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Mixed boundary integral methods for Helmholtz transmission problems

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Abstract

In this paper we propose a hybrid between direct and indirect boundary integral methods to solve a transmission problem for the Helmholtz equation in Lipschitz and smooth domains. We present an exhaustive abstract study of the numerical approximation of the resulting system of boundary integral equations by means of Galerkin methods. Some particular examples of convergent schemes in the smooth case in two dimensions are given. Finally, we extend the results to a thermal scattering problem in a half plane with several obstacles and provide numerical results that illustrate the accuracy of our methods depending on the regularity of the interface.

MSC: 65R20; 65N38

Keywords: Galerkin methods; Boundary integral equations; Helmholtz transmission problems

1. Introduction

This work is devoted to the analytical and numerical study of transmission problems for the Helmholtz equation in two or three dimensions. The problem consists on a pair of Helmholtz equations with different wave numbers, one on a bounded domain and the other on its complement. The equations are coupled through continuity conditions of the unknown and the flux on the common interface. This kind of problems arises in the study of the scattering of acoustic or thermal waves by an obstacle and also in the study of electromagnetic waves.

Formulations based on boundary integral methods are powerful tools to deal with transmission problems in unbounded media. A very complete description of the use of single and double layer potentials for solving direct and related inverse problems for the Helmholtz equation can be found in [8,9]. Different formulations using boundary integral equations also appear in [11,19,21,27,37,38]. We propose here a mixed indirect–direct integral formulation, representing the solution as a single layer potential in the unbounded domain and recovering it from the representation formula in the bounded one, that is, our unknowns will be an exterior density and the interior Cauchy data. This formulation has the advantage of providing a simple far-field representation and a good near-field approximation. A good example of the application of this formulation for the solution of an inverse problem can be found in [15]. The numerical method

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proposed there for solving the inverse problem involves the iterative solution to direct problems computing both the normal derivative on the boundary of the obstacle and the solution of the problem on a measurement boundary located far from the obstacle.

This approach can fail when either the interior or the exterior parameter in the Helmholtz equation is a Dirichlet eigenvalue for the Laplace operator in the interior domain. It is anyway valid for a wide class of problems, for instance when adsorption is present. This situation can be overcome by using Brakhage–Werner potentials with the drawback of the occurrence of hypersingular operators. This was done in [28] (see also [29]) only for an indirect formulation both in the interior and the exterior domains. We will discuss on this at the end of Section 3.

Our formulation can be seen as a particular case of one proposed in [19] but the proofs in this work are more involved and neither the non-smooth case nor the numerical approach are considered. There is not much previous work concerning numerical aspects related with Helmholtz transmission problems and their related boundary integral formulations. Existing literature is mainly focused on questions involving conditions guaranteeing uniqueness of the original problem and unique solvability of the resulting system of boundary integral equations. To our best knowledge, only in [27], where an indirect formulation of a similar problem is proposed, a complete analysis of convergence for Petrov–Galerkin methods is performed. We present here, from an abstract point of view, an exhaustive analysis of Galerkin discretizations for solving the equivalent system of integral equations in the mixed formulation considering all the cases: two and three dimensions, smooth and Lipschitz interfaces. We give necessary and sufficient conditions for convergence of such methods in the natural norms for Lipschitz and Lyapunov interfaces and in the full range of the Sobolev spaces $H^s(\Gamma)$ for smooth boundaries. For the Lyapunov and smooth cases we will arrive at conditions that have to do with the ones appearing in [27] but they vary for Lipschitz regularity. In this case, they do not depend on the parameters of the problem as in [27] but, on the other hand, they are still not easy to check theoretically for concrete discrete spaces.

For the sake of clarity, we have centered our attention on problems with only one obstacle in the whole plane or space, but the results can be easily generalized to problems involving many interfaces. We give some hints on how to apply them for a concrete problem arising in photothermal science which not only involves a finite number of inclusions contained in a half-plane, but also a homogeneous Neumann boundary condition on the interface separating the medium and the exterior.

This paper is organized as follows: we introduce the problem in Section 2 and give its mixed formulation in Section 3. Under some hypotheses on the parameters of the problem to avoid the characteristic frequencies of the obstacle, we show invertibility of the system of integral equations. Proofs are given for general Lipschitz interfaces, indicating how they can be simplified in the more restrictive situation of Lyapunov regularity. Section 4 is devoted to the numerical approximation of the resulting system of boundary integral equations by means of Galerkin methods. The analysis requires taking into account some uniform Babuška–Brezzi estimates and the different order of the operators. We deal here with mixed and generalized mixed operators. For an easy reference, an appendix gathers some general results concerning Petrov–Galerkin schemes for solving operator equations with an emphasis on operators with mixed and generalized mixed structures. In the two-dimensional case and assuming that the boundary of the obstacle is a parametrizable smooth curve, some convergent methods can be accomplished with periodic smoothest splines and trigonometric polynomials. This is shown in Section 5, where we also give some ideas for the three-dimensional case. The question on whether simple spaces satisfy the conditions guaranteeing convergence when the interface is only Lipschitz continuous is left open. In the last section we present some numerical results discussing the suitability of the spectral approximation for smooth but not $\mathscr{C}^\infty$-curves, showing that simpler finite element type spaces are to be preferred. We also illustrate how one can make use of the advantage of knowing information of the original unknown near the obstacle. We end with a numerical illustration for a Helmholtz transmission problem related with the scattering of thermal waves in a composite material occupying the half plane.

Notation: Throughout this work C (also $C', C'', \ldots$) will denote a general positive constant independent of the discretization parameter (n or h) and of any quantity that is multiplied by it, being possibly different in each occurrence.

We will use the Sobolev spaces $H^m(\Omega)$ with $m \geqslant 0$, where Ω is a general bounded or unbounded domain and the Sobolev spaces $H^s(\Gamma)$ for smooth and Lipschitz closed curves and surfaces. For a very detailed explanation of these spaces in the same setting we will be using them we refer to [23]. In the final sections we also use the 1-periodic Sobolev spaces for which we refer to [20,30]. We will use the notation $(\cdot, \cdot)$ for sesquilinear forms (linear in the left component and conjugate linear in the right one), both for inner products and antiduality products.

2. Formulation of the problem

Let Ω_{int} denote a bounded simply connected open set in $\mathbb{R}^d$, $d=2$ or 3, and $\Omega_{\text{ext}} := \mathbb{R}^d \backslash \overline{\Omega}_{\text{int}}$. The common interface Γ is assumed to be connected and Lipschitz. We are interested in solving the following Helmholtz transmission problem:

$$\Delta u + \lambda^2 u = 0 \quad \text{in } \Omega_{\text{ext}}, \tag{1a}$$

$$\Delta u + \mu^2 u = 0 \quad \text{in } \Omega_{\text{int}}, \tag{1b}$$

$$u|_{\Gamma}^{\text{int}} - u|_{\Gamma}^{\text{ext}} = g_0, \tag{1c}$$

$$\alpha \partial_{\nu} u|_{\Gamma}^{\text{int}} - \beta \partial_{\nu} u|_{\Gamma}^{\text{ext}} = g_1, \tag{1d}$$

$$\lim_{r\to\infty} r^{(d-1)/2}(\partial_r u - \imath\lambda u) = 0, \tag{1e}$$

where $\alpha \neq -\beta$ are non-zero given parameters and $-\lambda^2$, $-\mu^2$ are not Dirichlet eigenvalues of the Laplace operator in Ω_{int}. Condition (1e) is the well-known Sommerfeld radiation condition at infinity and has to be satisfied uniformly in all directions $\mathbf{x}/|\mathbf{x}| \in \mathbb{R}^d$, $r := |\mathbf{x}|$, (see [3,8]). In principle, $g_0 \in H^{1/2}(\Gamma)$ and $g_1 \in H^{-1/2}(\Gamma)$ are given and unrelated. In applications that have to do with acoustic or thermal problems in time-harmonic regime, g_0 plays the role of the trace on Γ of an incident wave and g_1 is β times the normal derivative on Γ of it.

Throughout this paper we will assume that parameters λ, μ, α and β are such that (1) has a unique solution belonging to $H^1(\Omega_{\text{int}})$ and $H^1_{\text{loc}}(\overline{\Omega}_{\text{ext}})$. Conditions on these parameters for existence and uniqueness of solution can be found in [11,19,21,27,37,38]. Two particular examples are:

(a) $\alpha, \beta > 0$, $\lambda^2, \mu^2 \notin \mathbb{R}_+$ and $\lambda/\mu \in \mathbb{R}$. This case is of interest when solving transmission problems for the heat equation via Laplace transform methods (see [16,31,32]).
(b) $\alpha, \beta > 0$ and $\lambda, \mu \in \{(1+\imath)\xi, \xi > 0\}$. Problem (1) for parameters satisfying the conditions above arises in the study of time-harmonic solutions to the heat equation (see [22,27,29] and Section 6.2 of this work).

3. Boundary integral formulation

In this section we give an equivalent mixed boundary integral formulation of (1). We begin by collecting some classical results on representation formulas and boundary integral operators in the context of the Sobolev spaces $H^{\pm 1/2}(\Gamma)$.

Consider the fundamental solution to the Helmholtz equation $\Delta u + \rho^2 u = 0$,

$$\phi_\rho(\mathbf{x}, \mathbf{y}) := \begin{cases} \dfrac{\imath}{4} H_0^{(1)}(\rho|\mathbf{x}-\mathbf{y}|) & \text{for } d=2, \\[2mm] \dfrac{\exp(\imath\rho|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} & \text{for } d=3, \end{cases} \tag{2}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and order zero. For $\varphi \in H^{-1/2}(\Gamma)$ we define the classical single layer potential

$$\mathscr{S}^\rho \varphi := \int_\Gamma \phi_\rho(\cdot, \mathbf{y})\varphi(\mathbf{y})\,\mathrm{d}\gamma_{\mathbf{y}} : \mathbb{R}^d \longrightarrow \mathbb{C},$$

and for $\eta \in H^{1/2}(\Gamma)$ the double layer potential

$$\mathscr{D}^\rho \eta := \int_\Gamma \partial_{\nu(\mathbf{y})}\phi_\rho(\cdot, \mathbf{y})\eta(\mathbf{y})\,\mathrm{d}\gamma_{\mathbf{y}} : \mathbb{R}^d \backslash \Gamma \longrightarrow \mathbb{C}.$$

We also consider boundary integral operators

$$V^\rho\varphi := \int_\Gamma \phi_\rho(\cdot,\mathbf{y})\varphi(\mathbf{y})\,\mathrm{d}\gamma_\mathbf{y} : \Gamma \longrightarrow \mathbb{C},$$

$$J^\rho\varphi := \int_\Gamma \partial_{\nu(\cdot)}\phi_\rho(\cdot,\mathbf{y})\varphi(\mathbf{y})\,\mathrm{d}\gamma_\mathbf{y} : \Gamma \longrightarrow \mathbb{C},$$

$$K^\rho\eta := \int_\Gamma \partial_{\nu(\mathbf{y})}\phi_\rho(\cdot,\mathbf{y})\eta(\mathbf{y})\,\mathrm{d}\gamma_\mathbf{y} : \Gamma \longrightarrow \mathbb{C}.$$

It is well-known (see [10]) that

$$\mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{int}} = \mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{ext}} = V^\rho\varphi, \quad \mathscr{D}^\rho\varphi|_\Gamma^{\mathrm{int}} = -\tfrac{1}{2}\eta + K^\rho\eta,$$

$$\partial_\nu\mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{int}} = \tfrac{1}{2}\varphi + J^\rho\varphi, \quad \partial_\nu\mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{ext}} = -\tfrac{1}{2}\varphi + J^\rho\varphi.$$

Notice that since the solution u to (1) is in particular a solution to (1b), by a classical result, known as the representation formula (or Green's third formula), it satisfies

$$u = \mathscr{S}^\mu\partial_\nu u|_\Gamma^{\mathrm{int}} - \mathscr{D}^\mu u|_\Gamma^{\mathrm{int}} \quad \text{in } \Omega_{\mathrm{int}}. \tag{3}$$

Our proposal is to take as boundary unknowns the interior Cauchy data $\eta := u|_\Gamma^{\mathrm{int}} \in H^{1/2}(\Gamma)$ and $\varphi := \partial_\nu u|_\Gamma^{\mathrm{int}} \in H^{-1/2}(\Gamma)$ and a exterior density $\psi \in H^{-1/2}(\Gamma)$ such that $u = \mathscr{S}^\lambda\psi$ in Ω_{ext}. In the bounded domain the solution will be recovered from (3), that is,

$$u = \left| \begin{array}{ll} \mathscr{S}^\lambda\psi & \text{in } \Omega_{\mathrm{ext}}, \\ \mathscr{S}^\mu\varphi - \mathscr{D}^\mu\eta & \text{in } \Omega_{\mathrm{int}}. \end{array} \right. \tag{4}$$

This choice guarantees that u satisfies the Helmholtz equations (1a) and (1b) as well as the Sommerfeld radiation condition (1e). In terms of the new unknowns, the transmission conditions (1c) and (1d) are

$$\eta - V^\lambda\psi = g_0, \quad \alpha\varphi - \beta(-\tfrac{1}{2}I + J^\lambda)\psi = g_1. \tag{5}$$

Notice also that from the jump relations of the layer potentials and the representation formula it follows that the interior Cauchy data have to satisfy the identity

$$(\tfrac{1}{2}I + K^\mu)\eta - V^\mu\varphi = 0. \tag{6}$$

Eqs. (5) and (6) can now be collected as

$$\mathscr{H}\begin{bmatrix}\psi\\ \varphi\\ \eta\end{bmatrix} := \begin{bmatrix} -V^\lambda & 0 & I \\ 0 & -V^\mu & \tfrac{1}{2}I + K^\mu \\ \beta(\tfrac{1}{2}I - J^\lambda) & \alpha I & 0 \end{bmatrix}\begin{bmatrix}\psi\\ \varphi\\ \eta\end{bmatrix} = \begin{bmatrix} g_0\\ 0\\ g_1\end{bmatrix}. \tag{7}$$

If we show that (7) has a unique solution belonging to $H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$, then it is straightforward that (4) provides the unique solution u to (1) and moreover, $\eta = u|_\Gamma^{\mathrm{int}}$ and $\varphi = \partial_\nu u|_\Gamma^{\mathrm{int}}$. We will analyze the invertibility of $\mathscr{H}$ in the remaining part of this section. By the properties of the integral operators involved (see [10]),

$$\mathscr{H} : H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$$

is a bounded operator. When Γ is a $\mathscr{C}^\infty$ parametrizable curve or surface, $\mathscr{H}$ can be written as a matrix of pseudodifferential operators, with principal symbol (see [3])

$$\begin{bmatrix} -|\xi|^{-1} & 0 & 1 \\ 0 & -|\xi|^{-1} & \tfrac{1}{2} \\ \tfrac{\beta}{2} & \alpha & 0 \end{bmatrix}.$$

The order of the operators is therefore (the symbol $\times$ is written on the zero positions which are operators of order $-\infty$)

$$\begin{bmatrix} -1 & \times & 0 \\ \times & -1 & 0 \\ 0 & 0 & \times \end{bmatrix}.$$

For explicit expressions of these operators in the circular case, see [5,30].

Proposition 1. *$\mathscr{H}$ is injective.*

Proof. Since $-\lambda^2$ and $-\mu^2$ are not Dirichlet eigenvalues of the Laplace operator in Ω_{int}, V^λ, $V^\mu : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ are invertible (see [10]), which justifies the decomposition

$$\mathscr{H} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\beta(\frac{1}{2}I - J^\lambda)(V^\lambda)^{-1} & -\alpha(V^\mu)^{-1} & I \end{bmatrix} \begin{bmatrix} -V^\lambda & 0 & I \\ 0 & -V^\mu & \frac{1}{2}I + K^\mu \\ 0 & 0 & H_{\lambda\mu} \end{bmatrix}, \tag{8}$$

where

$$H_{\lambda\mu} := \beta(\tfrac{1}{2}I - J^\lambda)(V^\lambda)^{-1} + \alpha(V^\mu)^{-1}(\tfrac{1}{2}I + K^\mu). \tag{9}$$

Thus, to prove the result we only have to show that $H_{\lambda\mu}$ is one-to-one. If $H_{\lambda\mu}\xi = 0$ with $\xi \in H^{1/2}(\Gamma)$, then

$$u := \left| \begin{array}{ll} \mathscr{S}^\lambda(V^\lambda)^{-1}\xi & \text{in } \Omega_{\text{ext}}, \\ \mathscr{S}^\mu(V^\mu)^{-1}\xi & \text{in } \Omega_{\text{int}}, \end{array} \right.$$

satisfies (1a), (1b) and (1e). Moreover, by the jump relations of the single layer potential,

$$u|_\Gamma^{\text{ext}} = \xi = u|_\Gamma^{\text{int}},$$

$$\partial_\nu u|_\Gamma^{\text{ext}} = (-\tfrac{1}{2}I + J^\lambda)(V^\lambda)^{-1}\xi.$$

Finally, since u is a solution to (1b), by the representation formula and the jump relations of the layer potentials we find that

$$\partial_\nu u|_\Gamma^{\text{int}} = (V^\mu)^{-1}(\tfrac{1}{2}I + K^\mu)u|_\Gamma^{\text{int}} = (V^\mu)^{-1}(\tfrac{1}{2}I + K^\mu)\xi,$$

and therefore, $\alpha\partial_\nu u|_\Gamma^{\text{int}} - \beta\partial_\nu u|_\Gamma^{\text{ext}} = H_{\lambda\mu}\xi = 0$. Hence, u is a solution to the homogeneous transmission problem, that is, to (1) with $g_0 = g_1 = 0$, so $u = 0$ and $\xi = u|_\Gamma^{\text{int}} = 0$. $\square$

Unique solvability of (7) will be proven by using Fredholm theory. With this aim, we introduce the operators:

$$V_0\varphi := \int_\Gamma \phi_0(\cdot, \mathbf{y})\varphi(\mathbf{y})\,\mathrm{d}\gamma_{\mathbf{y}} : \Gamma \longrightarrow \mathbb{C},$$

$$J_0\varphi := \int_\Gamma \partial_{\nu(\cdot)}\phi_0(\cdot, \mathbf{y})\varphi(\mathbf{y})\,\mathrm{d}\gamma_{\mathbf{y}} : \Gamma \longrightarrow \mathbb{C},$$

$$K_0\eta := \int_\Gamma \partial_{\nu(\mathbf{y})}\phi_0(\cdot, \mathbf{y})\eta(\mathbf{y})\,\mathrm{d}\gamma_{\mathbf{y}} : \Gamma \longrightarrow \mathbb{C}, \tag{10}$$

where

$$\phi_0(\mathbf{x}, \mathbf{y}) := \left| \begin{array}{ll} -\dfrac{1}{2\pi}\log|r(\mathbf{x}-\mathbf{y})| & \text{for } d = 2, \\ \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|} & \text{for } d = 3. \end{array} \right. \tag{11}$$

In the two-dimensional case, r is chosen such that $0 < r < 1/\text{diam}(\Omega_{\text{int}})$, which ensures ellipticity of $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ (see [23, Theorem 8.16]). It is well known (see for instance [10]) that, in both cases $d = 2$ or 3, the operators

$V^\rho - V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $J^\rho - J_0 : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $K^\rho - K_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ are compact and $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an elliptic bounded isomorphism. It is clear then that the principal part of the operator $\mathscr{H}$ is

$$\mathscr{H}_0^L := \begin{bmatrix} -V_0 & 0 & I \\ 0 & -V_0 & \frac{1}{2}I + K_0 \\ \beta(\frac{1}{2}I - J_0) & \alpha I & 0 \end{bmatrix}. \tag{12}$$

Superscript L means that $\mathscr{H}_0^L$ is the principal part of the operator $\mathscr{H}$ in the Lipschitz setting. In some forthcoming results we will deal with the principal part of this operator for smoother curves or surfaces, for which we will use the notation $\mathscr{H}_0^S$.

Proposition 2. *$\mathscr{H}$ is Fredholm of index zero.*

Proof. As $\mathscr{H} - \mathscr{H}_0^L$ is compact, we can equivalently show that $\mathscr{H}_0^L$ is Fredholm of index zero. We proceed as in the proof of Proposition 1, decomposing

$$\mathscr{H}_0^L = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\beta(\frac{1}{2}I - J_0)V_0^{-1} & -\alpha V_0^{-1} & I \end{bmatrix} \begin{bmatrix} -V_0 & 0 & I \\ 0 & -V_0 & \frac{1}{2}I + K_0 \\ 0 & 0 & H_0 \end{bmatrix}$$

with

$$H_0 := \beta(\tfrac{1}{2}I - J_0)V_0^{-1} + \alpha V_0^{-1}(\tfrac{1}{2}I + K_0).$$

Now we can use the identity $K_0 V_0 = V_0 J_0$, which is a consequence of the idempotence of Calderon's projection (see for instance [38, Definition 3.6 and Theorem 3.7]) to write

$$H_0 = (\beta(\tfrac{1}{2}I - J_0) + \alpha V_0^{-1}(\tfrac{1}{2}I + K_0)V_0)V_0^{-1} = (\beta(\tfrac{1}{2}I - J_0) + \alpha(\tfrac{1}{2}I + J_0))V_0^{-1}.$$

Since $\beta(\frac{1}{2}I - J_0) + \alpha(\frac{1}{2}I + J_0) : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is Fredholm of index zero (see [27, Lemma 9.1]), the result follows readily. □

If Γ is a Lyapunov boundary (see [25] for a detailed definition and properties), then J^ρ and K^ρ are compact operators and the proof of Proposition 2 can be simplified somewhat since there is not need of introducing J_0 and K_0. In this case we can make use of the decomposition (8). We simply have to notice that $V^\lambda(V^\mu)^{-1} - I = (V^\lambda - V_0 + V_0 - V^\mu)(V^\mu)^{-1}$ is compact and therefore

$$V^\lambda H_{\lambda\mu} - \frac{\alpha+\beta}{2} I = -\beta V^\lambda J^\lambda (V^\lambda)^{-1} + \frac{\alpha}{2}(V^\lambda (V^\mu)^{-1} - I) + \alpha V^\lambda (V^\mu)^{-1} K^\mu,$$

$H_{\lambda\mu}$ being the operator introduced in (9), is also compact. Since $\alpha + \beta \neq 0$, $H_{\lambda\mu}$ is Fredholm of index zero. By (8) this implies that also $\mathscr{H}$ is Fredholm of index zero.

In all cases, by Propositions 1 and 2 we have the following result:

Corollary 3. *$\mathscr{H}$ is an isomorphism.*

When Γ is a smooth interface, for brevity $\mathscr{C}^\infty$, we can study the invertibility of $\mathscr{H}$ in the full range of the Sobolev spaces $H^s(\Gamma)$. With the same ideas as before we can prove the following result.

Proposition 4. *If Γ is smooth, $\mathscr{H} : H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma) \to H^{s+1}(\Gamma) \times H^{s+1}(\Gamma) \times H^s(\Gamma)$ is an isomorphism for all $s \in \mathbb{R}$.*

Proof. By the well-known properties of the boundary operators in $\mathscr{H}$, this operator is bounded for all $s \in \mathbb{R}$. The operators

$$V^\rho - V_0 : H^s(\Gamma) \to H^{s+1}(\Gamma), \quad J^\rho, K^\rho : H^s(\Gamma) \to H^s(\Gamma),$$

are compact and $V_0 : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is invertible (see [11] and references therein). Therefore, the principal part of $\mathscr{H}$ is

$$\mathscr{H}_0^S := \begin{bmatrix} -V_0 & 0 & I \\ 0 & -V_0 & \frac{1}{2}I \\ \frac{\beta}{2}I & \alpha I & 0 \end{bmatrix}, \tag{13}$$

which is invertible since

$$\mathscr{H}_0^S = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\frac{\beta}{2}V_0^{-1} & -\alpha V_0^{-1} & I \end{bmatrix} \begin{bmatrix} -V_0 & 0 & I \\ 0 & -V_0 & \frac{1}{2}I \\ 0 & 0 & \frac{\alpha+\beta}{2}V_0^{-1} \end{bmatrix}.$$

Moreover, not only $\mathscr{K}_0^S := \mathscr{H} - \mathscr{H}_0^S : H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma) \to H^{s+1}(\Gamma) \times H^{s+1}(\Gamma) \times H^s(\Gamma)$ is compact, but also $\mathscr{K}_0^S : H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma) \to H^{s+2}(\Gamma) \times H^{s+2}(\Gamma) \times H^{s+1}(\Gamma)$ is bounded (see [17] and [13,34] for the two and three dimensional cases respectively). Thus, $\mathscr{H}$ is a Fredholm operator of index zero. By the Fredholm alternative, we just have to show that $\mathscr{H}$ is one-to-one. Assume then that $\mathscr{H}\xi = 0$ with $\xi \in H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma)$. Then,

$$0 = \mathscr{H}\xi = \mathscr{H}_0^S(I + (\mathscr{H}_0^S)^{-1}\mathscr{K}_0^S)\xi$$

implies that

$$\xi = -(\mathscr{H}_0^S)^{-1}\mathscr{K}_0^S\xi \in H^{s+1}(\Gamma) \times H^{s+1}(\Gamma) \times H^{s+2}(\Gamma).$$

By induction we prove now that $\xi \in H^{s+n}(\Gamma) \times H^{s+n}(\Gamma) \times H^{s+n+1}(\Gamma)$ for all $n \in \mathbb{N}$, and in particular, $\xi \in H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. The result follows then from Proposition 1. □

Remark. The use of single layer potentials for the exterior component introduces spurious eigenmodes in the problem (see for instance [4] for this question in the 2D circular case). A possibility to circumvent the problem of spurious eigenfrequencies is using a mixed or Brakhage–Werner potential. In a purely indirect approach (indirect method for both the interior and the exterior domains), this has been done in [28]. The techniques of this paper can be adapted to the new situation, admitting however the drawback of having to use hypersingular operators.

4. Numerical approximation

We take two families of finite dimensional spaces depending on the parameter $n \to \infty$ satisfying

$$X_n \subset H^{-1/2}(\Gamma), \quad Y_n \subset H^{1/2}(\Gamma), \quad \dim X_n = \dim Y_n.$$

We will study Galerkin methods with a discrete space of the form $X_n \times X_n \times Y_n$. That is, we consider the discretization

$$\left| \begin{array}{ll} \text{Find } \psi_n, \varphi_n \in X_n, \eta_n \in Y_n \text{ s.t.} & \\ (-V^\lambda\psi_n + \eta_n, \phi_n) = (g_0, \phi_n) & \forall\, \phi_n \in X_n, \\ (-V^\mu\varphi_n + (\frac{1}{2}I + K^\mu)\eta_n, \phi_n) = 0 & \forall\, \phi_n \in X_n, \\ (\beta(\frac{1}{2}I - J^\lambda)\psi_n + \alpha\varphi_n, \mu_n) = (g_1, \mu_n) & \forall\, \mu_n \in Y_n, \end{array} \right. \tag{14}$$

for solving (7). Once (14) is solved, the solution to (1) can be approximated by replacing in (4) the boundary unknowns by the approximate ones,

$$u_n := \left| \begin{array}{ll} \mathscr{S}^\lambda\psi_n & \text{in } \Omega_{\text{ext}}, \\ \mathscr{S}^\mu\varphi_n - \mathscr{D}^\mu\eta_n & \text{in } \Omega_{\text{int}}. \end{array} \right. \tag{15}$$

Although we are interested in solving (7), the analysis we carry out here is valid for any right-hand side, i.e., the second component could be non-zero. For the sake of clarity we have collected in the appendix at the end of this paper some general properties of Petrov–Galerkin methods in the exact setting we will be using them. We refer to [1,2,18,20,39] for more details. In the analysis of the discretization we will deal with Galerkin schemes for operational equations where the operator has a mixed structure or with Petrov–Galerkin methods for generalized mixed operators (we specify the definition of this kind of operators at the Appendix).

In the next two subsections we discuss from an abstract point of view necessary and sufficient conditions for the convergence of (14) without giving any particular examples of discretizations. We will start by assuming that Γ is Lyapunov or smoother. After that, we will show the adjustments that have to be performed in order to extend the results to the more general case of Lipschitz boundaries. We will see that in this case the results are more involved. In a forthcoming section we will focus our attention in the particular case of smooth boundaries in $\mathbb{R}^2$, for which we know some couples of discrete spaces which provide convergent methods and say some words about the three-dimensional case. The question on whether concrete discrete spaces satisfy the requirements for convergence in the Lipschitz setting is left open.

4.1. Lyapunov and smooth boundaries

We begin by giving necessary and sufficient conditions for the convergence of (14) in the natural space $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. At this stage we just assume that Γ is Lyapunov. For notational simplicity instead of considering the usual norm in $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ we take the non-Hilbertian norm $\|\cdot\|_{-1/2} + \|\cdot\|_{-1/2} + \|\cdot\|_{1/2}$ which is equivalent to it.

Proposition 5. *Convergence of* (14) *in* $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *is equivalent to the following conditions*:

(a) X_n *satisfies the approximation property in* $H^{-1/2}(\Gamma)$ *as well as* Y_n *in* $H^{1/2}(\Gamma)$,
(b) *the Petrov–Galerkin* $\{X_n; Y_n\}$ *method for* $I : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is stable.*

Proof. In both cases (a) holds and we can equivalently show stability of the discretization when applied to operational equations for $\mathscr{H}_0^S$, the operator introduced in (13), because convergence of Petrov–Galerkin methods is preserved by compact perturbations (see [20, Theorem3.7]). Since $\alpha\beta \neq 0$, we can decompose $\mathscr{H}_0^S$ as

$$\mathscr{H}_0^S = \begin{bmatrix} I & 0 & 0 \\ 0 & \frac{1}{2}I & 0 \\ 0 & 0 & I \end{bmatrix} \left[\begin{array}{cc|c} -\frac{2}{\beta}V_0 & 0 & I \\ 0 & -\frac{2}{\alpha}V_0 & I \\ \hline I & I & 0 \end{array}\right] \begin{bmatrix} \frac{\beta}{2}I & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Thus, convergence of (14) is equivalent to the convergence of the same method for the operator in the middle in the decomposition above. Notice that it has a mixed structure since

$$[I \quad I] : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma),$$

$$\begin{bmatrix} I \\ I \end{bmatrix} : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$$

are adjoint operators. Moreover, as $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is elliptic, the operator on the left-upper corner is elliptic as well. Stability is equivalent to the uniform inf-sup condition (see the Appendix)

$$\sup_{r_n, s_n \in X_n} \frac{|(r_n + s_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma} + \|s_n\|_{-1/2,\Gamma}} \geqslant \gamma \|\mu_n\|_{1/2,\Gamma} \quad \forall \mu_n \in Y_n. \tag{16}$$

Finally, by the relation (we take $s_n = 0$ in the first inequality)

$$\sup_{r_n \in X_n} \frac{|(r_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma}} \leqslant \sup_{r_n, s_n \in X_n} \frac{|(r_n + s_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma} + \|s_n\|_{-1/2,\Gamma}} \leqslant 2 \sup_{r_n \in X_n} \frac{|(r_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma}}, \tag{17}$$

condition (16) is equivalent to (b). □

When dealing with smooth boundaries, we can also study convergence properties of (14) in the full range of the Sobolev spaces $H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma)$. To ensure a correct statement of the method the discrete subspaces have to satisfy

$$X_n \subset H^s(\Gamma) \cap H^{-s-1}(\Gamma), \quad Y_n \subset H^{s+1}(\Gamma) \cap H^{-s}(\Gamma).$$

Proposition 6. *For a given value of $s \in \mathbb{R}$, convergence of* (14) *in $H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma)$ is equivalent to*

(a) *X_n satisfies the approximation property in $H^s(\Gamma)$ as well as Y_n in $H^{s+1}(\Gamma)$,*
(b) *the Petrov–Galerkin $\{X_n; Y_n\}$ method for $I : H^s(\Gamma) \to H^s(\Gamma)$ is stable,*
(c) *the Petrov–Galerkin $\{Y_n; X_n\}$ method for $I : H^{s+1}(\Gamma) \to H^{s+1}(\Gamma)$ is stable,*
(d) *the Galerkin $\{X_n; X_n\}$ method for $V_0 : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is stable.*

Proof. We start as in Proposition 5 to deduce that, when the approximation properties hold, stability of (14) in $H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma)$ is equivalent to the stability of the Galerkin $\{X_n \times X_n \times Y_n; X_n \times X_n \times Y_n\}$ scheme for the operator

$$\left[\begin{array}{cc|c} -\frac{2}{\beta}V_0 & 0 & I \\ 0 & -\frac{2}{\alpha}V_0 & I \\ \hline I & I & 0 \end{array}\right] : H^s(\Gamma) \times H^s(\Gamma) \times H^{s+1}(\Gamma) \to H^{s+1}(\Gamma) \times H^{s+1}(\Gamma) \times H^s(\Gamma).$$

Since

$$[I \ \ I] : H^s(\Gamma) \times H^s(\Gamma) \to H^s(\Gamma), \qquad \begin{bmatrix} I \\ I \end{bmatrix} : H^{s+1}(\Gamma) \to H^{s+1}(\Gamma) \times H^{s+1}(\Gamma),$$

are only adjoint operators for $s = -\frac{1}{2}$, the difference with respect to Proposition 5 is that for a general value of $s \in \mathbb{R}$, this operator has a generalized mixed structure. Furthermore, the operator on the left-upper corner is not elliptic anymore. Stability is equivalent to the existence of $\gamma_1, \gamma_2, \gamma_3 > 0$, independent of n, such that for n large enough

$$\sup_{r_n, s_n \in X_n} \frac{|(r_n + s_n, \mu_n)|}{\|r_n\|_{s,\Gamma} + \|s_n\|_{s,\Gamma}} \geqslant \gamma_1 \|\mu_n\|_{-s,\Gamma} \quad \forall \mu_n \in Y_n, \tag{18}$$

$$\sup_{\mu_n \in Y_n} \frac{|(r_n + s_n, \mu_n)|}{\|\mu_n\|_{s+1,\Gamma}} \geqslant \gamma_2 (\|r_n\|_{-s-1,\Gamma} + \|s_n\|_{-s-1,\Gamma}) \quad \forall r_n, s_n \in X_n, \tag{19}$$

$$\sup_{(s_n,t_n) \in V_n^0} \frac{|(\alpha V_0 r_n, s_n) + (\beta V_0 p_n, t_n)|}{\|s_n\|_{-s-1,\Gamma} + \|t_n\|_{-s-1,\Gamma}} \geqslant \gamma_3 (\|r_n\|_{s,\Gamma} + \|p_n\|_{s,\Gamma}) \quad \forall (r_n, p_n) \in V_n^0, \tag{20}$$

where $V_n^0 := \{(r_n, s_n) \in X_n \times X_n | (r_n + s_n, \mu_n) = 0, \forall \mu_n \in Y_n\}$ (see the results collected in the Appendix).

With the same argument as in (17), we find that (18) is equivalent to

$$\sup_{r_n \in X_n} \frac{|(r_n, \mu_n)|}{\|r_n\|_{s,\Gamma}} \geqslant \gamma \|\mu_n\|_{-s,\Gamma} \quad \forall \mu_n \in Y_n,$$

that is, to the stability of the $\{Y_n; X_n\}$ method for $I : H^{-s}(\Gamma) \to H^{-s}(\Gamma)$ and, by duality, to (b). Also by duality, (19) is equivalent to

$$\sup_{r_n, s_n \in X_n} \frac{|(r_n + s_n, \mu_n)|}{\|r_n\|_{-s-1,\Gamma} + \|s_n\|_{-s-1,\Gamma}} \geqslant \gamma \|\mu_n\|_{s+1,\Gamma} \quad \forall \mu_n \in Y_n,$$

and therefore to (c). Finally we note that when (14) is convergent as well as when (b) holds, the discrete operator $B_n : X_n \times X_n \to X_n$ defined by

$$(B_n(r_n, s_n)^T, \mu_n) = (r_n + s_n, \mu_n) \quad \forall r_n, s_n \in X_n, \ \ \mu_n \in Y_n,$$

is surjective. Thus, $V_n^0 = \{(r_n, -r_n),\ r_n \in X_n\}$ since this space is obviously contained in V_n^0 and both have the same finite dimension (notice that $V_n^0 = \operatorname{Ker} B_n$). Then, (20) can be written as

$$\sup_{s_n \in X_n} \frac{|(V_0 r_n, s_n)|}{\|s_n\|_{-s-1,\Gamma}} \geqslant \gamma \|r_n\|_{s,\Gamma} \quad \forall r_n \in X_n,$$

which is equivalent to (d). □

Notice that conditions (a)–(d) in the proposition above are also equivalent to (b), (c) and (d) changing the word "stable" by "convergent".

4.2. Lipschitz boundaries

When Γ is Lipschitz we can also give sufficient and necessary conditions for convergence in the natural space $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Proposition 7. *Convergence of* (14) *in* $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *is equivalent to the following conditions*:

(a) X_n *satisfies the approximation property in* $H^{-1/2}(\Gamma)$ *as well as* Y_n *in* $H^{1/2}(\Gamma)$.
(b) *There exists* $\gamma_1 > 0$, *independent of* n, *such that for* n *large enough*

$$\sup_{(u_n,v_n)\in W_n^0} \frac{|\alpha(V_0 r_n, u_n) + \beta(V_0 s_n, v_n)|}{\|u_n\|_{-1/2,\Gamma} + \|v_n\|_{-1/2,\Gamma}} \geqslant \gamma_1 (\|r_n\|_{1/2,\Gamma} + \|s_n\|_{1/2,\Gamma}) \quad \forall (r_n, s_n) \in V_n^0,$$

where

$$V_n^0 := \{(r_n, s_n) \in X_n \times X_n | ((\tfrac{1}{2} I - J_0) r_n + s_n, \mu_n) = 0, \forall \mu_n \in Y_n\},$$

$$W_n^0 := \{(u_n, v_n) \in X_n \times X_n | (u_n + (\tfrac{1}{2} I + J_0) v_n, \mu_n) = 0, \forall \mu_n \in Y_n\}.$$

(c) *There exists* $\gamma_2 > 0$, *independent of* n, *such that for* n *large enough*

$$\sup_{r_n, s_n \in X_n} \frac{|(r_n + (\tfrac{1}{2} I - J_0) s_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma} + \|s_n\|_{-1/2,\Gamma}} \geqslant \gamma_2 \|\mu_n\|_{1/2,\Gamma} \quad \forall \mu_n \in Y_n. \tag{21}$$

(d) *There exists* $\gamma_3 > 0$, *independent of* n, *such that for* n *large enough*

$$\sup_{r_n, s_n \in X_n} \frac{|(r_n + (\tfrac{1}{2} I + J_0) s_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma} + \|s_n\|_{-1/2,\Gamma}} \geqslant \gamma_3 \|\mu_n\|_{1/2,\Gamma} \quad \forall \mu_n \in Y_n. \tag{22}$$

Proof. We follow the same ideas as in the proofs of Propositions 5 and 6. Now the principal part of $\mathscr{H}$ is the operator $\mathscr{H}_0^L$ defined in (12), that can be decomposed as

$$\mathscr{H}_0^L = \left[\begin{array}{cc|c} -\frac{1}{\beta} V_0 & 0 & I \\ 0 & -\frac{2}{\alpha} V_0 & \frac{1}{2} I + K_0 \\ \hline \frac{1}{2} I - J_0 & I & 0 \end{array}\right] \begin{bmatrix} \beta I & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Thus, convergence of (14) is equivalent to the convergence of the Galerkin $\{X_n \times X_n \times Y_n; X_n \times X_n \times Y_n\}$ method for the left-most operator in the decomposition above. Notice that the operators

$$[\tfrac{1}{2} I - J_0 \;\; I], \quad \begin{bmatrix} I \\ \frac{1}{2} + K_0 \end{bmatrix},$$

are not adjoint of each other and therefore we are dealing again with a generalized mixed structure. The result follows now from the abstract theory in the Appendix, taking into account that the adjoint operator of $K_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is $J_0 : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. □

Although $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is elliptic, it is not clear how (b) can be accomplished. For a concrete pair of discrete spaces X_n, Y_n, the necessary and sufficient conditions in the proposition above will be in general complicated to check. We propose in the next result sufficient conditions which are less involved, since we can give a stronger but easy to check condition implying (c) and (d). However, it remains the question on choosing X_n, Y_n satisfying (b).

Proposition 8. *Assume that conditions* (a) *and* (b) *in Proposition* 7 *hold and that there exists* $\gamma > 0$, *independent of* n, *such that for* n *large enough*

$$\sup_{r_n \in X_n} \frac{|(r_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma}} \geqslant \gamma \|\mu_n\|_{1/2,\Gamma} \quad \forall \mu_n \in Y_n.$$

Then (14) *converges in* $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Proof. Taking $s_n = 0$,

$$\sup_{r_n, s_n \in X_n} \frac{|(r_n + (\frac{1}{2} I \pm J_0) s_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma} + \|s_n\|_{-1/2,\Gamma}} \geqslant \sup_{r_n \in X_n} \frac{|(r_n, \mu_n)|}{\|r_n\|_{-1/2,\Gamma}} \geqslant \gamma \|\mu_n\|_{1/2,\Gamma} \quad \forall \mu_n \in Y_n.$$

The result is then a straightforward consequence of Proposition 7. □

The uniform inf–sup condition in Proposition 8 is equivalent to the stability of the Petrov–Galerkin $\{Y_n; X_n\}$ method for $I : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, or by duality, to the stability of the $\{X_n; Y_n\}$ method for $I : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. Notice that this condition is necessary for convergence when dealing with Lyapunov boundaries.

An indirect method for solving a similar problem to (1) is proposed in [27]. The solution is found as a single layer potential in both, the interior and the exterior domains. When studying Petrov–Galerkin methods for solving the equivalent system of integral equations, the simpler sufficient condition guaranteeing convergence involves the inf–sup condition of Proposition 8 but γ not only has to be positive, but also greater than a constant that depends on the size of $|\alpha/\beta|$. On the other hand, with that formulation the problem of checking condition (b) in Proposition 7 is avoided.

5. Parameterizable curves in two dimensions

This section is devoted to indicating some concrete discretizations providing convergent methods in the two-dimensional case. At the end of it we will give some ideas for the three-dimensional setting.

Throughout this section we assume that Γ is a $\mathscr{C}^\infty$-curve in $\mathbb{R}^2$ which admits a 1-periodic parameterization $\mathbf{x} : \mathbb{R} \to \Gamma$ satisfying therefore $|\mathbf{x}'| > 0$, $\mathbf{x}(t) \neq \mathbf{x}(s)$, $s - t \notin \mathbb{Z}$. We will substitute the traditional Sobolev spaces $H^s(\Gamma)$ by the corresponding periodic ones,

$$H^s := \left\{ \phi \in \mathscr{D}' \;\middle|\; |\widehat{\phi}(0)|^2 + \sum_{0 \neq k \in \mathbb{Z}} |k|^{2s} |\widehat{\phi}(k)|^2 < \infty \right\},$$

$\mathscr{D}'$ being the space of 1-periodic distributions at the real line and $\widehat{\phi}(k)$ the Fourier coefficients of ϕ. We refer to [20, Chapter 8] or [30, Chapter 5] for a detailed description and properties of these spaces.

We keep the same notation for the parameterized versions of the single and double layer potentials,

$$\mathscr{S}^\rho \varphi := \int_0^1 \phi_\rho(\cdot, \mathbf{x}(t)) \varphi(t) \, \mathrm{d}t : \mathbb{R}^2 \longrightarrow \mathbb{C},$$

$$\mathscr{D}^\rho \eta := \int_0^1 |\mathbf{x}'(t)| \partial_{\nu(t)} \phi_\rho(\cdot, \mathbf{x}(t)) \eta(t) \, \mathrm{d}t : \mathbb{R}^2 \backslash \Gamma \longrightarrow \mathbb{C},$$

and for their related operators

$$V^\rho\varphi := \int_0^1 \phi_\rho(\mathbf{x}(\cdot), \mathbf{x}(t))\varphi(t)\,\mathrm{d}t : \mathbb{R} \longrightarrow \mathbb{C},$$

$$J^\rho\varphi := \int_0^1 |\mathbf{x}'(\cdot)|\partial_{\nu(\cdot)}\phi_\rho(\mathbf{x}(\cdot), \mathbf{x}(t))\varphi(t)\,\mathrm{d}t : \mathbb{R} \longrightarrow \mathbb{C},$$

$$K^\rho\eta := \int_0^1 |\mathbf{x}'(t)|\partial_{\nu(t)}\phi_\rho(\mathbf{x}(\cdot), \mathbf{x}(t))\eta(t)\,\mathrm{d}t : \mathbb{R} \longrightarrow \mathbb{C}.$$

Here $\nu(t)$ is the exterior normal derivative at $\mathbf{x}(t)$. Recall that in the two-dimensional space the fundamental solution ϕ_ρ is given by the Hankel function $H_0^{(1)}$, see (2).

All the continuity, compactness and invertibility properties of these operators remain valid when replacing the traditional Sobolev spaces $H^s(\Gamma)$ by the 1-periodic spaces H^s (see [30]). In this parameterized form, the jump relations of the layer potentials read as follows:

$$\mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{int}} \circ \mathbf{x} = \mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{ext}} \circ \mathbf{x} = V^\rho\varphi, \quad \mathscr{D}^\rho\varphi|_\Gamma^{\mathrm{int}} \circ \mathbf{x} = -\tfrac{1}{2}\eta + K^\rho\eta,$$

$$|\mathbf{x}'|\partial_\nu\mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{int}} \circ \mathbf{x} = \tfrac{1}{2}\varphi + J^\rho\varphi, \quad |\mathbf{x}'|\partial_\nu\mathscr{S}^\rho\varphi|_\Gamma^{\mathrm{ext}} \circ \mathbf{x} = -\tfrac{1}{2}\varphi + J^\rho\varphi.$$

It is straightforward to check now that if u is a solution to (1b) and we define

$$\eta := u|_\Gamma^{\mathrm{int}} \circ \mathbf{x}, \quad \varphi := |\mathbf{x}'|\partial_\nu u|_\Gamma^{\mathrm{int}} \circ \mathbf{x},$$

then the representation formula reads

$$u = \mathscr{S}^\mu\varphi - \mathscr{D}^\mu\eta \quad \text{in } \Omega_{\mathrm{int}}.$$

Therefore, we can look for the solution to (1) in the form

$$u = \left| \begin{array}{ll} \mathscr{S}^\lambda\psi & \text{in } \Omega_{\mathrm{ext}}, \\ \mathscr{S}^\mu\varphi - \mathscr{D}^\mu\eta & \text{in } \Omega_{\mathrm{int}}, \end{array} \right.$$

where $\psi, \varphi \in H^{-1/2}$ and $\eta \in H^{1/2}$ have to be determined. Proceeding as in Section 3, we arrive at the system of equations

$$\mathscr{H}\begin{bmatrix}\psi\\ \varphi\\ \eta\end{bmatrix} := \begin{bmatrix} -V^\lambda & 0 & I \\ 0 & -V^\mu & \tfrac{1}{2}I + K^\mu \\ \beta(\tfrac{1}{2}I - J^\lambda) & \alpha I & 0 \end{bmatrix}\begin{bmatrix}\psi\\ \varphi\\ \eta\end{bmatrix} = \begin{bmatrix} g_0 \circ \mathbf{x} \\ 0 \\ |\mathbf{x}'| g_1 \circ \mathbf{x} \end{bmatrix}, \tag{23}$$

that has the same structure as (7). Indeed the solutions to (7) and (23) satisfy the simple relation

$$\psi \longleftrightarrow \psi \circ \mathbf{x}, \quad \varphi \longleftrightarrow |\mathbf{x}'|\varphi \circ \mathbf{x}, \quad \eta \longleftrightarrow \eta \circ \mathbf{x}.$$

We can follow step by step the proof of Proposition 4 to show that for all $s \in \mathbb{R}$ the operator $\mathscr{H} : H^s \times H^s \times H^{s+1} \to H^{s+1} \times H^{s+1} \times H^s$ is an isomorphism. Now the operator V_0 introduced in (10)–(11) is replaced by the Bessel operator

$$V_0\varphi := -\frac{1}{4\pi}\int_0^1 \log(4\mathrm{e}^{-1}\sin^2\pi(\cdot - t))\varphi(t)\,\mathrm{d}t \tag{24}$$

which satisfies the same properties as the operator in (10)–(11) when replacing the traditional Sobolev spaces by the 1-periodic ones (see [30, Chapter 5]).

For numerical purposes, a parallel approximation for Galerkin methods for subspaces $X_n \subset H^s \cap H^{-s-1}$, $Y_n \subset H^{s+1} \cap H^{-s}$ with $\dim X_n = \dim Y_n$ can be performed. The resultant method has exactly the same structure as (14) replacing in the right-hand side g_0 and g_1 by their parameterized versions indicated in (23). But again, the analysis can be carried out without taking into account its particular form. The abstract necessary and sufficient conditions are now:

- Convergence of the Galerkin $\{X_n \times X_n \times Y_n; X_n \times X_n \times Y_n\}$ method for $\mathscr{H}$ in $H^{-1/2} \times H^{-1/2} \times H^{1/2}$ is equivalent to the convergence of the $\{X_n; Y_n\}$ method for $I : H^{-1/2} \to H^{-1/2}$ plus the approximation property of Y_n in $H^{1/2}$.
- For a given value of $s \in \mathbb{R}$, convergence of the Galerkin $\{X_n \times X_n \times Y_n; X_n \times X_n \times Y_n\}$ method for $\mathscr{H}$ in $H^s \times H^s \times H^{s+1}$ is equivalent to the simultaneous convergence of the $\{X_n; Y_n\}$ method for $I : H^s \to H^s$, of the $\{Y_n; X_n\}$ method for $I : H^{s+1} \to H^{s+1}$ and of the $\{X_n; X_n\}$ method for $V_0 : H^s \to H^{s+1}$.

As we have said before, an analogous analysis for general Petrov–Galerkin methods for solving a similar problem to (1) is developed in [27]. By the results given there with straightforward adaptations to our formulation, it can be proven that the following concrete couples of discrete spaces provide convergent methods satisfying the convergence estimates that we indicate:

Spectral approximation: Taking the space of trigonometric polynomials

$$X_n = Y_n = \text{span}\{\exp(2\pi k \imath \cdot) \mid -n/2 \leqslant k \leqslant n/2\},$$

method (14) converges in $H^s \times H^s \times H^{s+1}$ for all $s \in \mathbb{R}$. Moreover, if $\psi, \varphi \in H^t$ and $\eta \in H^{t+1}$, then for all $s \leqslant t$,

$$\|\psi - \psi_n\|_s + \|\varphi - \varphi_n\|_s + \|\eta - \eta_n\|_{s+1} \leqslant C_{s,t}(1/n)^{t-s}(\|\psi\|_t + \|\varphi\|_t + \|\eta\|_{t+1}),$$

with C depending only on s and t. We can also define a pointwise approximation to u as in (15), which inherits the same order:

$$|u(\mathbf{z}) - u_n(\mathbf{z})| \leqslant C_{\mathbf{z},s,t}(1/n)^{t-s}(\|\psi\|_t + \|\varphi\|_t + \|\eta\|_{t+1}), \quad \mathbf{z} \notin \Gamma.$$

Therefore this method has superalgebraic convergence order. We want to point out that, although the constant $C_{\mathbf{z},s,t}$ depends on $\mathbf{z}$, it is uniformly bounded in the exterior of any ball enclosing Γ and only blows up when we approach to Γ. Moreover, we are computing numerical approximations to $u|_\Gamma^{\text{int}}$ and $\partial_\nu u|_\Gamma^{\text{int}}$ and it is straightforward to recover from them approximations to $u|_\Gamma^{\text{ext}}$ and $\partial_\nu u|_\Gamma^{\text{ext}}$ from the transmission conditions (1c) and (1d). We can use them to compute the solution on Γ, or in a small neighbourhood of it, by using for instance an order one Taylor expansion. As we will see in a numerical example in Section 6, it provides a substantial improvement for small values of the discretization parameter. Of course, as n increases we deal with the intrinsic error of the Taylor approximation.

In some very favourable conditions, requiring analyticity of the boundary, the very subtle analysis of periodic integral equations with analytic data in [30] can be adapted to show exponential order of convergence of the method. Analyticity may seem to be a very demanding hypothesis on the boundary, but circles and ellipses satisfy it, as well as some boundaries constructed with trigonometric polynomials that can be used in inverse problems to approximate the true unknown boundary. The three-dimensional case is, however, much more intricate. Let us remark again that the circular case produces in a natural way a numerical approximation with trigonometric polynomials where the diagonal operators can be analytically inverted. This is the origin of the well-known method of Z-matrices, very much employed in the physical literature. For recent applications of those ideas see [6,7].

Periodic smoothest splines on staggered meshes: We define $x_i := i/n$ and $x_{i+1/2} = (i + 1/2)/n$ for $i \in \mathbb{Z}$. For $m \geqslant 0$ we consider the spaces of periodic smoothest splines of degrees m and $m + 1$,

$$X_n = \{p_n \in H^m \mid p_n|_{[x_{i-1/2}, x_{i+1/2}]} \in \mathbb{P}_m, \forall i\},$$

$$Y_n = \{q_n \in H^{m+1} \mid q_n|_{[x_{i-1}, x_i]} \in \mathbb{P}_{m+1}, \forall i\}.$$

Then, scheme (14) converges in $H^s \times H^s \times H^{s+1}$ for all $-m - 3/2 \leqslant s \leqslant m + 1/2$. Furthermore, for couples (s, t) satisfying

$$-m - 2 \leqslant s < m + 1/2, \quad -m - 3/2 < t \leqslant m + 1, \quad s \leqslant t,$$

we have the error estimate

$$\|\psi - \psi_n\|_s + \|\varphi - \varphi_n\|_s + \|\eta - \eta_n\|_{s+1} \leqslant C_{s,t} h^{t-s}(\|\psi\|_t + \|\varphi\|_t + \|\eta\|_{t+1}),$$

provided that $\psi, \varphi \in H^t$ and $\eta \in H^{t+1}$. Pointwise convergence properties are also preserved for $\mathbf{z} \notin \Gamma$. Notice that, for smooth data, the optimal convergence rate is $2m + 3$.

Piecewise constant and linear functions on staggered non-uniform meshes: We take now a couple of staggered meshes $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ and $x_{i+1/2} := (x_i + x_{i+1})/2$ and define $h_i := x_i - x_{i-1}$, $h := \max_i h_i$. We assume that there exist $C_1, C_2 > 0$, independent of h, such that

$$\frac{1}{C_1} \leqslant \frac{h_{i-1}}{h_i} \leqslant C_1, \quad 6 - \left(\frac{h_i + h_{i-1}}{h_i + h_{i+1}} + \frac{h_i + h_{i+1}}{h_i + h_{i-1}} \right) \geqslant C_2 > 0.$$

We take then the discrete spaces

$$X_n = \{p_n \in H^0 | p_n|_{[x_{i-1/2}, x_{i+1/2}]} \in \mathbb{P}_0, \forall i\},$$

$$Y_n = \{q_n \in H^1 | q_n|_{[x_{i-1}, x_i]} \in \mathbb{P}_1, \forall i\}.$$

With this choice, method (14) is convergent in $H^{-1/2} \times H^{-1/2} \times H^{1/2}$. Besides, for all $-2 \leqslant s \leqslant -1/2 \leqslant t \leqslant 1$,

$$\|\psi - \psi_n\|_s + \|\varphi - \varphi_n\|_s + \|\eta - \eta_n\|_{s+1} \leqslant C_{s,t} h^{t-s} (\|\psi\|_t + \|\varphi\|_t + \|\eta\|_{t+1}),$$

when $\psi, \varphi \in H^t$ and $\eta \in H^{t+1}$. Again we can use this estimation to control the pointwise error in the approximation of the solution to (1). With this method we have cubic convergence order for smooth data, as well as when we consider the same spaces for uniform meshes.

In the three-dimensional case if Ω_{int} is sphere-like and Γ can be parameterized in polar coordinates, then a method with superalgebraic convergence order can be constructed with spherical harmonics playing the role of the trigonometric polynomials in two dimensions.

If Γ is a polyhedron, then the method with piecewise constant and linear functions on non-uniform meshes can also be generalized following the ideas of [33]. The dual mesh now is constructed as in finite volume methods (see [14]). The mesh size will have to satisfy some local conditions uniformly as in the two-dimensional setting. However, it is not clear how to choose the meshes in order to obtain a convergent method since it remains to check whether condition (b) in Proposition 7 holds.

6. Numerical examples

We illustrate in this section how our numerical methods work for solving some Helmholtz transmission problems in the two-dimensional case. We will use the Galerkin methods with trigonometric polynomials and with piecewise linear and constant functions defined on staggered uniform grids that were introduced in Section 5. The fully discrete methods we use here are due adaptations of methods proposed in some previous works. For a detailed description and properties of the fully discretization concerning the spectral approximation we refer to [24] and for the method with spline functions we refer to [27].

We construct a pointwise approximation of the solution to (1) from the approximate solution to (23) as a discrete version of (15) by applying simple midpoint rules to compute numerically the integrals appearing in the layer potentials. It can be shown that when the functions on the right-hand side in (23) are smooth, the pointwise approximation has superalgebraic convergence order with the spectral method and quadratic order with the method with piecewise linear and constant functions. On the other hand, when the boundaries of the obstacles are smooth but not $\mathscr{C}^\infty$-curves, then the additional implementation effort in the spectral approximation is wasted as we will see in the next numerical example.

6.1. A test problem

We compare both discretizations on two problems with known solution. In order to show that depending on the regularity of the boundary of the obstacle not always spectral approximations are preferred, we solve two different problems that only differ slightly in the chosen domain. In the first one, we consider the bounded domain Ω_1 represented in Fig. 1 whose boundary is given by the $\mathscr{C}^\infty$-parameterization $\mathbf{x}_1(t) := (r_1(t)\cos(2\pi t), -1 + r_1(t)\sin(2\pi t))$, with $r_1(t) := \frac{1}{2} + \sin(2\pi t)/(4\cos(2\pi t) - 5)$.

We take the parameters $\lambda = 1 + 2\imath$, $\mu = \imath$, $\alpha = 3$, $\beta = 1$ and choose for the right hand side in (23) the functions g_0 and g_1 such that the solution to (1) is

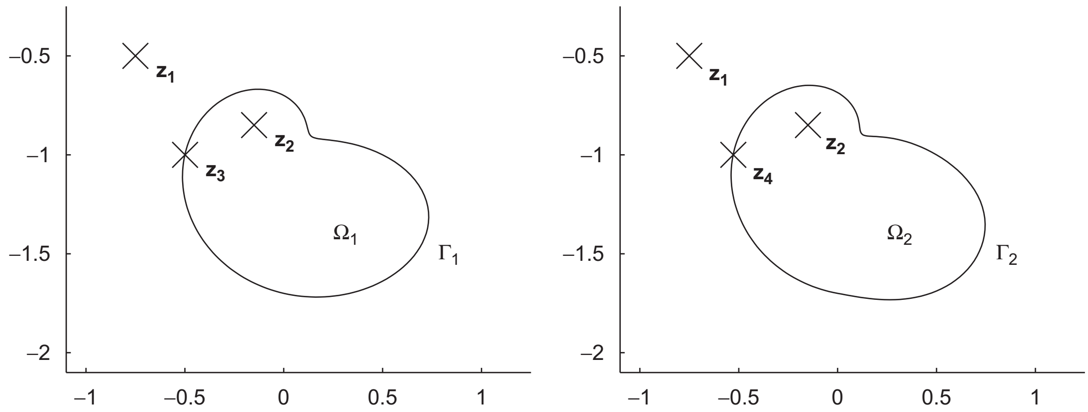

Fig. 1. Geometry of the test problems, Γ_1 is a $\mathscr{C}^\infty$-curve and Γ_2 is a $\mathscr{C}^2$-curve. The crosses mark points where errors are measured.

Table 1
Relative errors and estimated convergence rates for a $\mathscr{C}^\infty$-boundary

n	E_1^{spline}	e.c.r.	E_2^{spline}	e.c.r.	E_3^{spline}	e.c.r.	$E_3^{\text{spline,T}}$	e.c.r.
32	3.116(−4)		1.844(−3)		2.225		1.270(−3)	
64	7.596(−5)	2.03	4.638(−4)	1.99	8.365(−1)	1.41	3.107(−4)	2.03
128	1.878(−5)	2.01	1.163(−4)	1.99	2.920(−1)	1.51	7.821(−5)	1.99
256	4.670(−6)	2.00	2.915(−5)	1.99	8.959(−2)	1.70	2.207(−5)	1.82
512	1.164(−6)	2.00	7.296(−6)	1.99	2.183(−2)	2.03	1.142(−5)	0.95
n	E_1^{spec}	e.c.r.	E_2^{spec}	e.c.r.	E_3^{spec}	e.c.r.	$E_3^{\text{spec,T}}$	e.c.r.
32	3.276(−7)		1.264(−6)		2.781		1.875(−3)	
64	6.259(−14)	22.31	3.923(−13)	21.61	1.126	1.30	1.032(−5)	7.50
128	1.259(−15)	5.63	5.492(−15)	6.15	4.342(−1)	1.37	1.032(−5)	0.00
256	3.893(−15)	−1.62	1.460(−14)	−1.41	1.551(−1)	1.48	1.032(−5)	0.00
512	6.754(−15)	−0.79	4.825(−15)	1.59	4.905(−2)	1.66	1.032(−5)	0.00

$$u(\mathbf{z}) = \left| \begin{array}{ll} \dfrac{\imath}{4} H_0^{(1)}(\lambda|\mathbf{z}_0 - \mathbf{z}|) & \text{if } \mathbf{z} \in \mathbb{R}^2 \setminus \overline{\Omega}_1, \\ e^{\imath\mu \mathbf{d}\cdot\mathbf{z}} & \text{if } \mathbf{z} \in \Omega_1, \end{array} \right.$$

where $\mathbf{z}_0 := (0, -1.25)$ and $\mathbf{d} := (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. We consider the three points $\mathbf{z}_1 := (-0.75, -0.5)$, $\mathbf{z}_2 := (-0.15, -0.85)$ and $\mathbf{z}_3 := (-0.499, -1.0002)$, that are represented in Fig. 1, and compute the relative errors E_i^{spline} and E_i^{spec}, $i = 1, 2, 3$, for a sequence of uniform grids with n nodes. In Table 1 are written these errors as well as the estimated convergence rates (e.c.r.) which are computed by comparing them on the consecutive grids in the usual way. Notice that both the exterior point $\mathbf{z}_1$ and the interior point $\mathbf{z}_2$ are far form Γ_1 and at them the behavior of the methods clearly fits with the theoretical results. We can also see that at the interior point $\mathbf{z}_3$, which is close to Γ_1, the relative error is significantly higher than at $\mathbf{z}_1$ or $\mathbf{z}_2$, being even bigger or comparable with the value of the exact solution for small values of the parameter and not satisfactory as n increases.

Since we are computing pointwise approximations of $u|_{\Gamma_1}^{\text{int}}$ and $\partial_\nu u|_{\Gamma_1}^{\text{int}}$, an alternative way of approximating the solution near Γ_1 is to use Taylor approximations. We have chosen the values of the discretization parameter n in such a way that always $(-0.5, -1)$ is one of the points lying on Γ_1 where we compute $u|_{\Gamma_1}^{\text{int}}$ and $\partial_\nu u|_{\Gamma_1}^{\text{int}}$. Specifically it corresponds with $\mathbf{x}_1(t_{n/2})$. Notice that the distance between this point and $\mathbf{z}_3$ is 0.001 in the normal direction. From the approximate values of $u|_{\Gamma_1}^{\text{int}}(t_{n/2})$ and $\partial_\nu u|_{\Gamma_1}^{\text{int}}(t_{n/2})$ we have constructed the Taylor polynomial of degree one to approximate the value of $u(\mathbf{z}_3)$. The corresponding relative errors $E_3^{\text{spline,T}}$ and $E_3^{\text{spec,T}}$ are also written in Table 1. It is clear that for both methods and for small values of the discretization parameter this approximation is more suitable.

Table 2
Relative errors and estimated convergence rates for a $\mathscr{C}^2$-boundary

n	E_1^{spline}	e.c.r.	E_2^{spline}	e.c.r.	E_4^{spline}	e.c.r.	$E_4^{\text{spline,T}}$	e.c.r.
32	3.579(−4)		1.737(−3)		2.326		1.902(−3)	
64	8.642(−5)	2.05	4.359(−4)	1.99	8.766(−1)	1.40	4.754(−4)	1.99
128	2.131(−5)	2.01	1.092(−4)	1.99	3.073(−1)	1.51	1.122(−4)	2.08
256	5.194(−6)	2.03	2.722(−5)	2.00	9.500(−2)	1.69	2.353(−5)	2.25
512	1.230(−6)	2.07	6.790(−6)	2.00	2.346(−2)	2.01	7.085(−6)	1.73
n	E_1^{spec}	e.c.r.	E_2^{spec}	e.c.r.	E_4^{spec}	e.c.r.	$E_4^{\text{spec,T}}$	e.c.r.
32	2.136(−3)		5.991(−3)		3.082		3.690(−3)	
64	1.067(−3)	1.00	2.998(−3)	0.99	1.265	1.28	1.831(−3)	1.01
128	5.336(−4)	1.00	1.499(−3)	0.99	4.976(−1)	1.34	8.585(−4)	1.09
256	2.667(−4)	1.00	7.502(−4)	0.99	1.835(−1)	1.43	4.158(−4)	1.04
512	1.334(−4)	1.00	3.753(−4)	0.99	6.118(−2)	1.58	2.040(−4)	1.02

As expected, with the spectral method the best approximation that can be achieved by the Taylor polynomial only requires a coarse mesh.

In the second example we replace the obstacle Ω_1 by an approximation of it. The new obstacle Ω_2 is defined by the $\mathscr{C}^2$-parameterization

$$\mathbf{x}_2(t) := (r_2(t)\cos(2\pi t), -1 + r_2(t)\sin(2\pi t)),$$

r_2 being the 1-periodic cubic spline that interpolates r_1 at 0.1, 0.175, 0.75, 0.95 and 1 (see Fig. 1). The problem we solve is exactly the same as in the previous example. The only difference now is that we compute the relative errors at the same points $\mathbf{z}_1$ and $\mathbf{z}_2$, and at the new interior point $\mathbf{z}_4 := (-0.5274, -1.0002)$, which is at a distance of 0.001 in the normal direction of $\mathbf{x}_2(t_{n/2})$. The results are written in Table 2. In this case we obtain better approximations with the method with piecewise linear and constant functions than with the spectral one. Indeed, although in [26] only quadratic convergence order for $\mathscr{C}^3$-boundaries is proved, numerical experiments show that $\mathscr{C}^2$-regularity is enough. Notice also that the errors with the spline method are almost the same in all points when considering Ω_1 or Ω_2, that is, this method is not very sensitive to small changes on the domain involving not only small deformations, but also changes on the regularity.

An interesting application of the use of smooth but not $\mathscr{C}^\infty$-curves is when solving inverse problems via iteration schemes where one has to use parametric representations of the obstacles. Trigonometric polynomial representations are not as flexible as the type of curve in polar coordinates we use here or as spline approximations and also are not as simple to handle.

6.2. A scattering problem

A relevant field where the results in this work apply is in the study of the scattering of thermal waves in composite materials by means of non-destructive evaluations. We are interested in a photothermal technique which consists on heating one of the surfaces of the material by a defocused laser beam modulated at a given frequency ω. The incident wave induced by the heating spreads into the material and is scattered by the different materials placed under the surface. The goal is to reconstruct the size, depth, orientation and physical properties of the subsurface features from the measurement of the temperature at the thermically excited surface. We refer to [36,35] for a more detailed description of this kind of techniques.

We show in the next example how our method can be applied for solving the direct heat diffusion problem related with the photothermal technique specified above in a simplified model. We assume that all the inclusions in the composite material are cylindrical with parallel axis and that the thermal excitement preserves the symmetry. This allows to formulate the problem with a two-dimensional model. The material will occupy the half plane $\mathbb{R}^2_- := \{(x, y) \in \mathbb{R}^2 \,|\, y < 0\}$ and $\Pi := \{(x, 0) \,|\, x \in \mathbb{R}\}$ will be the heated side. The boundaries of the obstacles $\Omega_1, \ldots, \Omega_k$ will be denoted by $\Gamma_1, \ldots, \Gamma_k$ and are assumed to be $\mathscr{C}^2$-curves. When the heating is periodic in time, we can look for time-harmonic solutions of the diffusion equation, i.e., solutions of the form $T(\mathbf{x},t) := \mathrm{Re}(v(\mathbf{x})\exp(-\imath\omega t))$. We will also assume adiabaticity, which corresponds to a condition on $\partial_\nu v|_\Pi$ that we will shortly specify. Then, the function

$$u := \left| \begin{array}{ll} v - u_{\mathrm{inc}} & \text{in } \Omega := \mathbb{R}^2 \Big\backslash \bigcup_{j=1}^{k} \overline{\Omega}_j, \\ v & \text{in } \Omega_j, \ \ j = 1, \ldots, k, \end{array} \right.$$

where u_{inc} is the complex amplitude of the incident wave, is a solution to the following Helmholtz transmission problem:

$$\left| \begin{array}{ll} \Delta u + \lambda^2 u = 0 & \text{in } \Omega, \\ \Delta u + \mu_j^2 u = 0 & \text{in } \Omega_j, \ \ j = 1, \ldots, k, \\ u|_{\Gamma_j}^{\mathrm{int}} - u|_{\Gamma_j}^{\mathrm{ext}} = u_{\mathrm{inc}}|_{\Gamma_j}, & j = 1, \ldots, k, \\ \alpha_j \partial_\nu u|_{\Gamma_j}^{\mathrm{int}} - \beta \partial_\nu u|_{\Gamma_j}^{\mathrm{ext}} = \beta \partial_\nu u_{\mathrm{inc}}|_{\Gamma_j}, & j = 1, \ldots, k, \\ \partial_\nu u|_\Pi = 0, & \\ \lim_{r\to\infty} r^{1/2}(\partial_r u - \imath\lambda u) = 0. & \end{array} \right. \tag{25}$$

Here the wave numbers $\mu_j, \lambda \in \{(1+\imath)\xi | \xi > 0\}$ and the thermal conductivities $\alpha_j, \beta > 0$ are given parameters.

We briefly explain now the required adjustments in the preceding work for this new situation. For a detailed study of this problem we refer to [26,27]. We consider 1-periodic regular parameterizations $\mathbf{x}_j : \mathbb{R} \to \Gamma_j$ of the boundaries of the obstacles and the layer potentials

$$\mathscr{S}_j^\rho \varphi := \int_0^1 \phi_\rho(\cdot, \mathbf{x}_j(t))\varphi(t)\,\mathrm{d}t : \mathbb{R}^2 \longrightarrow \mathbb{C},$$

$$\mathscr{D}_j^\rho \eta := \int_0^1 |\mathbf{x}_j'(t)| \partial_{\nu(t)} \phi_\rho(\cdot, \mathbf{x}_j(t))\eta(t)\,\mathrm{d}t : \mathbb{R}^2 \backslash \Gamma \longrightarrow \mathbb{C}.$$

To deal with the boundary condition on Π we define the new single layer potentials

$$\widetilde{\mathscr{S}}_j^\rho \varphi := \int_0^1 (\phi_\rho(\cdot, \mathbf{x}_j(t)) + \phi_\rho(\cdot, \widetilde{\mathbf{x}}_j(t)))\varphi(t)\,\mathrm{d}t : \mathbb{R}^2 \longrightarrow \mathbb{C},$$

where $\widetilde{\mathbf{x}} := (x, -y)$ is the reflected point of $\mathbf{x} := (x, y)$. We can give now an equivalent boundary integral formulation of (25) where the unknowns are k exterior densities $\psi_1, \ldots, \psi_k \in H^{-1/2}$, and $2k$ parameterized versions of the Cauchy data of the solution on the boundaries of the obstacles, $\eta_1, \ldots, \eta_k \in H^{1/2}$ and $\varphi_1, \ldots, \varphi_k \in H^{-1/2}$. The solution to (25) can be recovered then as

$$u = \left| \begin{array}{ll} \sum_{j=1}^{k} \widetilde{\mathscr{S}}_j^\lambda \psi_j & \text{in } \Omega, \\ \mathscr{S}_j^{\mu_j} \varphi_j - \mathscr{D}_j^{\mu_j} \eta_j & \text{in } \Omega_j, \ \ j = 1, \ldots, k. \end{array} \right.$$

It is not difficult to adapt all the analysis developed in this paper to the new problem. In particular, the equivalent system of boundary integral equations has the same structure as (23), having now instead of each operator related with the interior problem a diagonal matrix of operators of size k, and instead of the boundary operators related to the exterior problem a full square matrix of size k associated to the reflected fundamental solution to the Helmholtz equation.

For a numerical illustration, we consider the composite material represented in Fig. 2. The obstacle Ω_2 is exactly the same as in the previous example and the boundaries of Ω_1 and Ω_3 are given by the $\mathscr{C}^\infty$-parameterizations

$$\mathbf{x}_1(t) := (-1.5 + r_1(t)\cos(2\pi t), -0.65 + r_1(t)\sin(2\pi t)),$$

$$\mathbf{x}_3(t) := (1.75 + r_3(t)\cos(2\pi t), -0.75 + r_3(t)\sin(2\pi t)),$$

where

$$r_1(t) := 1/2 + 1/6(\cos(4\pi(t + 1/3)) + \sin(2\pi(t - 1/3))),$$

$$r_3(t) := 1/2 + 1/8(\sin(2\pi(t - 1/3)) + \sin(6\pi t)).$$

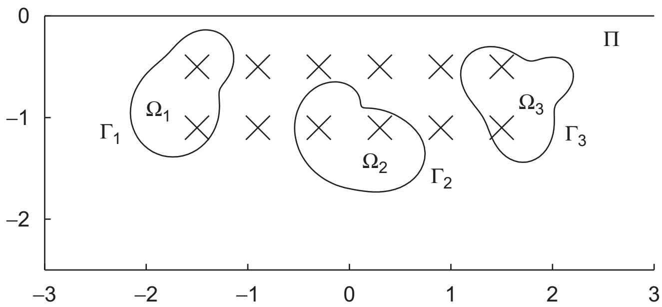

Fig. 2. Geometry of the test problem, Γ_1 and Γ_3 are $\mathscr{C}^\infty$-curves and Γ_2 is $\mathscr{C}^2$. The crosses mark points where errors are measured.

Table 3
Mean relative error and estimated convergence rate

n	E^{spline}	e.c.r.
32	2.084(−3)	
64	5.204(−4)	2.002
128	1.293(−4)	2.009
256	3.225(−5)	2.003
512	8.211(−6)	1.974

We choose the parameters

$$\lambda = (1+\imath)/4, \quad \alpha = 5, \quad \mu_j = j(1+\imath), \quad \alpha_j = j, \quad j = 1, 2, 3,$$

and take $u_{\text{inc}} := (\imath/4)H_0^{(1)}(\lambda|\cdot - (0.5, 0)|)$, which physically corresponds to a periodic heating at the source point $(0.5, 0)$.

Since the exact solution is unknown, we have taken as the exact solution to compare with an approximation computed by a quadrature method based on the ideas of qualocation methods that was proposed and analyzed in [12] on a very fine grid.

As we have seen before, since Γ_2 is only a $\mathscr{C}^2$-curve, the spaces of piecewise linear and constant functions provide a simple method with quadratic convergence order.

In Table 3 we write the mean relative error E^{spline} and the estimated convergence rate (e.c.r.) at the twelve points represented in Fig. 2. The discretization parameter n is the number of nodes per domain. It is clear again that the pointwise approximation has order two.

Acknowledgments

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Appendix

In this section we recall some general concepts on Petrov–Galerkin schemes for operational equations which are of use in the preceding work and state them in the precise form that is need. We refer to [18,20,39] for a detailed study of Petrov–Galerkin methods applied to general operational equations and to [1,2] for mixed or generalized mixed operators (we will shortly specify the definition this kind of operators).

Given two Hilbert spaces V, W and an isomorphism $\mathscr{A} : V \to W'$, a Petrov–Galerkin method consists of two sequences of finite-dimensional subspaces

$$V_n \subset V, \quad W_n \subset W, \quad \dim V_n = \dim W_n,$$

and the discretization scheme:

$$\left| \begin{array}{l} v_n \in V_n, \\ (\mathscr{A} v_n, w_n) = (\mathscr{A} v, w_n) \quad \forall\, w_n \in W_n. \end{array} \right.$$

For easy reference we will call this method the Petrov–Galerkin $\{V_n; W_n\}$ scheme for $\mathscr{A} : V \to W'$. The method is said to be stable if we can find $\gamma > 0$ independent of n (at least for n large enough) such that the discrete *Babuška–Brezzi condition*

$$\sup_{0 \neq w_n \in W_n} \frac{|(\mathscr{A} v_n, w_n)|}{\|w_n\|} \geqslant \gamma \|v_n\| \quad \forall v_n \in V_n, \tag{26}$$

holds. This condition is equivalent to unique solvability of the discrete equations plus the inequality

$$\|v_n\| \leqslant (\|\mathscr{A}\|/\gamma)\|v\|$$

and to also the *Céa estimate*

$$\|v - v_n\| \leqslant (\|\mathscr{A}\|/\gamma) \inf_{u_n \in V_n} \|v - u_n\|.$$

Then convergence is equivalent to stability plus the *approximation property*:

$$\inf_{u_n \in V_n} \|v - u_n\| \to 0 \quad \forall v \in V.$$

When $W = V$, the $\{V_n; W_n\}$ scheme with $W_n = V_n$ is called Galerkin or Bubnov–Galerkin method. If $\mathscr{A} : V \to V'$ and there exists $\alpha > 0$ such that

$$\mathrm{Re}(\mathscr{A} v, v) \geqslant \alpha \|v\|^2 \quad \forall v \in V,$$

i.e., $\mathscr{A}$ is an elliptic operator, then the Galerkin $\{V_n; V_n\}$ method is stable.

We will say that $\mathscr{A}$ has a *mixed structure* if

$$\mathscr{A} = \begin{bmatrix} A & B' \\ B & 0 \end{bmatrix} : V \times M \to V' \times M',$$

$A : V \to V'$ and $B : V \to M'$ being bounded operators and V, M Hilbert spaces. For this kind of operators we take two sequences of finite-dimensional subspaces

$$V_n \subset V, \quad M_n \subset M, \quad \dim V_n = \dim M_n,$$

and define $V_n^0 := \{v_n \in V_n \,|\, (B v_n, \mu_n) = 0, \forall \mu_n \in M_n\}$. Then, the Galerkin $\{V_n \times M_n; V_n \times M_n\}$ method for $\mathscr{A}$ is well defined if and only if there exist $\gamma_{A,n}, \gamma_{B,n} > 0$ such that

$$\sup_{0 \neq w_n^0 \in V_n^0} \frac{|(A v_n^0, w_n^0)|}{\|w_n^0\|} \geqslant \gamma_{A,n} \|v_n^0\| \quad \forall v_n^0 \in V_n^0, \tag{27}$$

$$\sup_{0 \neq v_n \in V_n} \frac{|(B v_n, \mu_n)|}{\|v_n\|} \geqslant \gamma_{B,n} \|\mu_n\| \quad \forall \mu_n \in M_n. \tag{28}$$

Moreover, stability is equivalent to (27) and (28) with $\gamma_{A,n}, \gamma_{B,n}$ independent of n, for n large enough. If A is elliptic in V, then the method is stable if and only if (28) holds with $\gamma_{B,n}$ independent of n, for n large enough.

Remark. The discrete space V_n^0 can be seen as the kernel of the operator $B_n : V_n \to M_n$ given by

$$(B_n v_n, \mu_n) = (B v_n, \mu_n) \quad \forall v_n \in V_n, \quad \mu_n \in M_n.$$

Besides, the inf–sup condition (28) is equivalent to the surjectivity of B_n. We can also define the discrete operator $A_n^0 : V_n^0 \to V_n^0$ by

$$(A_n^0 v_n^0, w_n^0) = (A v_n^0, w_n^0) \quad \forall v_n^0,\ w_n^0 \in V_n^0.$$

Then, the Galerkin $\{V_n \times M_n; V_n \times M_n\}$ method for $\mathscr{A}$ is well defined if and only if both A_n^0 and B_n are surjective.

We will say that $\mathscr{A}$ has a *generalized mixed structure* if

$$\mathscr{A} = \begin{bmatrix} A & C' \\ B & 0 \end{bmatrix} : V \times M \to W' \times N',$$

with $A : V \to W'$, $B : V \to N'$, $C : W \to M'$ bounded (V, M, W, N being Hilbert spaces). For this kind of operators we consider four families of finite-dimensional spaces, $V_n \subset V$, $M_n \subset M$, $W_n \subset W$ and $N_n \subset N$ satisfying $\dim V_n + \dim M_n = \dim W_n + \dim N_n$, and the Petrov–Galerkin $\{V_n \times M_n; W_n \times N_n\}$ method for $\mathscr{A}$. In this case we introduce

$$V_n^0 := \{v_n \in V_n | (B v_n, \nu_n) = 0, \forall \nu_n \in N_n\},$$

$$W_n^0 := \{w_n \in W_n | (C w_n, \mu_n) = 0, \forall \mu_n \in M_n\}.$$

The Petrov–Galerkin method is well defined if and only if there exist three sequences $\gamma_{A,n}, \gamma_{B,n}, \gamma_{C,n} > 0$ such that

$$\sup_{0 \neq w_n^0 \in W_n^0} \frac{|(A v_n^0, w_n^0)|}{\|w_n^0\|} \geqslant \gamma_{A,n} \|v_n^0\| \quad \forall v_n^0 \in V_n^0, \tag{29}$$

$$\sup_{0 \neq v_n \in V_n} \frac{|(B v_n, \nu_n)|}{\|v_n\|} \geqslant \gamma_{B,n} \|\nu_n\| \quad \forall \nu_n \in N_n, \tag{30}$$

$$\sup_{0 \neq w_n \in W_n} \frac{|(C w_n, \mu_n)|}{\|w_n\|} \geqslant \gamma_{C,n} \|\mu_n\| \quad \forall \mu_n \in M_n, \tag{31}$$

and stability is equivalent to (29)–(31) with n-independent constants for n large enough.

Remark. Defining $A_n^0 : V_n^0 \to W_n^0$, $B_n : V_n \to N_n$ and $C_n : W_n \to M_n$ as before, invertibility of the Petrov–Galerkin $\{V_n \times M_n; W_n \times N_n\}$ equations for $\mathscr{A}$ is equivalent to the simultaneous subjectivity of A_n^0, B_n and C_n. Notice also that V_n^0 and W_n^0 are the kernels of the discrete operators B_n and C_n respectively.

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