence results. In section 4, we give several numerical examples to demonstrate the effectiveness of our proposed method. Finally, we give a conclusion in section 5.

2. Formulation of the problem and transformation to a moment problem

Let Ω be a simply connected and bounded domain in $\mathbb{R}^2$ with a sufficiently regular boundary $\partial\Omega$ and Γ be an open part of boundary $\partial\Omega$. Without loss of generality, we assume that Γ is connected.

Consider the following Cauchy problem for the Helmholtz equation:

$$\Delta u(x, y) + k^2 u(x, y) = 0, \qquad (x, y) \in \Omega, \tag{2.1}$$

$$u(x, y) = f(x, y), \qquad (x, y) \in \Gamma, \tag{2.2}$$

$$\frac{\partial u(x, y)}{\partial n} = g(x, y), \qquad (x, y) \in \Gamma, \tag{2.3}$$

where $f \in H^{3/2}(\Gamma)$, $g \in H^{1/2}(\Gamma)$, n is the outer unit normal with respect to $\partial\Omega$ and constant $k > 0$ is the wave number. In this paper, we assume that $-k^2$ is not an eigenvalue of the Laplacian operator with the homogenous Neumann boundary condition.

Suppose the Cauchy problem (2.1)–(2.3) has a solution u in $H^2(\Omega)$, then for any $\phi \in H^1(\Omega)$, we know u satisfies the following formulation:

$$\int_\Omega \nabla u \nabla \phi \, \mathrm{d}x \, \mathrm{d}y - k^2 \int_\Omega u\phi \, \mathrm{d}x \, \mathrm{d}y = \int_\Gamma g\phi \, \mathrm{d}s + \int_{\partial\Omega\setminus\Gamma} \frac{\partial u}{\partial n} \phi \, \mathrm{d}s, \qquad \forall \phi \in H^1(\Omega), \tag{2.4}$$

where ds is the curve element.

For any $q \in L^2(\Gamma)$, let $v_q \in H^1(\Omega)$ be a weak solution of the following problem:

$$\Delta v(x, y) + k^2 v(x, y) = 0, \qquad (x, y) \in \Omega, \tag{2.5}$$

$$\frac{\partial v(x, y)}{\partial n} = 0, \qquad (x, y) \in \partial\Omega\setminus\Gamma, \tag{2.6}$$

$$\frac{\partial v(x, y)}{\partial n} = q, \qquad (x, y) \in \Gamma, \tag{2.7}$$

then by theorem A.5 in the appendix, v_q exists and satisfies

$$\int_\Omega \nabla v_q \nabla \phi \, \mathrm{d}x \, \mathrm{d}y - k^2 \int_\Omega v_q \phi \, \mathrm{d}x \, \mathrm{d}y = \int_\Gamma q\phi \, \mathrm{d}s, \qquad \forall \phi \in H^1(\Omega). \tag{2.8}$$

Denote

$$\mathcal{H} = \{v(x, y) \in H^1(\Omega) \mid v \text{ satisfies (2.8) for all } q \in L^2(\Gamma)\}.$$

For any $v \in \mathcal{H}$, take $\phi = v$ in (2.4) and $\phi = u$ in (2.8) with $v_q = v$, minus (2.8) by (2.4), note that $u|_\Gamma = f$, then we have the following equation:

$$\int_{\partial\Omega\setminus\Gamma} v \frac{\partial u}{\partial n} \, \mathrm{d}s = \int_\Gamma \left(f \frac{\partial v}{\partial n} - vg \right) \mathrm{d}s. \tag{2.9}$$

Proposition 2.1. *If the Cauchy problem (2.1)–(2.3) has a solution* $u \in H^2(\Omega)$ *such that* $\frac{\partial u}{\partial n}\big|_{\partial\Omega\backslash\Gamma} \in H^{1/2}(\partial\Omega\backslash\Gamma)$*, then* $\beta = \frac{\partial u}{\partial n}\big|_{\partial\Omega\backslash\Gamma}$ *satisfies the following moment problem:*

$$\int_{\partial\Omega\backslash\Gamma} v\beta ds = \int_{\Gamma}\left(f\frac{\partial v}{\partial n} - gv\right) ds \equiv \mu_v(f, g) \tag{2.10}$$

where $v \in \mathcal{H}$.

Conversely if $\beta \in L^2(\partial\Omega\backslash\Gamma)$ *is a solution of (2.10), then there exists a solution* $u \in H^1(\Omega)$ *of the Cauchy problem (2.1)–(2.3) such that* $\frac{\partial u}{\partial n}\big|_{\partial\Omega\backslash\Gamma} = \beta$.

Proof. From the above deduction, we have known that if u is a solution of the Cauchy problem (2.1)–(2.3) in $H^2(\Omega)$ and $\frac{\partial u}{\partial n}\big|_{\partial\Omega\backslash\Gamma} \in H^{1/2}(\partial\Omega\backslash\Gamma)$, then $\beta = \frac{\partial u}{\partial n}\big|_{\partial\Omega\backslash\Gamma}$ is a solution of the moment problem (2.10).

In the following, we verify that if $\beta \in L^2(\partial\Omega\backslash\Gamma)$ is a solution of the moment problem (2.10), then we can get a solution for the Cauchy problem (2.1)–(2.3) in $H^1(\Omega)$. Consider the following Neumann boundary value problem:

$$\Delta w + k^2 w = 0, \qquad \text{in} \quad \Omega, \tag{2.11}$$

$$\frac{\partial w}{\partial n}\bigg|_{\partial\Omega\backslash\Gamma} = \beta, \tag{2.12}$$

$$\frac{\partial w}{\partial n}\bigg|_{\Gamma} = g. \tag{2.13}$$

By theorem A.5 in the appendix, we know that there exists a unique weak solution $w \in H^1(\Omega)$ for the Neumann boundary value problem (2.11)–(2.13) when $g \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and $\beta \in L^2(\partial\Omega\backslash\Gamma)$. In the following, we will show that $w|_\Gamma = f$.

By definition A.1 in the appendix, we know that w satisfies

$$\int_\Omega \nabla w \nabla\phi \,\mathrm{d}x\,\mathrm{d}y - k^2\int_\Omega w\phi\,\mathrm{d}x\,\mathrm{d}y = \int_\Gamma g\phi\,\mathrm{d}s + \int_{\partial\Omega\backslash\Gamma}\beta\phi\,\mathrm{d}s, \qquad \forall\phi \in H^1(\Omega). \tag{2.14}$$

For any $v \in \mathcal{H}$, we have

$$\int_\Omega \nabla v\nabla\phi\,\mathrm{d}x\,\mathrm{d}y - k^2\int_\Omega v\phi\,\mathrm{d}x\,\mathrm{d}y = \int_\Gamma \frac{\partial v}{\partial n}\phi\,\mathrm{d}s, \qquad \forall\phi\in H^1(\Omega). \tag{2.15}$$

Let $\phi = v$ in (2.14) and $\phi = w$ in (2.15), minus (2.15) by (2.14), it is easy to obtain

$$\int_{\partial\Omega\backslash\Gamma}\beta v\,\mathrm{d}s = \int_\Gamma\left(w\frac{\partial v}{\partial n} - gv\right)\mathrm{d}s. \tag{2.16}$$

Since β is a solution of the moment problem (2.10), by (2.16), we know

$$\int_\Gamma (w-f)\frac{\partial v}{\partial n}\,\mathrm{d}s = 0. \tag{2.17}$$

Now by theorem A.5 in the appendix, there exists a function $v \in H^1(\Omega)$ which satisfies (2.8) and

$$\frac{\partial v}{\partial n}\bigg|_\Gamma = w - f. \tag{2.18}$$

Thus (2.17) becomes

$$\int_\Gamma (w-f)^2\,\mathrm{d}s = 0. \tag{2.19}$$

Thus $w|_\Gamma = f$ and w is a solution of the Cauchy problem (2.1)–(2.3). The proof is completed. □

In the following, we choose $\{v_n\}_{n=1}^{\infty} \subset \mathcal{H}$, such that

$$\overline{\text{span}\{v_n|_{\partial\Omega\backslash\Gamma}\}_{n=1}^{\infty}} = L^2(\partial\Omega\backslash\Gamma).$$

Then the moment problem (2.10) becomes

$$\int_{\partial\Omega\backslash\Gamma} v_n \frac{\partial u}{\partial n}\,\mathrm{d}s = \int_\Gamma \left(f\frac{\partial v_n}{\partial n} - v_n g\right)\mathrm{d}s := \mu_n, \qquad n = 1, 2, \ldots, \tag{2.20}$$

where μ_n is determined by f, g, v_n. It is noted that there is at most one solution to the moment problem (2.10).

3. A numerical method for solving the moment problem

In this section, we choose a basis of $L^2(\partial\Omega\backslash\Gamma)$ in space $\mathcal{H}$ for a special domain and then the moment problem (2.10) will become a Hausdorff moment problem. Further, we use a numerical method to solve it. Error estimate and convergence analysis will be given in the following.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected and bounded domain and hereafter $\partial\Omega\backslash\Gamma = \{(x, y) \mid y = 0, 0 \leqslant x \leqslant 1\}$ and Γ is a smooth curve in half plane $\{(x, y) \mid y \geqslant 0\}$ which connects two points $(0, 0)$ and $(1, 0)$. Note that here the boundary $\partial\Omega\backslash\Gamma$ is supposed to be a special shape. For general cases, usually the basis functions satisfying (2.5)–(2.6) cannot be given by the analytic formulae, which will lead to a difficulty of the use of the proposed method.

Choose a basis of $L^2(\partial\Omega\backslash\Gamma)$ in space $\mathcal{H}$ as follows:

$$v_n(x, y) = \frac{1}{n^2k^2}\cos\left(\sqrt{n^2+1}ky\right)\mathrm{e}^{nkx}, \qquad n = 1, 2, \ldots. \tag{3.1}$$

It is easy to verify that v_n satisfy

$$\Delta v_n(x, y) + k^2 v_n(x, y) = 0, \qquad (x, y) \in \mathbb{R}^2, \tag{3.2}$$

$$\frac{\partial v_n(x, 0)}{\partial y} = 0, \qquad x \in \mathbb{R}. \tag{3.3}$$

Then the Cauchy problem for the Helmholtz equation can be transformed to be the following moment problem:

$$\int_0^1 \frac{1}{n^2k^2}\mathrm{e}^{nkx}\beta(x)\,\mathrm{d}x = \mu_n, \qquad n = 1, 2, \ldots, \tag{3.4}$$

where

$$\mu_n = \int_\Gamma \left(f\frac{\partial v_n}{\partial n} - g v_n\right)\mathrm{d}s. \tag{3.5}$$

Assume that $z = \frac{\mathrm{e}^{kx}-1}{\mathrm{e}^k-1}$, then the moment problem (3.4) becomes

$$\int_0^1 \frac{\mathrm{e}^k-1}{n^2k^3}(1+(\mathrm{e}^k-1)z)^{n-1}\beta\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right)\mathrm{d}z = \mu_n, \qquad n = 1, 2, \ldots, \tag{3.6}$$

furthermore, we have

$$\sum_{m=0}^{n-1}\frac{1}{n^2k^3}C_{n-1}^m(\mathrm{e}^k-1)^{(m+1)}\int_0^1 z^m\beta\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right)\mathrm{d}z = \mu_n, \qquad n = 1, 2, \ldots. \tag{3.7}$$

Remark 3.1. If $\overline{\text{span}\{v_n(x,0)\}_{n=1}^{\infty}} \neq L^2(0,1)$, then there exists a function $\beta_0(x) \in L^2(0,1)$ and $\beta_0(x) \neq 0$ satisfy

$$\int_0^1 \frac{1}{n^2k^2}\, \mathrm{e}^{nkx} \beta_0(x)\,\mathrm{d}x = 0, \qquad n = 1, 2, \ldots. \tag{3.8}$$

From (3.7), it is easy to know

$$\int_0^1 z^m \beta_0\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right)\mathrm{d}z = 0, \qquad m = 1, 2, \ldots. \tag{3.9}$$

Note that $\beta_0(x) \in L^2(0,1)$, then $\beta_0\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right) \in L^2(0,1)$, due to $\overline{\text{span}\{1, z, z^2, \ldots\}} = L^2(0,1)$, we know $\beta_0\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right) = 0$, further $\beta_0(x) = 0$, which leads to a contradiction. Thus $\overline{\text{span}\{v_n(x,0)\}_{n=1}^{\infty}} = L^2(0,1)$.

In the following, we will consider a finite moment problem for (3.6), i.e. take index n from 1 to $N+1$. Then we obtain a linear system of equations

$$Ba = \mu, \tag{3.10}$$

where B is a matrix $B = (b_{i,j})_{N+1,N+1}$ with (i, j) element

$$b_{i,j} = \begin{cases} \dfrac{C_{i-1}^{j-1}(\mathrm{e}^k-1)^j}{i^2k^3}, & i \geqslant j, \\ 0, & i < j. \end{cases} \tag{3.11}$$

and μ is a vector

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{N+1})^T;$$

a is a vector to be determined by solving (3.10)

$$a = (a_1, a_2, \ldots, a_{N+1})^T$$

with $a_j = \int_0^1 z^{j-1}\beta\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right)\mathrm{d}z$.

Denote $\rho(z) = \beta\left(\frac{\ln(1+(\mathrm{e}^k-1)z)}{k}\right)$. By solving equations (3.10), we get a finite Hausdorff moment problem as follows:

$$\int_0^1 z^{j-1}\rho(z)\,\mathrm{d}z = a_j, \qquad j = 1, 2, \ldots, N+1. \tag{3.12}$$

The numerical computation for the Hausdorff moment problem has been proposed in [1, 16, 18]. In this paper, we employ the Talenti's method [16] to solve (3.12) and the basic idea comes from paper [4].

Note that the solution of the finite Hausdorff moment problem (3.12) is not unique, so we try to find an approximate solution with minimum L^2-norm. That is, solving the following optimal problem:

$$\min_{\rho} \int_0^1 |\rho(z)|^2\,\mathrm{d}z \tag{3.13}$$

subject to the constraints

$$\int_0^1 \rho(z) z^j\,\mathrm{d}z = a_{j+1}, \qquad j = 0, 1, \ldots, N.$$

According to [4, 16], the minimizer of (3.13) can be obtained by the following steps:

Step 1. Calculate the coefficients of the shifted Legendre polynomials

$$C_{0,0} = 1, \qquad C_{j,0} = (2j+1)^{\frac{1}{2}}, \qquad C_{j,k} = -C_{j,k-1}\left(\frac{j}{k}+1\right)\left(\frac{j+1}{k}-1\right).$$

$$j = 1, 2, 3, \ldots, \qquad k = 1, 2, \ldots, j.$$

Step 2. Calculate the coefficients of the solution

$$\lambda_j = \sum_{k=0}^{j} C_{jk} a_{k+1}$$

Step 3. Calculate an approximation solution

$$\rho_N(z) = \sum_{j=0}^{N} \lambda_j L_j(z),$$

where the shifted Legendre polynomials are defined by

$$L_j(z) = \sum_{k=0}^{j} C_{jk} z^k, \qquad j = 0, 1, 2, \ldots.$$

Then $\rho_N(z)$ given in step 3 approximates $\rho(z)$.

Due to the ill-posedness of the Cauchy problem for the Helmholtz equation, we need to assume that Cauchy data f and g contain some noises. Let $f_\delta \in L^2(\Gamma)$ and $g_\delta \in L^2(\Gamma)$ be measured noisy data satisfying

$$\|f - f_\delta\|_{L^2(\Gamma)} + \|g - g_\delta\|_{L^2(\Gamma)} \leqslant \delta. \tag{3.14}$$

Moments corresponding to f_δ and g_δ in (3.5) are

$$\mu_n^\delta = \int_\Gamma \left(f_\delta \frac{\partial v_n}{\partial n} - g_\delta v_n \right) \mathrm{d}s, \qquad n = 1, 2, \ldots. \tag{3.15}$$

From (3.5), (3.14) and (3.15), by the Hölder inequality, the error between the noisy moment and the exact moment is bounded by

$$\left|\mu_n^\delta - \mu_n\right| \leqslant \left[\int_\Gamma \left(v_n^2 + \left(\frac{\partial v_n}{\partial n}\right)^2 \right) \mathrm{d}s \right]^{\frac{1}{2}} \delta.$$

By the definition of function v_n in (3.1), we have

$$|v_n(x, y)| \leqslant \frac{\mathrm{e}^{nkx}}{n^2 k^2} \leqslant \frac{(M^k)^n}{k^2},$$

where $M = \sup_{(x,y)\in\Omega} |\mathrm{e}^x| > 1$ is a constant depending on Ω. Similarly, we can obtain that $\left|\frac{\partial v_n}{\partial x}\right| \leqslant \frac{(M^k)^n}{k}$, $\left|\frac{\partial v_n}{\partial y}\right| \leqslant \frac{\sqrt{2}(M^k)^n}{k}$. Then the error bound between the noisy moment and the exact moment is

$$\sum_{n=1}^{N+1} |\mu_n^\delta - \mu_n|^2 \leqslant c(M^k)^{2N+2}\delta^2. \tag{3.16}$$

where $c > 0$ is a constant which only depends on Ω, Γ and k.

According to (3.10), we obtain

$$a^\delta = B^{-1}\mu^\delta, \tag{3.17}$$

where

$$\mu^\delta = \left(\mu_1^\delta, \ldots, \mu_{N+1}^\delta\right)^T, \qquad a^\delta = \left(a_1^\delta, \ldots, a_{N+1}^\delta\right)^T.$$

Therefore the difference between a^δ and a in 2-norm is bounded by

$$\|a^\delta - a\|_2 = \|B^{-1}(\mu^\delta - \mu)\|_2 \leqslant \|B^{-1}\|_2\|\mu^\delta - \mu\|_2 \leqslant \sqrt{c}\|B^{-1}\|_2(M^k)^{N+1}\delta. \tag{3.18}$$

In the following, we estimate $\|B^{-1}\|_2$. By (3.11), we note that matrix

$$B = \frac{1}{k^3}QDP,$$

where diagonal matrices $Q = (q_{ij})_{N+1,N+1}$ and $P = (p_{ij})_{N+1,N+1}$, matrix $D = (d_{ij})_{N+1,N+1}$ with (i, j) element respectively are

$$q_{ij} = \begin{cases} \frac{1}{i^2}, & i = j, \\ 0, & i \neq j; \end{cases} \tag{3.19}$$

$$p_{ij} = \begin{cases} (\mathrm{e}^k - 1)^i, & i = j, \\ 0, & i \neq j; \end{cases} \tag{3.20}$$

$$d_{ij} = \begin{cases} C_{i-1}^{j-1}, & i \geqslant j, \\ 0, & i < j. \end{cases} \tag{3.21}$$

The inverse matrix of B is then

$$B^{-1} = k^3 P^{-1} D^{-1} Q^{-1}. \tag{3.22}$$

It is not hard to obtain that the 2-norm of matrix P^{-1} and Q^{-1} are as follows:

$$\|P^{-1}\|_2 = \begin{cases} (\mathrm{e}^k - 1)^{-(N+1)}, & 0 < k < \ln 2, \\ (\mathrm{e}^k - 1)^{-1}, & k \geqslant \ln 2, \end{cases} \tag{3.23}$$

$$\|Q^{-1}\|_2 = (N+1)^2. \tag{3.24}$$

In the following we estimate $\|D^{-1}\|_2$. Consider the linear system of equations

$$D\alpha = \gamma, \tag{3.25}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_{N+1})^T, \qquad \gamma = (\gamma_1, \ldots, \gamma_{N+1})^T$$

and

$$D = \begin{pmatrix} C_0^0 & 0 & 0 & \cdots & 0 \\ C_1^0 & C_1^1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_N^0 & C_N^1 & C_N^2 & \cdots & C_N^N \end{pmatrix}.$$

Since $C_N^0 + C_N^1 + \cdots + C_N^{N-1} + C_N^N = 2^N$, the maximum element d in matrix D is bounded by

$$1 \leqslant d \leqslant 2^N. \tag{3.26}$$

According to (3.25), it is easy to see

$$\alpha_{i+1} = \gamma_{i+1} - C_i^0\alpha_1 - C_i^1\alpha_2 - \cdots - C_i^{i-1}\alpha_i, \qquad i = 0, 1, \ldots, N. \tag{3.27}$$

Note that for every γ_i, $|\gamma_i| \leqslant \|\gamma\|_2$, $i = 1, 2, \ldots, N+1$. For $i = 0$, $\alpha_1 = \gamma_1$, thus we have $|\alpha_1| \leqslant \|\gamma\|_2$. Suppose that the inequality

$$|\alpha_i| \leqslant (1+d)^{i-1}\|\gamma\|_2 \tag{3.28}$$

is satisfied, then we can prove

$$|\alpha_{i+1}| \leqslant \|\gamma\|_2 + d(1+d)^0\|\gamma\|_2 + \cdots + d(1+d)^{i-1}\|\gamma\|_2 = (1+d)^i\|\gamma\|_2.$$

Therefore, by the induction, the estimate 3.28 is satisfied for all $i = 1, 2, \ldots, N+1$.

From (3.25), we know

$$\alpha = D^{-1}\gamma. \tag{3.29}$$

According to (3.28), (3.29), we can obtain

$$\|\alpha\|_2^2 = \sum_{i=1}^{N+1}\alpha_i^2 \leqslant \left(\sum_{i=1}^{N+1}(1+d)^{2(i-1)}\right)\|\gamma\|_2^2 \leqslant 2^{2(N+1)}d^{2N}\|\gamma\|_2^2, \tag{3.30}$$

thus

$$\|D^{-1}\|_2 \leqslant 2^{N+1}d^N. \tag{3.31}$$

Further, consider (3.26), it can be obtained

$$\|D^{-1}\|_2 \leqslant 2^{N^2+N+1}. \tag{3.32}$$

Therefore, by (3.22)–(3.24) and (3.32), we have

$$\|B^{-1}\|_2 = \|k^3P^{-1}D^{-1}Q^{-1}\|_2 \leqslant k^3\|P^{-1}\|_2\|D^{-1}\|_2\|Q^{-1}\|_2 \tag{3.33}$$

$$\leqslant \begin{cases} \dfrac{k^3(N+1)^2 2^{N^2+N+1}}{(\mathrm{e}^k-1)^{N+1}}, & 0 < k < \ln 2, \\ \dfrac{k^3(N+1)^2 2^{N^2+N+1}}{(\mathrm{e}^k-1)}, & k \geqslant \ln 2. \end{cases} \tag{3.34}$$

Further,

$$\|B^{-1}\|_2 \leqslant \begin{cases} \dfrac{k^3 2^{N^2+N+1}\,\mathrm{e}^{N+1}}{(\mathrm{e}^k-1)^{N+1}}, & 0 < k < \ln 2, \\ k^3 2^{N^2+N+1}\,\mathrm{e}^{N+1}, & k \geqslant \ln 2. \end{cases} \tag{3.35}$$

In the following, the right terms in (3.35) are denoted by F_N, i.e.,

$$F_N = \begin{cases} \dfrac{k^3 2^{N^2+N+1}\,\mathrm{e}^{N+1}}{(\mathrm{e}^k-1)^{N+1}}, & 0 < k < \ln 2, \\ k^3 2^{N^2+N+1}\,\mathrm{e}^{N+1}, & k \geqslant \ln 2. \end{cases} \tag{3.36}$$

By (3.18), we have

$$\|a_\delta - a\|_2 \leqslant K_N\delta, \tag{3.37}$$

where

$$K_N = \sqrt{c}F_N(M^k)^{N+1}. \tag{3.38}$$

For noisy Cauchy data, the coefficients in step 2 are

$$\lambda_j^\delta = \sum_{k=0}^{j} C_{jk}a_{k+1}^\delta. \tag{3.39}$$

The approximate solution in step 3 with the noisy Cauchy data will be

$$\rho_N^\delta(z) = \sum_{j=0}^{N} \lambda_j^\delta L_j(z). \tag{3.40}$$

Denote $\beta_N^\delta(x) = \rho_N^\delta\left(\frac{e^{kx}-1}{e^k-1}\right)$. We can obtain the following error estimate:

Theorem 3.2. *Suppose that u is a solution of the Cauchy problem (2.1)–(2.3) satisfying*

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega\backslash\Gamma} \in C^1(\partial\Omega\backslash\Gamma), \tag{3.41}$$

then

$$\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_N^\delta(x)\right|^2 \mathrm{d}x \leqslant K_N^2\delta^2\,\mathrm{e}^{3.5(N+1)} + \frac{1}{4}(N+1)^{-2}E^2, \tag{3.42}$$

where

$$\int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial u}{\partial n}(x,0)\right)\right|^2 \mathrm{d}x \leqslant E^2 \tag{3.43}$$

and K_N is given by (3.38).

Proof. By proposition 2.1, we know that the Cauchy problem (2.1)–(2.3) is equivalent to the moment problem (3.4). By [4, theorem 5] and (3.37), we have

$$\begin{aligned}\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_N^\delta(x)\right|^2 \mathrm{d}x &\leqslant \mathrm{e}^{3.5(N+1)}\|a^\delta - a\|_2^2 + \frac{1}{4}(N+1)^{-2}E^2\\ &\leqslant K_N^2\delta^2\,\mathrm{e}^{3.5(N+1)} + \frac{1}{4}(N+1)^{-2}E^2.\end{aligned}$$

The proof is completed. □

In the following theorem, we give an *a priori* choice of *N* such that the convergence is satisfied.

Theorem 3.3. *Let*

$$N(\delta) = \begin{cases} \left[\left(\dfrac{\ln\frac{1}{\delta}}{2k\ln M + \frac{11}{2} + 4\ln 2 - 2\ln(\mathrm{e}^k - 1)}\right)^{\frac{1}{2}}\right], & 0 < k < \ln 2,\\[2ex] \left[\left(\dfrac{\ln\frac{1}{\delta}}{2k\ln M + \frac{11}{2} + 4\ln 2}\right)^{\frac{1}{2}}\right], & k \geqslant \ln 2,\end{cases}$$

then there exist constants $C_1 > 0$ and $C_2 > 0$ which depend on E, k, Γ and Ω such that

$$\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_{N(\delta)}^\delta(x)\right|^2 \mathrm{d}x \leqslant C_1\delta + \frac{C_2}{|\ln\delta|},$$

where [·] denotes the nearest integer towards minus infinity of a real number.

Proof. By theorem 3.2, for $0 < k < \ln 2$, we have

$$\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_N^\delta(x)\right|^2 \mathrm{d}x \leqslant K_N^2\delta^2\,\mathrm{e}^{3.5(N+1)} + \frac{1}{4}(N+1)^{-2}E^2$$
$$= \frac{ck^6 2^{2(N^2+N+1)}\,\mathrm{e}^{2(N+1)}(M^k)^{2(N+1)}\,\mathrm{e}^{3.5(N+1)}\delta^2}{(\mathrm{e}^k-1)^{2(N+1)}} + \frac{1}{4}(N+1)^{-2}E^2$$
$$= \frac{c_1 2^{2(N^2+N)}\,\mathrm{e}^{2N}(M^k)^{2N}\,\mathrm{e}^{3.5N}\delta^2}{(\mathrm{e}^k-1)^{2N}} + \frac{1}{4}(N+1)^{-2}E^2$$
$$\leqslant \frac{c_1 2^{4N^2}\,\mathrm{e}^{2N^2}(M^k)^{2N^2}\,\mathrm{e}^{3.5N^2}\delta^2}{(\mathrm{e}^k-1)^{2N^2}} + \frac{1}{4}N^{-2}E^2,$$

where $c_1 = 4ck^6\,\mathrm{e}^{5.5}M^{2k}(\mathrm{e}^k-1)^{-2}$.

Let

$$\frac{2^{4N^2}\,\mathrm{e}^{2N^2}(M^k)^{2N^2}\,\mathrm{e}^{3.5N^2}}{(\mathrm{e}^k-1)^{2N^2}} = \frac{1}{\delta},$$

then we can choose

$$N = N(\delta) = \left[\left(\frac{\ln\frac{1}{\delta}}{2k\ln M + \frac{11}{2} + 4\ln 2 - 2\ln(\mathrm{e}^k-1)}\right)^{\frac{1}{2}}\right].$$

When $k \geqslant \ln 2$, we have

$$\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_N^\delta(x)\right|^2 \mathrm{d}x \leqslant K_N^2\delta^2\,\mathrm{e}^{3.5(N+1)} + \frac{1}{4}(N+1)^{-2}E^2$$
$$= ck^6 2^{2(N^2+N+1)}\,\mathrm{e}^{2(N+1)}(M^k)^{2(N+1)}\,\mathrm{e}^{3.5(N+1)}\delta^2 + \frac{1}{4}(N+1)^{-2}E^2$$
$$= c_2 2^{2(N^2+N)}\,\mathrm{e}^{2N}(M^k)^{2N}\,\mathrm{e}^{3.5N}\delta^2 + \frac{1}{4}(N+1)^{-2}E^2$$
$$\leqslant c_2 2^{4N^2}\,\mathrm{e}^{2N^2}(M^k)^{2N^2}\,\mathrm{e}^{3.5N^2}\delta^2 + \frac{1}{4}N^{-2}E^2,$$

where $c_2 = 4ck^6\,\mathrm{e}^{5.5}M^{2k}$.

Let

$$2^{4N^2}\,\mathrm{e}^{2N^2}(M^k)^{2N^2}\,\mathrm{e}^{3.5N^2} = \frac{1}{\delta},$$

then we can choose

$$N = N(\delta) = \left[\left(\frac{\ln\frac{1}{\delta}}{2k\ln M + \frac{11}{2} + 4\ln 2}\right)^{\frac{1}{2}}\right].$$

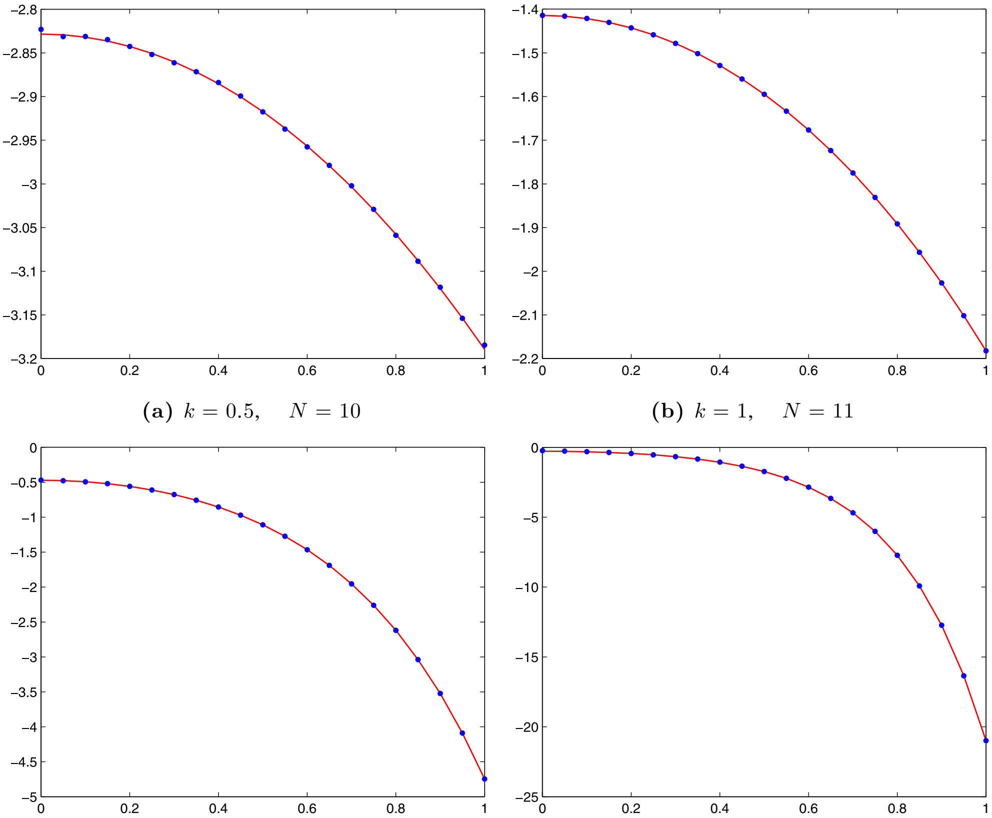
For the special chosen N, we have the following convergence result:

$$\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_{N(\delta)}^\delta(x)\right|^2 \mathrm{d}x \leqslant C_1\delta + \frac{C_2}{|\ln\delta|},$$

where $C_1 > 0$ and $C_2 > 0$ which depend on E, k, Γ and Ω. □

Consider the following Neumann boundary value problem:

$$\Delta u_N^\delta + k^2 u_N^\delta = 0, \qquad in \quad \Omega, \tag{3.44}$$



(a) $k = 0.5, \quad N = 10$ **(b)** $k = 1, \quad N = 11$

(c) $k = 3, \quad N = 13$ **(d)** $k = 5, \quad N = 10$

Figure 1. The exact β (solid lines) and its approximation β_N^δ (dotted lines) by using the exact Cauchy data.

$$\left.\frac{\partial u_N^\delta}{\partial n}\right|_\Gamma = g_\delta, \tag{3.45}$$

$$\left.\frac{\partial u_N^\delta}{\partial n}\right|_{\partial\Omega\setminus\Gamma} = \beta_N^\delta, \tag{3.46}$$

where we assume that $g_\delta \in L^2(\Gamma)$.

Suppose that u is a solution of the Cauchy problem (2.1)–(2.3), by theorem A.7 in the appendix, the following error estimate is satisfied:

$$\|u_N^\delta - u\|_{L^2(\Omega)}^2 \leqslant C\left(\int_0^1 \left|\frac{\partial u}{\partial n}(x,0) - \beta_N^\delta(x)\right|^2 \mathrm{d}x + \delta^2\right),$$

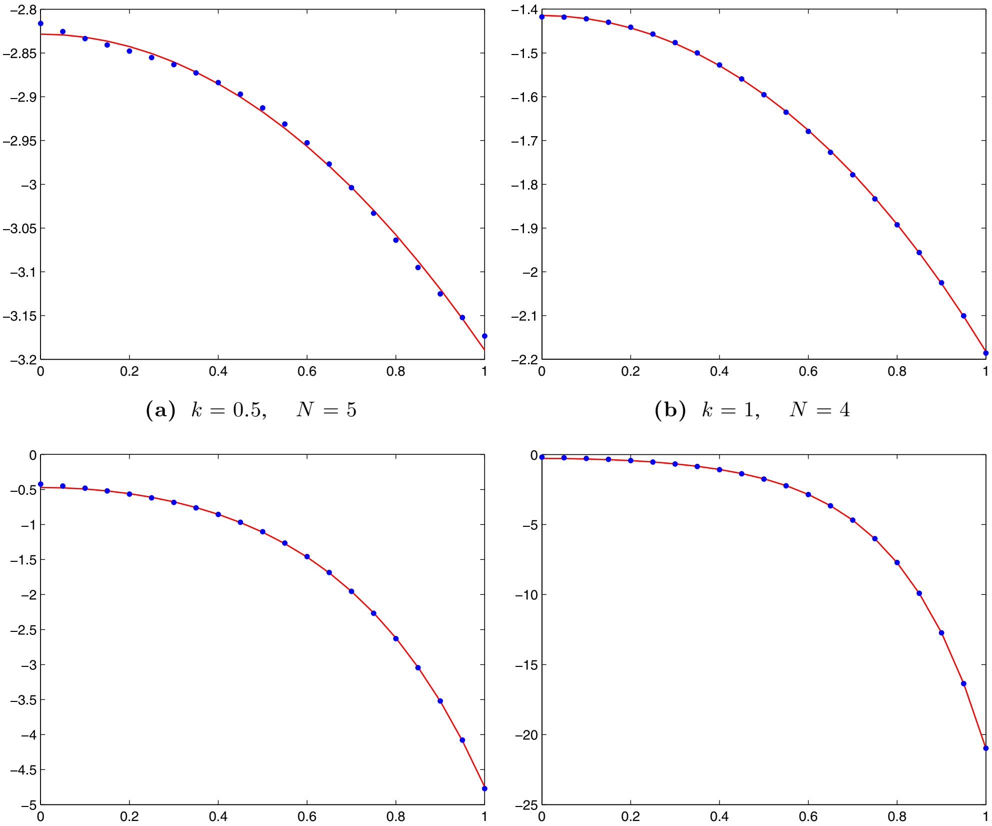
where $C > 0$ is a constant depending on Ω, Γ and k.

Therefore, we have the following main result in this paper.

Theorem 3.4. *Under the assumptions given in theorem 3.2, we have the following error estimate:*

$$\left\|u_N^\delta - u\right\|_{L^2(\Omega)}^2 \leqslant C\left\{\mathrm{e}^{3.5(N+1)}K_N^2\delta^2 + \tfrac{1}{4}(N+1)^{-2}E^2 + \delta^2\right\}$$

where K_N is given by (3.38), constant $C > 0$ depends on Ω, Γ and k.



(a) $k = 0.5, \quad N = 5$ **(b)** $k = 1, \quad N = 4$

(c) $k = 3, \quad N = 4$ **(d)** $k = 5, \quad N = 4$

Figure 2. The exact β (solid lines) and its approximation β_N^δ (dotted lines) by using the noisy data $\varepsilon = 0.0001$.

Theorem 3.5. *If we take*

$$N(\delta) = \begin{cases} \left[\left(\dfrac{\ln \frac{1}{\delta}}{2k \ln M + \frac{11}{2} + 4 \ln 2 - 2 \ln(\mathrm{e}^k - 1)} \right)^{\frac{1}{2}} \right], & 0 < k < \ln 2, \\ \left[\left(\dfrac{\ln \frac{1}{\delta}}{2k \ln M + \frac{11}{2} + 4 \ln 2} \right)^{\frac{1}{2}} \right], & k \geqslant \ln 2, \end{cases}$$

then we have the following convergence estimate:

$$\left\| u_{N(\delta)}^\delta - u \right\|_{L^2(\Omega)}^2 \leqslant C\delta^2 + C_3\delta + \frac{C_4}{|\ln \delta|},$$

where constant $C_3 = CC_1$, $C_4 = CC_2$ *depend on* E, Ω, Γ *and* k.

4. Numerical examples

Let $\Omega = \{(x, y) | 0 < x < 1, 0 < y < 1\}$ and $\partial\Omega \backslash \Gamma = \{(x, y) | y = 0, 0 \leqslant x \leqslant 1\}$.

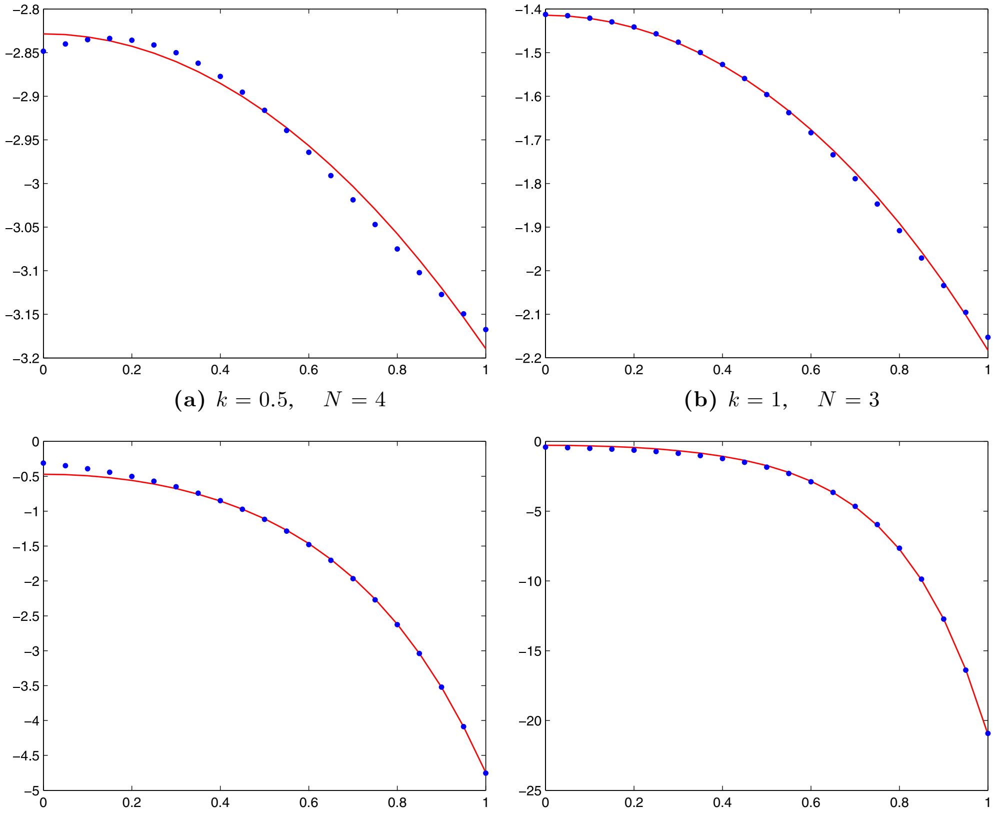


Figure 3. The exact β (solid lines) and its approximation β_N^δ (dotted lines) by using the noisy data $\varepsilon = 0.001$.

We choose $u(x, y) = \frac{1}{2k^2}\sin(\sqrt{2}ky)(e^{kx} + e^{-kx})$ as the exact solutions of (2.1)–(2.3) for various wave number $k = 0.5, 1, 3, 5$. The numerical results for the approximate solution $\beta_N^\delta(x)$ and the exact solution $\frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma}(x)$ are presented in figures 1–3 in which the solid line represents the exact solution and the dotted line is its approximation. There is no noise to f and g in figure 1. The same examples with noisy data $f^\delta = f + \varepsilon\, e^x \sin y$ and $g^\delta = g + \varepsilon\, e^x \cos y$ are given in figures 2–3 with $\varepsilon = 0.0001$ and $\varepsilon = 0.001$ respectively. It is observed that our proposed algorithm is effective and stable to the noises.

5. Conclusions

In this paper, we proposed a numerical method for solving the Cauchy problem for the Helmholtz equation. The error estimate and convergence analysis have been presented. The numerical examples demonstrate that our proposed method is accurate and effective.

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Appendix A

Let Ω be a simply connected and bounded open set in $\mathbb{R}^2$ with a sufficiently regular boundary $\partial\Omega$. In this appendix, we always denote $u = u(x)$, $x = (x_1, x_2) \in \mathbb{R}^2$ and $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2$.

Definition A.1. *Suppose $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, the weak solution of the Neumann boundary value problem*

$$-\Delta u + cu = f, \qquad \text{in} \quad \Omega, \tag{A.1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g \tag{A.2}$$

is defined as a solution of the following variational problem:

$$u \in H^1(\Omega), \qquad \int_\Omega \nabla u \nabla v\,\mathrm{d}x + \int_\Omega cuv\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x + \int_{\partial\Omega} gv\,\mathrm{d}s, \qquad \forall\, v \in H^1(\Omega), \tag{A.3}$$

where c is a real number.

Proposition A.2. *The variational problem (A.3) with $c > 0$ has a unique solution in $H^1(\Omega)$.*

Proof. Define $a(u, v) = \int_\Omega(\nabla u\nabla v + cuv)\,\mathrm{d}x$, $\ell(v) = \int_\Omega fv\,\mathrm{d}x + \int_{\partial\Omega} gv\,\mathrm{d}s$. Then (A.3) becomes $a(u, v) = \ell(v), \forall\, v \in H^1(\Omega)$. By the Lax–Milgram theorem from chapter VII, section 1 of book [5], the variational problem (A.3) has a unique solution $u \in H^1(\Omega)$. □

Define

$$M = \left\{u \in H^1(\Omega);\, -\Delta u \in L^2(\Omega) \text{ and } \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\right\}.$$

Proposition A.3. *For $g = 0$, $c = 1$ and any $f \in L^2(\Omega)$, the Neumann boundary value problem (A.1)–(A.2) admits a unique weak solution $u \in M$. Further, we have*

$$\|u\|_{H^1(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}, \tag{A.4}$$

where $C > 0$ is a constant.

Proof. See page 96 in chapter VIII of book [6] and pages 69–78 in chapter IV of book [12]. □

Furthermore, we have the following proposition.

Proposition A.4. *The boundary value problem*

$$-\Delta u - k^2 u = f, \qquad in \quad \Omega, \tag{A.5}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \tag{A.6}$$

has a unique weak solution $u \in M$ for each $f \in L^2(\Omega)$ if and only if $-k^2$ is not the eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition.

Proof. From proposition A.3, we know that $L := (-\Delta + I)^{-1} : L^2(\Omega) \mapsto M \subset H^1(\Omega)$ is a bounded linear operator. Note that $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compactly embedded. Thus, L is a linear compact operator from $L^2(\Omega) \to L^2(\Omega)$.

Note that if $\varphi = -\Delta u + u$, $\psi = -\Delta v + v$, with $\forall u, v \in M$, we have

$$\begin{aligned}(L\varphi, \psi)_{L^2(\Omega)} &= (L(-\Delta + I)u, -\Delta v + v)_{L^2(\Omega)} \\ &= (u, -\Delta v + v)_{L^2(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)} \\ &= (-\Delta u + u, v)_{L^2(\Omega)} = (\varphi, L\psi)_{L^2(\Omega)}.\end{aligned}$$

Thus, $L = L^*$, i.e. L is self-adjoint.

The boundary value problem (A.5)–(A.6) is equivalent to

$$-\Delta u + u = (k^2 + 1)u + f, \qquad \text{in} \quad \Omega, \tag{A.7}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.8}$$

Thus, we can rewrite (A.5)–(A.6) as

$$u - (k^2 + 1)Lu = Lf. \tag{A.9}$$

According to the Fredholm alternative theorem from chapter VIII, section 2 of book [6], the boundary value problem (A.9) exists a solution in $L^2(\Omega)$ for every $f \in L^2(\Omega)$ if its homogeneous problem $v - (k^2+1)Lv = 0$ has a unique solution $v = 0$. Further, there exists a set of real numbers $\Lambda = \{k_1, k_2, \ldots\}$ where $\frac{1}{k_j^2+1}$ are the eigenvalues of problem $\lambda v - Lv = 0$. For $k \notin \Lambda$, the problem $v - (k^2 + 1)Lv = 0$ has a unique solution $v = 0$. Thus, the problem (A.9) has a unique solution in $L^2(\Omega)$ if $k \notin \Lambda$.

Let $u_j \in L^2(\Omega)$ be the eigenfunction of problem $\lambda v - Lv = 0$ corresponding to eigenvalue $\frac{1}{k_j^2+1}$, i.e.,

$$u_j - \left(k_j^2 + 1\right)Lu_j = 0.$$

Note that $Lu_j \in M$, so we know $u_j \in M$. Further, we have $(-\Delta u_j + u_j) - (k_j^2 + 1)u_j = 0$, i.e., $\Delta u_j = -k_j^2 u_j$. Therefore, $-k_j^2$ is the eigenvalue of the Laplace operator with the homogeneous Neumann boundary condition. Thus, the proof is completed. □

Theorem A.5. *The boundary value problem*

$$-\Delta u - k^2 u = f, \qquad in \quad \Omega, \tag{A.10}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g \tag{A.11}$$

admits a unique weak solution in $H^1(\Omega)$ provided that $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $-k^2$ is not the eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition.

Proof. For $g \in L^2(\partial\Omega)$, by proposition A.2, there exists a unique weak solution $w \in H^1(\Omega)$ for the following Neumann boundary value problem:

$$-\Delta w + w = 0, \qquad \text{in} \quad \Omega, \tag{A.12}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = g, \tag{A.13}$$

i.e. w satisfies the following variational problem:

$$\int_\Omega \nabla w \nabla \phi \, \mathrm{d}x + \int_\Omega w\phi \, \mathrm{d}x = \int_{\partial\Omega} g\phi \, \mathrm{d}s, \qquad \forall \phi \in H^1(\Omega). \tag{A.14}$$

The variational formulation of the problem (A.10)–(A.11) is

$$\int_\Omega \nabla u \nabla \phi \, \mathrm{d}x - \int_\Omega k^2 u \phi \, \mathrm{d}x = \int_\Omega f \phi \, \mathrm{d}x + \int_{\partial\Omega} g \phi \, \mathrm{d}s, \qquad \forall \phi \in H^1(\Omega). \tag{A.15}$$

Let $v = w - u$, from (A.14) and (A.15), we have

$$\int_\Omega \nabla v \nabla \phi \, \mathrm{d}x - \int_\Omega k^2 v \phi \, \mathrm{d}x = -\int_\Omega f \phi \, \mathrm{d}x - \int_\Omega (k^2 + 1) w \phi \, \mathrm{d}x, \qquad \forall \phi \in H^1(\Omega). \tag{A.16}$$

Note that (A.16) is the variational formulation of the following Neumann boundary value problem:

$$-\Delta v - k^2 v = -(1 + k^2) w - f, \qquad \text{in} \quad \Omega, \tag{A.17}$$

$$\left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.18}$$

Then, by proposition A.4, the problem (A.17)–(A.18) has a unique solution $v \in M$ provided that $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition. Hence, $u = w - v \in H^1(\Omega)$ is the unique solution of problem (A.10)–(A.11) if $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition. □

Lemma A.6. *If $-k^2$ is not an eigenvalue of Laplacian operator with the homogeneous Neumann boundary condition and let $u \in M$ be the unique weak solution of problem*

$$-\Delta u - k^2 u = g, \qquad \text{in} \quad \Omega, \tag{A.19}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \tag{A.20}$$

where $g \in L^2(\Omega)$, then there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C\|g\|_{L^2(\Omega)}.$$

Proof. If the statement is not true, there exist sequences $\{g_j\}_{j=1}^\infty \subset L^2(\Omega)$ and $\{u_j\}_{j=1}^\infty \subset M$ are the weak solutions of problems

$$-\Delta u_j - k^2 u_j = g_j, \qquad \text{in} \quad \Omega, \tag{A.21}$$

$$\left.\frac{\partial u_j}{\partial n}\right|_{\partial\Omega} = 0, \tag{A.22}$$

with $\|u_j\|_{L^2(\Omega)} = 1$ and

$$\|u_j\|_{L^2(\Omega)} \geqslant j\|g_j\|_{L^2(\Omega)}, \qquad j = 1, 2, \ldots.$$

Then $g_j \to 0$ in $L^2(\Omega)$ when $j \to \infty$. The problem (A.21)–(A.22) has the following variational formulation:

$$\int_\Omega (\nabla u_j \nabla v + u_j v) \, \mathrm{d}x = \int_\Omega ((k^2 + 1) u_j v + g_j v) \, \mathrm{d}x, \qquad \text{for} \quad \forall v \in H^1(\Omega). \tag{A.23}$$

Choose $v = u_j$, then

$$\|u_j\|_{H^1(\Omega)}^2 \leqslant (k^2 + 1)\|u_j\|_{L^2(\Omega)}^2 + \|u_j\|_{L^2(\Omega)}\|g_j\|_{L^2(\Omega)} = (k^2 + 1) + \|g_j\|_{L^2(\Omega)},$$

i.e. $\{u_j\}_{j=1}^{\infty}$ is bounded in $H^1(\Omega)$. Then, there exists a subsequence $\{u_{j_m}\}_{m=1}^{\infty} \subset \{u_j\}_{j=1}^{\infty}$ such that $u_{j_m} \rightharpoonup u_0$ weakly in $H^1(\Omega)$, hence $u_{j_m} \to u_0$ in $L^2(\Omega)$.

Let $m \to \infty$, then from (A.23), we obtain

$$\int_\Omega (\nabla u_0 \nabla v + u_0 v)\, dx = \int_\Omega (k^2+1) u_0 v\, dx, \qquad \text{for} \quad \forall v \in H^1(\Omega).$$

Therefore, u_0 is a weak solution of the following Neumann boundary value problem:

$$-\Delta u_0 = k^2 u_0, \qquad \text{in} \quad \Omega, \tag{A.24}$$

$$\left.\frac{\partial u_0}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.25}$$

Since $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition, so we have $u_0 \equiv 0$. This leads to a contraction with $\|u_0\|_{L^2(\Omega)} = 1$. □

Theorem A.7. *If $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition and let $u \in H^1(\Omega)$ be the unique weak solution of problem*

$$-\Delta u - k^2 u = 0, \qquad \text{in} \quad \Omega, \tag{A.26}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g, \tag{A.27}$$

where $g \in L^2(\partial\Omega)$, then there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C \|g\|_{L^2(\partial\Omega)}.$$

Proof. Consider the following Neumann boundary value problem:

$$-\Delta w + w = 0, \qquad \text{in} \quad \Omega, \tag{A.28}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = g. \tag{A.29}$$

By proposition A.2, there exists a unique weak solution $w \in H^1(\Omega)$ such that

$$\int_\Omega \nabla w \nabla v\, dx + \int_\Omega w v\, dx = \int_{\partial\Omega} g v\, ds, \qquad \forall v \in H^1(\Omega). \tag{A.30}$$

Choose $v = w$, we have

$$\|w\|_{H^1(\Omega)}^2 \leqslant \|g\|_{L^2(\partial\Omega)} \|w\|_{L^2(\partial\Omega)} \leqslant C_1 \|g\|_{L^2(\partial\Omega)} \|w\|_{H^1(\Omega)}. \tag{A.31}$$

Consequently,

$$\|w\|_{H^1(\Omega)} \leqslant C_1 \|g\|_{L^2(\partial\Omega)}. \tag{A.32}$$

Note that the problem (A.26)–(A.27) has the following variational formulation:

$$u \in H^1(\Omega), \qquad \int_\Omega \nabla u \nabla v\, dx - \int_\Omega k^2 u v\, dx = \int_{\partial\Omega} g v\, ds, \qquad \forall v \in H^1(\Omega). \tag{A.33}$$

Let $\tilde{u} = u - w$, by (A.30) and (A.33), we have

$$\int_\Omega \nabla \tilde{u} \nabla v\, dx - \int_\Omega k^2 \tilde{u} v\, dx = \int_\Omega (k^2+1) w v\, dx, \qquad \forall v \in H^1(\Omega), \tag{A.34}$$

which is the variational formulation of the following Neumann boundary value problem:

$$-\Delta \tilde{u} - k^2 \tilde{u} = (1+k^2) w, \qquad \text{in} \quad \Omega, \tag{A.35}$$

$$\left.\frac{\partial \tilde{u}}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.36}$$

Then, by proposition A.4, there exists a unique solution $\tilde{u} \in H^1(\Omega)$ for the problem (A.35)–(A.36) provided that $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition. By lemma A.6, we have

$$\|\tilde{u}\|_{L^2(\Omega)} \leqslant C_2\|w + k^2 w\|_{L^2(\Omega)} \leqslant C_3\|w\|_{L^2(\Omega)},$$

where $C_3 > 0$ is a constant which depends on Ω, $\partial\Omega$ and k.

Since $\tilde{u} = u - w$, then

$$\|u\|_{L^2(\Omega)} \leqslant \|w\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Omega)} \leqslant C_4\|w\|_{L^2(\Omega)},$$

combining (A.32), we have $\|u\|_{L^2(\Omega)} \leqslant C\|g\|_{L^2(\partial\Omega)}$, where constant $C > 0$ depends on Ω, $\partial\Omega$ and k. □

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