Adaptive error estimation of desingular meshless method in conjunction with regularization methods for solving inverse problem with the overspecified-boundary condition

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ABSTRACT

In this paper, desingular meshless method (DMM) is applied to solve divergent problems which occurs in the Laplace equation with overspecified boundary conditions. The source points can be located on the real boundary by using the desingularization technique of adding-back and subtracting technique to regularize the singularity and hypersingularity of the kernel functions. The main contribution in this paper can be divided into two parts as follows. The first contribution: the accompanied ill-posed problem in the inverse problem has been remedied successfully by using the proposed regularization methods, truncated singular value decomposition method (TSVD), 0th-order and 1th-order Tikhonov regularization methods, respectively. The second contribution: in order to obtain the optimal parameter without comparing with analytical solution, L-curve and the novel developed adaptive error techniques are provided. The numerical evidences are given to verify the accuracy of the solutions after comparing with the results of analytical solution and discussed in the examples. Finally, the better regularization method and error estimation technique can be suggested in this paper.

Keywords: desingular meshless method, TSVD method, Tikhonov regularization method, L-curve technique, adaptive error technique, adding-back and subtracting technique.

1. INTRODUCTION

Inverse problems are presently becoming more important in many fields of science and engineering [1, 2, 3]. Sometimes, unreasonable results occur in the inverse problems subjected to the measured and contaminated errors on the over-specified boundary condition, because of the ill-posed behavior in the linear algebraic system [4, 5, 6]. Mathematically speaking, the influence matrix in the inverse problem is ill-posed since the solution is very sensitive to the given data. Such a divergent problem could be avoided by using regularization methods [2, 7]. For examples, truncated singular value decomposition method (TSVD) [7] and Tikhonov regularization technique [2] have been applied to deal with divergent problems.

For the inverse problem, the influence matrix is often ill-posed such that the regularization techniques which regularize the influence matrix is necessary. The TSVD transform the ill-posed matrix into a well-posed one by choosing an appropriate truncated number, *i*. Similarly, the 0th-order and 1th-order Tikhonov regularization techniques transform into a well-posed one by choosing an appropriate parameter for λ [8]. In this paper, the novel meshless method in conjunction with the two regularization techniques is employed to solve the inverse problem by using the error estimation techniques. To obtain a better regularization method, the comparison of two regularization techniques is made. For parameter λ (or *i*), if too much regularization, i.e. λ (or *i*) is larger, the solution will be too smoothing. If too little regularization, i.e. λ (or *i*) is small, the solution will be unreasonable by the contributions from the input data with perturbation error in measurements. The choice of the optimal parameter in regularization methods is vital for obtaining a reasonable and convergent solution.

In the literatures [8, 9], L-curve technique is often implemented to can be determine an appropriate parameter according to a compromise point between regularization errors (due to data smoothing) and perturbation errors (due to noise disturbance). It is an adaptive technique for error estimation without comparing with analytical solution. In this paper, we employ the L-curve technique and a new developed technique, respectively, to obtain the optimal parameter. The new technique is called adaptive error estimation technique. Also, it belongs to an adaptive technique and does not need to compare the results with analytical solution. This new estimation technique will be elaborated latterly.

During the last decade, scientific researchers have paid attention to the method of fundamental solutions (MFS) for solving engineering problems [10, 11, 12] in which the mesh or element is free. The desingular meshless method (DMM) is one kind of modified MFS and has been extensively applied to solve some potential problems [13, 14, 15]. By employing the desingularization technique of subtracting and adding-back technique to regularize the singularity and hypersingularity of the kernel functions, the proposed method can distributes the observation and source points on the coincident locations of the real boundary and still maintains the spirit of the MFS.

In this paper, we will employ the developed DMM [13, 14, 15] in conjunction with the TSVD, 0^{th}-order and 1^{th}-order Tikhonov regularization methods, respectively, to circumvent the ill-posed problems. The better regularization method is discussed in this paper. Also a better error estimation technique without comparing with analytical solution will be discussed in this paper. The results of the examples contaminated with artificial noises on the over-specified boundary condition are given to illustrate the validity of the proposed technique.

2. FORMULATION

2.1 Govening equation subject to over-specified boundary conditions

The inverse problem for the Laplace equation subject to over-specified boundary conditions as shown in Fig. 1 can be modeled by:

$$\nabla^2\phi(\mathrm{x})=0\,,\quad \mathrm{x}\in \mathrm{D}\,, \tag{1}$$

subjected to the boundary condition on B_1 as

$$\phi(\mathrm{x})=\overline{\phi}\,,\quad x\in B_1\,, \tag{2}$$

$$\phi_y(\mathrm{x})=\overline{\phi_y}\,,\quad x\in B_1\,, \tag{3}$$

where ∇^2 is the Laplacian operator, D is the domain of interest, $\phi_y(x)=\partial\phi(x)/\partial y$, B_1 is the known boundary (B_1) of B in which B is the whole boundary which consists of boundary (B_1) with specified BCs, and the boundary (B_2) with unknown BCs.

2.2 Method of fundamental solutions

2.2.1 Review of conventional method of fundamental solutions

By employing the radial basis function (RBFs) concept [10, 11, 12], the representation of the solution for interior problem can be approximated in terms of the strengths α_j of the singularities s_j as

$$\phi(x_i)=\sum_{j=1}^{N+M}A(s_j,x_i)\alpha_j\,, \tag{4}$$

$$\psi(x_i)=\sum_{j=1}^{N+M}B(s_j,x_i)\alpha_j\,, \tag{5}$$

where $A(s_j,x_i)$ is RBF, $B(s_j,x_i)=\partial A(s_j,x_i)/\partial n_{x_i}$, $\psi(x_i)=\partial\phi(x_i)/\partial n_{x_i}$ in which n_{x} is the normal vector at x, α_j is the jth unknown coefficient (strength of the singularity), s_j is the jth source point (singularity), x_i is the ith observation point. N and M are number of the boundary points on B_1 and B_2, respectively. The chosen RBFs of Eqs. (4) and (5) in this paper are the double-layer potentials in the potential theory as

$$A(s_j,x_i)=-<(x_i-s_j),n_j>/r_{ij}^2\,, \tag{6}$$

$$B(s_j,x_i)=2<(x_i-s_j),n_j><(x_i-s_j),\overline{n_i}>/r_{ij}^4 - <n_j,\overline{n_i}>/r_{ij}^2\,, \tag{7}$$

where < , > is the inner product of two vectors, r_{ij} is $\left|s_j-x_i\right|$, n_j is the normal vector at s_j, and $\overline{n_i}$ is the normal vector at x_i.

When the collocation point x_i approaches to the source point s_j, Eqs. (4) and (5) become singular. Eqs. (4) and (5) for the interior problems need to be regularized by using the subtracting and adding-back technique [13, 14] as follows:

$$\phi(x_i)=\sum_{j=1}^{N+M}A^{(I)}(s_j,x_i)\alpha_j-\sum_{j=1}^{N+M}A^{(E)}(s_j,x_i)\alpha_i\,, \tag{8}$$

in which

$$\sum_{j=1}^{N+M}A^{(E)}(s_j,x_i)\alpha_i=0\,,\quad \mathrm{x_i}\in\mathrm{B}\,, \tag{9}$$

where the superscript *(I)* and *(E)* denotes the inward and outward normal vectors, respectively. The detailed derivation of Eq. (9) is given in reference [13]. Therefore, we can obtain

$$\phi(x_i)=\sum_{j=1}^{i-1}A^{(I)}(s_j,x_i)\alpha_j+\sum_{j=i+1}^{N+M}A^{(I)}(s_j,x_i)\alpha_j +\left[\sum_{m=1}^{N+M}A^{(I)}(s_m,x_i)-A^{(I)}(s_i,x_i)\right]\alpha_i\,. \tag{10}$$

Similarly, the boundary flux is obtained as

$$\psi(x_i)=\sum_{j=1}^{N+M}B^{(I)}(s_j,x_i)\alpha_j-\sum_{j=1}^{N+M}B^{(E)}(s_j,x_i)\alpha_i\,, \tag{11}$$

in which

$$\sum_{j=1}^{N+M}B^{(E)}(s_j,x_i)\alpha_i=0\,,\quad \mathrm{x_i}\in\mathrm{B}\,. \tag{12}$$

The detailed derivation of Eq. (12) is given in the reference [10]. Therefore, we can obtain

$$\psi(x_i)=\sum_{j=1}^{i-1}B^{(I)}(s_j,x_i)\alpha_j+\sum_{j=i+1}^{N+M}B^{(I)}(s_j,x_i)\alpha_j -\left[\sum_{m=1}^{N+M}B^{(I)}(s_m,x_i)-B^{(I)}(s_i,x_i)\right]\alpha_i\,. \tag{13}$$

According to the dependence of the normal vectors for inner and outer boundaries [13, 14], their relationships are

$$\begin{cases} A^{(I)}(s_j,x_i)=-A^{(E)}(s_j,x_i), & i\neq j \\ A^{(I)}(s_j,x_i)=A^{(E)}(s_j,x_i), & i=j \end{cases} \tag{14}$$

$$\begin{cases} B^{(I)}(s_j,x_i)=B^{(E)}(s_j,x_i), & i\neq j \\ B^{(I)}(s_j,x_i)=B^{(E)}(s_j,x_i), & i=j \end{cases} \tag{15}$$

where the left-hand and right-hand sides of the equal sign in Eqs.(14) and (15) denote the kernels for observation and source point with the inward and outward normal vectors, respectively.

By using the proposed technique, the singular terms in Eqs. (4) and (5) have been transformed into regular terms $\left[\sum_{m=1}^{N+M}A^{(I)}(s_m,x_i)-A^{(I)}(s_i,x_i)\right]$ and $-\left[\sum_{m=1}^{N+M}B^{(I)}(s_m,x_i)-B^{(I)}(s_i,x_i)\right]$ in Eqs. (10) and (13), respectively. The terms of

$\sum_{m=1}^{N+M} A^{(I)}(s_m, x_i)$ and $\sum_{m=1}^{N+M} B^{(I)}(s_m, x_i)$ are the adding-back terms and the terms of $A^{(I)}(s_i, x_i)$ and $B^{(I)}(s_i, x_i)$ are the subtracting terms in two brackets for the special treatment technique. After using the abovementioned method of regularization of subtracting and adding-back technique [13, 14], we are able to remove the singularity and hypersingularity of the kernel functions.

2.2.2 Derivation of diagonal coefficients of influence matrices

We can obtain the following linear algebraic system after collocating N observation points on B_1 and M observation points on B_2, $\{x_i\}_{i=1}^{N+M}$, in Eq. (10) as

$$\left\{\begin{matrix}\left\{\begin{matrix}\overline{\phi}_1\\ \vdots \\ \overline{\phi}_N\end{matrix}\right\}\\ \left\{\begin{matrix}\phi_{N+1}\\ \vdots \\ \phi_{N+M}\end{matrix}\right\}\end{matrix}\right\}_{(N+M)\times 1} = \begin{bmatrix}[A_1]_{N\times(N+M)}\\ [A_2]_{M\times(N+M)}\end{bmatrix}\left\{\begin{matrix}\left\{\begin{matrix}\alpha_1\\ \vdots \\ \alpha_N\end{matrix}\right\}\\ \left\{\begin{matrix}\alpha_{N+1}\\ \vdots \\ \alpha_{N+M}\end{matrix}\right\}\end{matrix}\right\}_{(N+M)\times 1}, \tag{16}$$

where

$$[A_1] = \begin{bmatrix} \sum_{m=1}^{N+M} a_{1,m} - a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} & \cdots & a_{1,N+M} \\ a_{2,1} & \sum_{m=1}^{N+M} a_{2,m} - a_{2,2} & \cdots & \cdots & a_{2,N} & \cdots & a_{2,N+M} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & \cdots & \sum_{m=1}^{N+M} a_{N,m} - a_{N,N} & \cdots & a_{N,N+M} \end{bmatrix}_{N\times(N+M)}, \tag{17}$$

$$[A_2] = \begin{bmatrix} a_{N+1,1} & \cdots & \sum_{m=1}^{N+M} a_{N+1,m} - a_{N+1,N+1} & \cdots & \cdots & a_{N+1,N+M} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{N+M,1} & \cdots & a_{N+M,N+1} & \cdots & \cdots & \sum_{m=1}^{N+M} a_{N+M,m} - a_{N+M,N+M} \end{bmatrix}_{M\times(N+M)}, \tag{18}$$

in which

$$a_{ij} = A^{(I)}(s_j, x_i), \quad i, j = 1,2,\cdots,N+M. \tag{19}$$

In a similar way, Eq. (13) yield

$$\left\{\begin{matrix}\left\{\begin{matrix}\overline{\psi}_1\\ \vdots \\ \overline{\psi}_N\end{matrix}\right\}\\ \left\{\begin{matrix}\psi_{N+1}\\ \vdots \\ \psi_{N+M}\end{matrix}\right\}\end{matrix}\right\}_{(N+M)\times 1} = \begin{bmatrix}[B_1]_{N\times(N+M)}\\ [B_2]_{M\times(N+M)}\end{bmatrix}\left\{\begin{matrix}\left\{\begin{matrix}\alpha_1\\ \vdots \\ \alpha_N\end{matrix}\right\}\\ \left\{\begin{matrix}\alpha_{N+1}\\ \vdots \\ \alpha_{N+M}\end{matrix}\right\}\end{matrix}\right\}_{(N+M)\times 1}, \tag{20}$$

where

$$[B_1] = \begin{bmatrix} -\left[\sum_{m=1}^{N+M} b_{1,m} - b_{1,1}\right] & b_{1,2} & \cdots & \cdots & b_{1,N} & \cdots & b_{1,N+M} \\ b_{2,1} & -\left[\sum_{m=1}^{N+M} b_{2,m} - b_{2,2}\right] & \cdots & \cdots & b_{2,N} & \cdots & b_{2,N+M} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ b_{N,1} & b_{N,2} & \cdots & \cdots & -\left[\sum_{m=1}^{N+M} b_{N,m} - b_{N,N}\right] & \cdots & b_{N,N+M} \end{bmatrix}_{N\times(N+M)}, \tag{21}$$

$$[B_2] = \begin{bmatrix} b_{N+1,1} & \cdots & -\left[\sum_{m=1}^{N+M} b_{N+1,m} - b_{N+1,N+1}\right] & \cdots & \cdots & b_{N+1,N+M} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{N+M,1} & \cdots & b_{N+M,N+1} & \cdots & \cdots & -\left[\sum_{m=1}^{N+M} b_{N+M,m} - b_{N+M,N+M}\right] \end{bmatrix}_{M\times(N+M)}, \tag{22}$$

in which

$$b_{ij} = B^{(I)}(s_j, x_i), \quad i, j = 1,2,\ldots,N+M. \tag{23}$$

2.2.3 Derivation of influence matrices

Rearrange the influence matrices of Eqs. (16) and (20) into the linear algebraic system as

$$\left\{\begin{matrix}\{\overline{\phi}_1\}_{N\times 1}\\ \{\overline{\psi}_1\}_{N\times 1}\end{matrix}\right\} = \begin{bmatrix}[A_1]_{N\times(N+M)}\\ [B_1]_{N\times(N+M)}\end{bmatrix}\{\alpha\}_{(N+M)\times 1}. \tag{24}$$

The linear algebraic system in Eq. (24) can be generally written as

$$D = CX. \tag{25}$$

For the inverse Laplace problem, the influence matrix C is often ill-posed such that the regularization technique in section 2.3 elaborately which regularizes the influence matrix is necessary.

2.3 Regularization techniques

2.3.1 TSVD method

In the singular value decomposition (SVD), the matrix C is decomposed into

$$C = [U][\Sigma][V]^T, \tag{26}$$

where $[U] = [u_1, u_2, \ldots, u_m]$ and $[V] = [v_1, v_2, \ldots, v_m]$ are column orthonormal matrices, with column vectors called left and right singular vectors, respectively, T denote the matrix transposition, and $[\Sigma] = diag(\sigma_1, \sigma_2, \ldots, \sigma_m)$ is a diagonal matrix with nonnegative diagonal elements in nonincreasing order, which are the singular values of C.

A convenient measure of the conditioning of the matrix C is the condition number defined as

$$Cond = \sigma_1 / \sigma_m, \tag{27}$$

where σ_1 is the maximum singular value and σ_m is the minimum singular value i.e. the ratio between the largest singular value and the smallest singular value. By means of the SVD, the solution a^0 can be written as

$$a^0 = \sum_{i=1}^{k} (u_i^T d / \sigma_i) v_i, \tag{28}$$

where *k* is the rank of C, u_i is the element of the left singular vector and v_i is the element of the right singular vector. For an ill-conditioned matrix, there are small singular values, therefore the solution is dominated by contributions from small singular values when the noise is present in the data. One simple remedy to treat the difficulty is to leave out contributions from small singular values, i.e. taking a^p as an approximate solution, where a^p is defined as

$$a^p = \sum_{i=1}^{p}(u_i^T d / \sigma_i)v_i \ , \tag{29}$$

where $p \le k$ is the regularization parameter, which determines when one starts to leave out small singular values. Note that if $p = k$, the approximate solution is exactly the least squares solution. This method is known as TSVD in the inverse problem community [7].

2.3.2 Tikhonov regularization technique

Tikhonov [2, 9] proposed a method to transform an ill-posed problem into a well-posed one. Instead of solving Eq. (25) directly, the solution of Tikhonov regularized as following:

$$f_\lambda(X_\lambda) = \min_{X \in R^M} f_\lambda(X) \ , \tag{30}$$

where the λ is the regularization parameter and f_λ is the *k*th order Tikhonov function as given

$$f_\lambda(X) = \left\|CX - D\right\|^2 + \lambda^2 \left\|R^{(k)}X\right\|^2, \quad R^{(k)} \in R_{(M-k)\times M} \ . \tag{31}$$

Solving $\nabla f_\lambda(X) = 0$, we can obtain the Tikhonov regularized solution X_λ of the Eq. (30) is given as the solution of the regularized equation

$$\left(C^T C + \lambda^2 R^{(k)T} R^{(k)}\right) X = C^T D \ , \tag{32}$$

where *T* denotes matrix transposition.

In this paper, the zeroth-order and first-order Tikhonov regularization method are considered, respectively. The matrix $R^{(0)}$ and $R^{(1)}$ of zeroth-order and first-order Tikhonov regularization method is given by

$$R^{(0)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{M \times M} \ , \tag{33}$$

$$R^{(1)} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(M-1) \times M} \ . \tag{34}$$

An ill-posed matrix will be transformed into a well a one by employing the proposed regularization techniques. If too much regularization, i.e. λ is larger, the solution will be too smoothing. If too little regularization, i.e. λ is small, the solution will be unreasonable by the contributions from the input data with perturbation error in measurements. The choice of the parameter λ in Eq. (32) is vital for obtaining a reasonable and convergent solution and this is obtained on next section.

2.4 Error estimation techniques

2.4.1 L-curve technique

The L-curve [2, 5, 9] is a log-log plot of the norm of a regularized solution versus the norm of the corresponding residual norm. The norm of a regularized solution is defined as

$$Log\left\|CX - D\right\|^2 \ , \tag{35}$$

and norm of the corresponding residual norm as following

$$Log\left\|X\right\|^2 \ , \tag{36}$$

Because of the corner point of the L-curve is not a local minimum norm. Therefore, the optimal regularization parameter need carefully chosen.

2.4.2 Adaptive error technique

The DMM in conjunction with adaptive error technique can be obtained optimal parameter $\lambda_{opt}^{(ae)}$ without analytical solution. This technique is a log-log plot of error as the *y*-axis versus regularization parameter $\lambda^{(ae)}$ as the *x*-axis. The error is defined as

$$error = \left\|\overline{\phi}_1(x) - \overline{\overline{\phi}}_1(x)\right\| = \int \left|\overline{\phi}_1(x) - \overline{\overline{\phi}}_1(x)\right|^2 dB_1 \ , \quad x \in B_1 \ , \tag{37}$$

where $\overline{\phi}_1(x)$ is original known boundary condition on the boundary B_1 and $\overline{\overline{\phi}}_1(x)$ is result of calculated again by using DMM. The use steps of Eq. (37) are

1. $\overline{\phi}_1(x)$ and $\overline{\psi}_1(x)$ are original known boundary conditions on the boundary B_1.
2. Used boundary condition $\overline{\phi}_1(x)$, we can obtain the unknown boundary condition $\phi_2(x)$ on the boundary B_2 by using the DMM in conjunction with Tikhonov regularization method.
3. We can employ boundary conditions $\overline{\psi}_1(x)$ and $\phi_2(x)$ to obtain new boundary condition $\overline{\overline{\phi}}_1(x)$ on the boundary B_1 by using DMM and compute error by using Eq. (37).
4. Repeated step 1 to step 3 and drafting, we can obtain optimal parameter $\lambda_{opt}^{(ae)}$ on the corner of curved line.

To express the formulation in the section 2.1 to section 2.4, the flowchart of solution procedures is shown in Fig. 2.

3. NNMERICAL EXAMPLES

To show the accuracy and validity of the proposed method and obtain a better regularizationr technique and error estimation technique, two cases containing the square domain and infinite strip domain with finite thickness subjected to the overspecified boundary conditions, are considered, respectively.

Case 1: Square domain

The square domain of the inverse problem and overspecified boundary conditions are given as shown in Fig. 3. The length of square domain is $L = 1.0$. To found out a better regularization method, the L2 norm error estimation comparing with analytical solution is implemented to determine the optimal parameter in this case. The L2 norm error is defined as

$$norm \quad error = \left\|\phi - \phi_{exact}\right\| = \int_a^b \left|\phi - \phi_{exact}\right|^2 dx \ , \qquad (38)$$

where ϕ and ϕ_{exact} are the numerical result and analytical solution, respectively. The L2 norm against regularization parameter, λ (or i) is shown in Fig. 4 (a)~(c) by using the proposed regularization techniques, TSVD method, 0^{th}-order and 1^{th}-order Tikhonov regularization methods, respectively, after distributing 400 nodes. The optimal values are 194, 0.0018 and 11.7, respectively. The results for the three optimal values are plotted in Fig. 5. From Fig. 5 the result by using the 1^{th}-order Tikhonov regularization method is more accurate better than other methods. The result of absolute error with the exact solution is shown in Fig. 6. To see convergent analysis as shown in Fig. 7 and convergent result is obtained after over 200 points are distributed. In this case, the better method can be obtained to remedy ill-posed problems and it is the 1^{th}-order Tikhonov regularization method.

Case 2: Infinite strip domain

The infinite strip region of the inverse problem and overspecified boundary conditions are given as shown in Fig. 8 and the square wave is specified on the bottom of infinite strip region is given. To obtain the optimal parameter λ of the 1^{th}-order Tikhonov regularization method (the better regularization method in case 1), the analysis of error estimation is shown in Fig. 9 (a)~(b) by employing the proposed techniques after by distributing 200 nodes. The optimal parameters are 0.00025 and 0.00086, respectively, for the different error estimation. A better result is observed in Fig. 10. We can find the result of the adaptive error estimation technique is more accurate better then the L-curve technique. The results of absolute error with exact solution are shown in Fig. 11. To see convergent analysis as show in Fig. 12 and convergent result is obtained after over 200 points are distributed. In this case, the better error estimation technique is the adaptive error estimation technique to obtain the optimal parameter of the 1^{th}-order Tikhonov regularization method, if analytical solution is not employing.

4. CONCLUSION

In this paper, we successfully applied the desingular meshless method to solve inverse problems with Laplace equation. The source and collocation points can be located on real boundary at the same time by using the proposed desingularization technique. The better regularization method and error estimation technique are obtained by giving the numerical evidences.

5. ACKNOWLEDGEMENT

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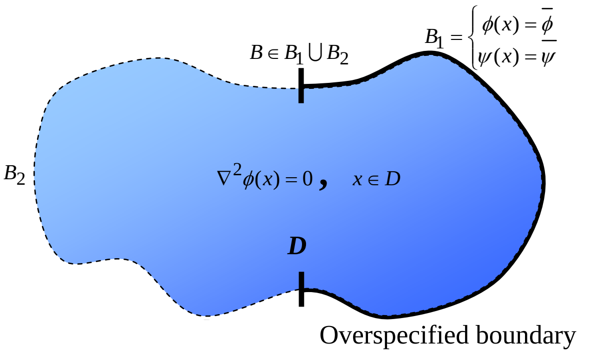

Figure 1 Problem sketch for inverse Laplace problem.

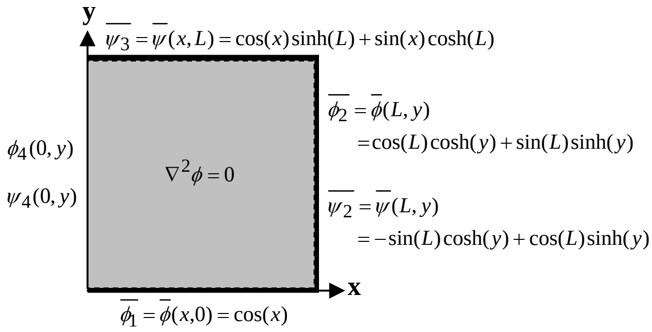

Figure 3 Problem sketch for the case 1.

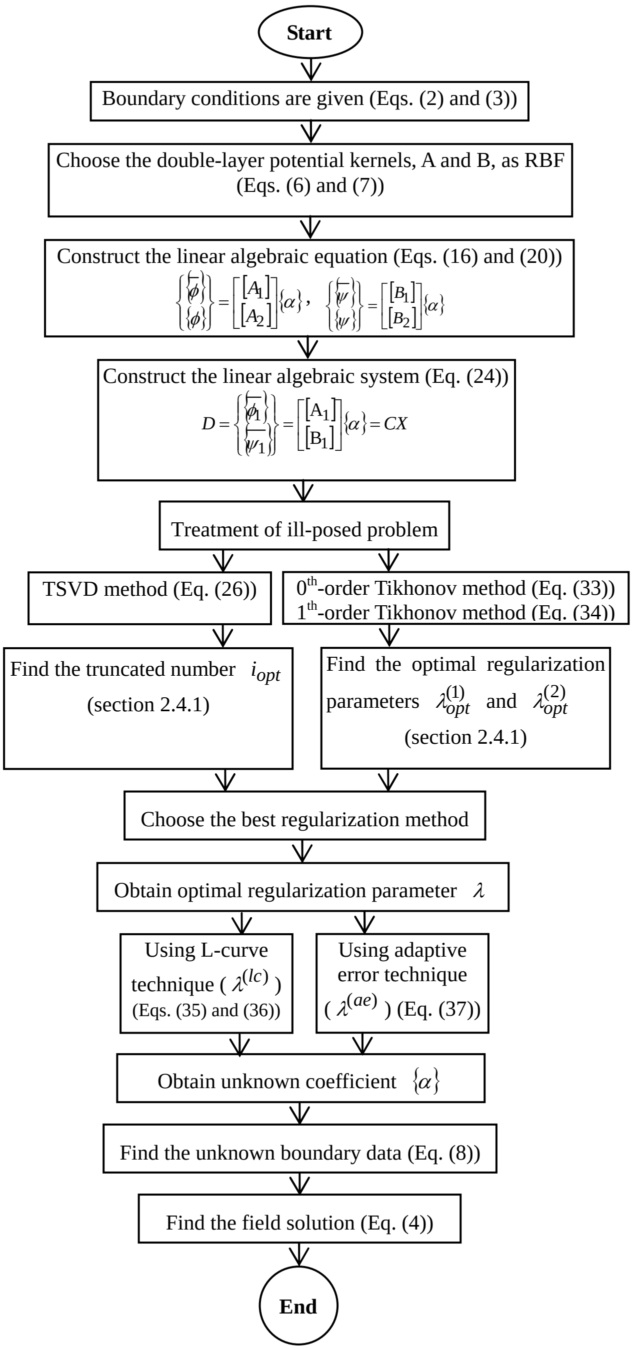

Figure 2 Flowchart of solution procedures.

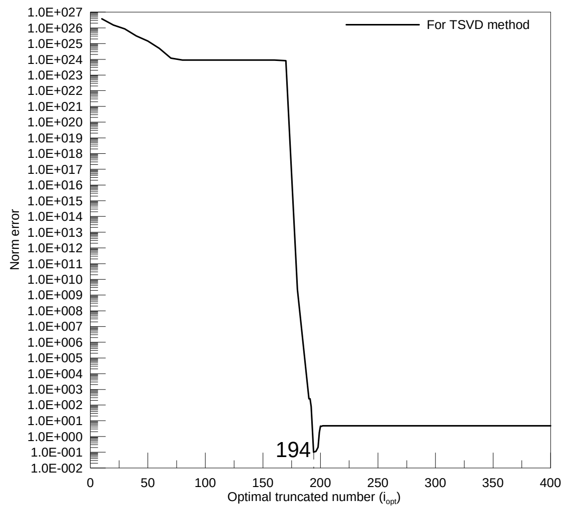

Figure 4 (a)

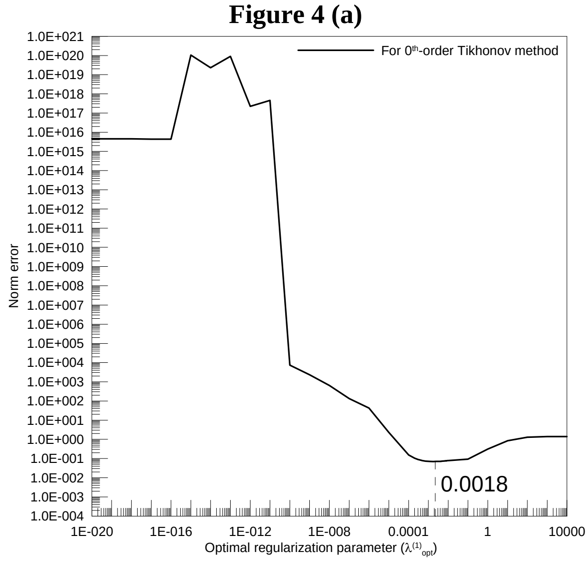

Figure 4 (b)

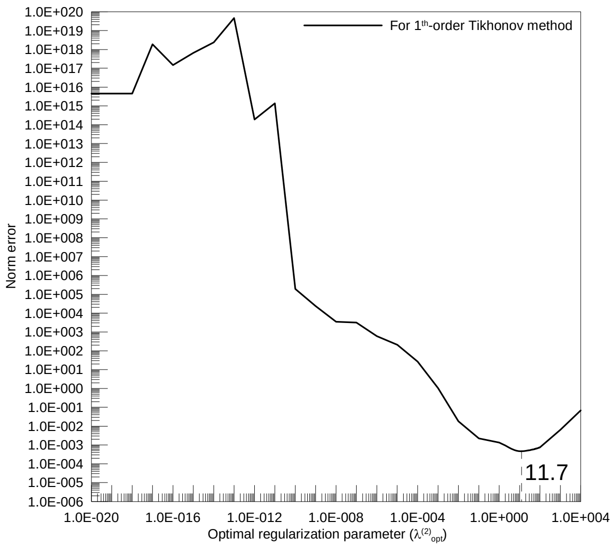

Figure 4 (c)

Figure 4 Optimal truncated number and regularization parameter for (a) TSVD method, (b) 0th-order Tikhonov method, (c) 1th-order Tikhonov method.

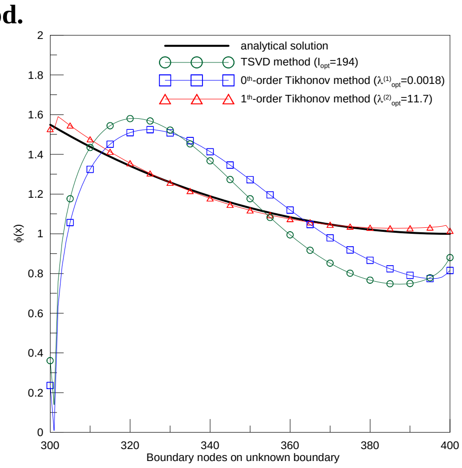

Figure 5 Numerical result of employing TSVD method, 0th-order and 1th-order Tikhonov methods, respectively, by using 400 nodes for the case 1.

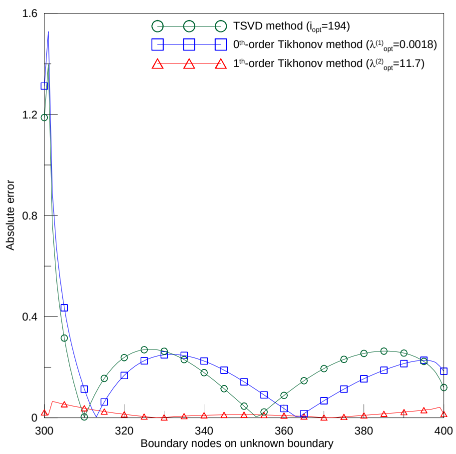

Figure 6 Absolute error with the exact solution of employing three regularization techniques by using 400 nodes for the case 1.

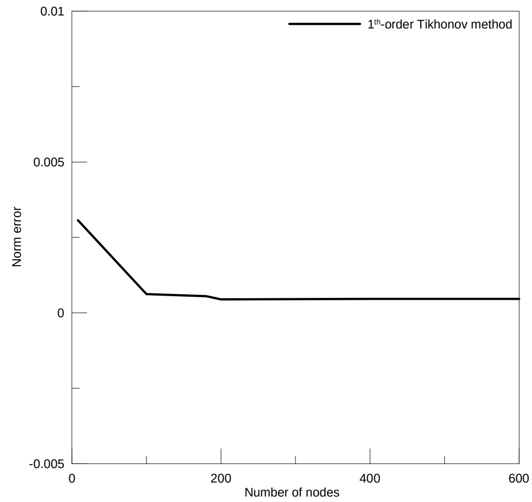

Figure 7 The norm error along the boundary versus the number of nodes by using 1th-order Tikhonov method for the case 1.

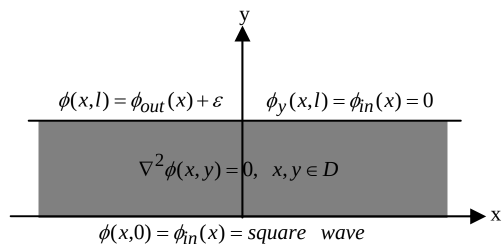

Figure 8 Problem sketch of infinite strip with finite thickness problem for the case 2.

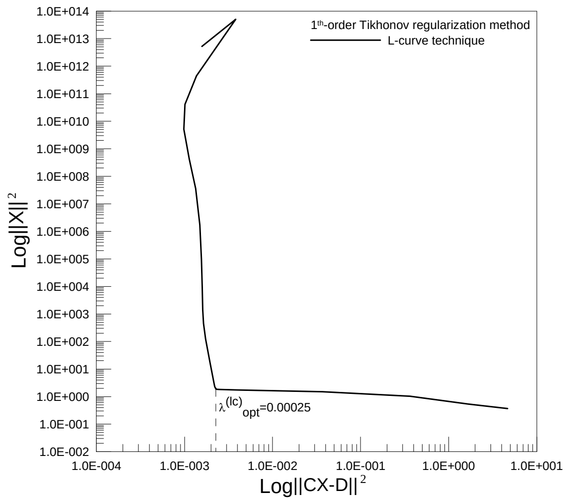

Figure 9 (a)

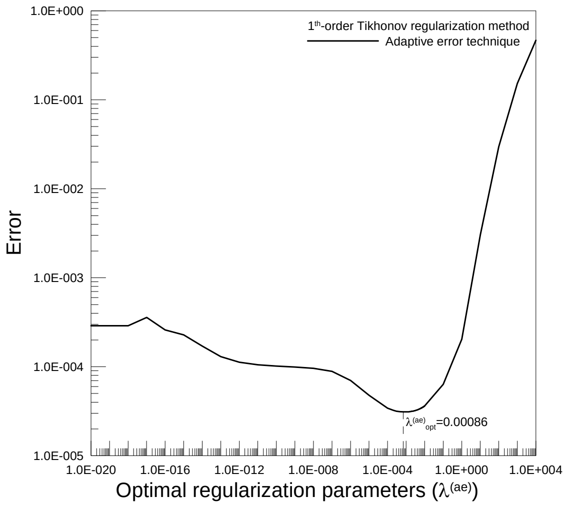

Figure 9 (b)

Figure 9 Optimal regularization parameters by using (a) L-curve technique, (b) adaptive error technique, for the case 2.

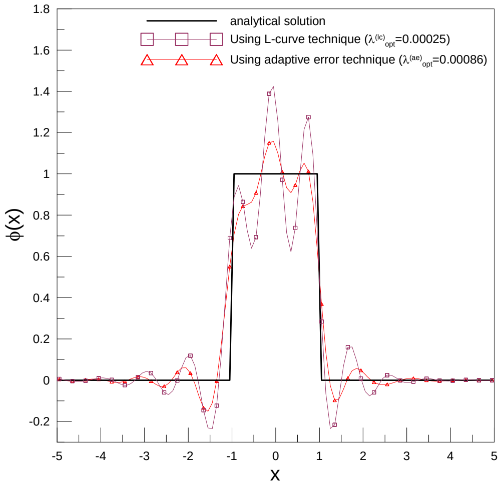

Figure 10 Numerical result of employing L-curve and adaptive error techniques, respectively, by using 200 nodes for the case 2.

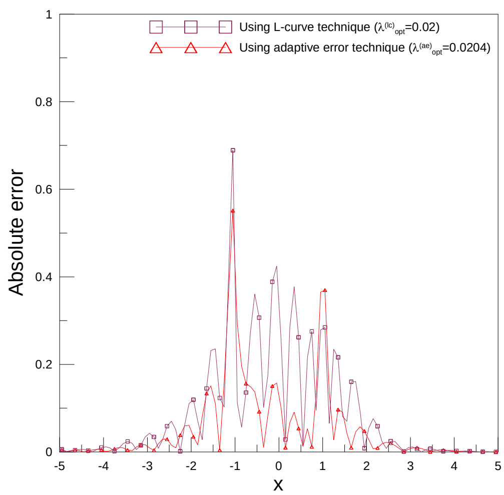

Figure 11 Absolute error with the exact solution of employing L-curve and adaptive error techniques, respectively, by using 200 nodes for the case 2.

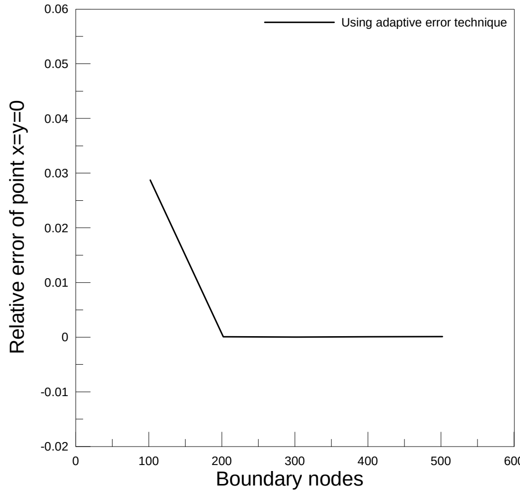

Figure 12 The convergent analysis by using adaptive error technique for the case 2.

去奇異無網格法結合正規化方法之自適性誤差評估於求解含過定邊界條件之反算問題

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摘要

本文是應用去奇異無網格法求解含過定邊界條件之拉普拉斯方程的發散問題。本法可將奇異源直接放在真實邊界上，藉由一加一減的技巧來正規化處理奇異及超奇異核函數。本文的主要貢獻是由兩部分所組成。第一貢獻：在反算問題中會發生的病態問題將被治療成功，分別使用建議的正規化方法－截取奇異值分解法、第零階與第一階 Tikhonov 正規化方法。第二貢獻：L 曲線技術與新發展的自適性誤差評估技術提供在不與解析解比較的情況下獲得最佳參數。數值結果在與解析解做比較及討論後將證明此結果的正確性。最後，最好的正規化方法與差評估技術將在本文中提出。

關鍵詞：去奇異無網格法，截取奇異值分解法，Tikhonov 正規化法，L 曲線技術，自適性誤差評估，一加一減法。