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Relaxation of Alternating Iterative Algorithms for the Cauchy Problem Associated with the Modified Helmholtz Equation

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Abstract: We propose two algorithms involving the relaxation of either the given Dirichlet data or the prescribed Neumann data on the over-specified boundary, in the case of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991) applied to Cauchy problems for the modified Helmholtz equation. A convergence proof of these relaxation methods is given, along with a stopping criterion. The numerical results obtained using these procedures, in conjunction with the boundary element method (BEM), show the numerical stability, convergence, consistency and computational efficiency of the proposed methods.

Keywords: Helmholtz Equation; Inverse Problem; Cauchy Problem; Alternating Iterative Algorithms; Relaxation Procedure; Boundary Element Method (BEM).

1 Introduction

Helmholtz-type equations characterise many physical applications related to wave propagation and vibration phenomena, as well as heat conduction problems. These equations are often used to describe, for example, the vibration of a structure [Beskos (1997)], the acoustic cavity problem [Chen and Wong (1998)], the radiation wave [Harari, Barbone, Slavutin and Shalom (1998)], the scattering of a wave [Hall and Mao (1995)], the problem of heat conduction in fins [Kraus, Aziz and Welty (2001)], Debye-Hückel theory [Debye and Hückel (1923)], the linearization of the Poisson-Boltzmann equation [Liang and Subramaniam (1997)]. In many engineering problems, either the boundary conditions are incomplete, or the geometry of the domain under investigation is not completely known, or the so-called wave number, $\kappa > 0$, that characterises the Helmholtz-type equation is unknown.

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These are *inverse problems* and it is well-known that they are generally ill-posed, in the sense that the existence, uniqueness and stability of their solutions are not always guaranteed, see e.g. Hadamard (1923).

Inverse problems are currently a very active research area, in particular for parameter, boundary and data reconstructions in solid mechanics and heat transfer, see e.g. Huang and Wu (2007), Qian, Cao, Zhang, Raabe, Yao and Fei (2008), Liu (2008a,b), Liu, Chang, Chang and Chen (2008), Liu, Chen and Chang (2009), Marin and Karageorghis (2009), Marin (2009b), Silieti, Divo and Kassab (2009). A classical example of an inverse boundary value problem associated with Helmholtz-type equations is represented by the *Cauchy problem*. In this case, the boundary conditions are incomplete, in the sense that a part of the boundary of the solution domain is over-specified by prescribing on it both the primary field and its normal derivative, while the remaining boundary is under-specified and the boundary conditions on the latter boundary have to be determined. The uniqueness of the Cauchy problem is guaranteed without the necessity of removing the eigenvalues for the Laplacian operator, as it happens in the case of direct problems for the Helmholtz equation, see e.g. Chen and Zhou (1992). However, the Cauchy problem suffers from the possible non-existence and instability of the solution.

Over the last decade, many theoretical and numerical studies have been devoted to the Cauchy problem associated with Helmholtz-type equations. DeLillo, Isakov, Valdivia and Wang (2001) detected the source of acoustical noise inside the cabin of a midsize aircraft from measurements of the acoustical pressure field inside the cabin by solving a linear Fredholm integral equation of the first kind and they extended this study to three-dimensional problems, see DeLillo, Isakov, Valdivia and Wang (2003). The alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991), which reduces the Cauchy problem to solving a sequence of well-posed boundary value problems, was implemented numerically using the boundary element method (BEM) for the two-dimensional modified Helmholtz equation by Marin, Elliott, Heggs, Ingham, Lesnic and Wen (2003a). Marin, Elliott, Heggs, Ingham, Lesnic and Wen (2003b) used the conjugate gradient method (CGM), in conjunction with the BEM, in order to solve the same inverse problem for both the Helmholtz and the modified Helmholtz equations. Four regularization methods for the stable solution of the Cauchy problem associated with Helmholtz-type equations, namely the Tikhonov regularization, the singular value decomposition (SVD), the CGM and the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991), were compared by Marin, Elliott, Heggs, Ingham, Lesnic and Wen (2004a). The Landweber-Fridman method and the BEM were used to solve the Cauchy problem for two-dimensional Helmholtz and modified Helmholtz equations with L^2-boundary data by Marin, Elliott, Heggs, Ingham, Lesnic and Wen

(2004b). Jin and Zheng (2005a) solved some inverse boundary value problems for the Helmholtz equation using the boundary knot method and a SVD regularization and they also extended this method to some inverse problems associated with the inhomogeneous Helmholtz equation Jin and Zheng (2005b). The numerical solution for the Cauchy problem for two- and three-dimensional Helmholtz-type equations by employing the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method and SVD, was investigated by Marin and Lesnic (2005) and Marin (2005), and Jin and Zheng (2006), respectively. Tumakov (2006) addressed the use of the Fourier transformation method for the Cauchy problem for the Helmholtz equation. Some spectral regularization methods and a modified Tikhonov regularization method to stabilize the Cauchy problem for the Helmholtz equation at fixed frequency were proposed by Xiong and Fu (2007), while Jin and Marin (2008) employed the plane wave method and the SVD to solve stably the same problem. Wei, Qin and Shi (2008), Qin and Wen (2009) and Qin, Wei and Shi (2009) reduced the Cauchy problem associated with Helmholtz-type equations to a moment problem and also provided an error estimate and convergence analysis for the latter. Qin and Wei (2009a, 2010) proposed two regularization methods, namely a modified Tikhonov regularization method and a truncation method, for the stable approximate solution to the Cauchy problem for the Helmholtz equation and they also presented convergence and stability results under suitable choices of the regularization parameter. The quasi-reversibility method and a truncation method were used to solve the Cauchy problem for the modified Helmholtz equation in a rectangular domain by Qin and Wei (2009b), who also analysed the stability and convergence of the proposed regularization procedures. Shi, Wei and Qin (2009) addressed a fourth-order modified method for the solution of the Cauchy problem associated with the modified Helmholtz equation in an infinite strip domain and they also provided convergence estimates under the suitable choices of regularization parameters and the a priori assumption on the bounds of the exact solution. Recently, the Cauchy problem for two-dimensional Helmholtz-type equations with L^2-boundary data was approached by combining the BEM with the minimal error method by Marin (2009a).

Jourhmane and Nachaoui (2002) and Jourhmane, Lesnic and Mera (2004) proposed the relaxation of the given Dirichlet data in the case of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991) applied to the Cauchy problem for steady-state heat conduction in isotropic and anisotropic media, respectively. This procedure drastically reduced the number of iterations required to achieve convergence for the inverse problems considered. Recently, a relaxation of the alternating method in elasticity was both numerically and theoretically investigated, see Ellabib and Nachaoui (2008) and Marin and Johansson (2010).

Encouraged by their results, we do further investigations and propose a relaxation of both the measured temperature and the prescribed normal heat flux on the over-specified boundary, in the case of the modified Helmholtz equation. Moreover, we also prove the convergence of these schemes and introduce appropriate optimal stopping rules.

The paper is organized as follows: Section 2 is devoted to the mathematical formulation of the inverse problem investigated, as well as the introduction of the function spaces used herein. The alternating iterative algorithms with relaxation for the Cauchy problem associated with the modified Helmholtz equation are then presented in Section 3, while the proof of the convergence theorem for this procedure is given in Section 4. The implementation of the proposed numerical method is realized using the BEM and this is briefly discussed in Section 5. In Section 6, the algorithms introduced in Section 3 are applied to solving three Cauchy problems with exact and noisy Cauchy data. Finally, some concluding remarks and possible future work are provided in Section 7.

2 Mathematical formulation

2.1 Notation and function spaces

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, where d is the dimension of the space where the problem is posed, usually $d \in \{1,2,3\}$. Let $\Gamma_0 \neq \varnothing$ be an arc of $\partial\Omega$ having non-zero length and set $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}_0$. Let $\mathrm{H}^1(\Omega)$ be the Sobolev space of real-valued functions in Ω endowed with the standard norm. We denote by $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}^1_{\Gamma_i}(\Omega)$, $i=0,1$, the subspaces of functions from $\mathrm{H}^1(\Omega)$ that vanish on $\partial\Omega$ and Γ_i, $i=0,1$, respectively.

The space of traces of functions from $\mathrm{H}^1(\Omega)$ to $\partial\Omega$ is denoted by $\mathrm{H}^{1/2}(\partial\Omega)$, while the restrictions of the functions belonging to the space $\mathrm{H}^{1/2}(\partial\Omega)$ to the subset $\Gamma_i \subset \partial\Omega$, $i=0,1$, define the space $\mathrm{H}^{1/2}(\Gamma_i)$, $i=0,1$. The set of real valued functions in $\partial\Omega$ with compact support in Γ_i, $i=0,1$, and bounded first-order derivatives are dense in $\mathrm{H}^{1/2}(\Gamma_i)$, $i=0,1$. Furthermore, we also define the space $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i=0,1$, that consists of functions from $\mathrm{H}^{1/2}(\partial\Omega)$ and vanishing on Γ_{1-i}, $i=0,1$. The space $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i=0,1$, is a subspace of $\mathrm{H}^{1/2}(\partial\Omega)$ with the norm given by:

$$\|f\|_{\mathrm{H}^{1/2}_{00}(\Gamma_i)} = \left(\int_{\Gamma_i} \frac{f^2(\mathbf{x})}{\operatorname{dist}(\mathbf{x},\Gamma_i)} \,\mathrm{d}\Gamma(\mathbf{x}) + \int_{\Gamma_i}\int_{\Gamma_i} \frac{|f(\mathbf{x})-f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^d} \,\mathrm{d}\Gamma(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{y}) \right)^{1/2}. \quad (1)$$

It should be mentioned that the space of restrictions from $\mathrm{H}^{1/2}_{00}(\Gamma_i)$ to Γ_i, $i=0,1$, is dense in $\mathrm{H}^{1/2}(\Gamma_i)$, $i=0,1$. Nonetheless, $\mathrm{H}^{1/2}_{00}(\Gamma_i) \neq \mathrm{H}^{1/2}(\Gamma_i)$. Finally, we denote by $\left(\mathrm{H}^{1/2}_{00}(\Gamma_i)\right)^*$ the dual space of $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i=0,1$.

2.2 The Cauchy problem

Helmholtz equations arise naturally in many physical applications, some examples being the vibration of a structure, the acoustic cavity problem, the radiation wave, the scattering of a wave, the heat conduction in fins, Debye-Hückel theory and the linearization of the Poisson-Boltzmann equation. In this work, in order to refer to a specific physical problem, we shall consider Helmholtz problems in the context of heat transfer problems, see e.g. Kraus, Aziz and Welty (2001). We therefore assume that the temperature field, $\mathrm{u}(\mathbf{x})$, satisfies the modified Helmholtz equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, namely

$$\mathscr{L}\mathrm{u}(\mathbf{x}) \equiv \left(\Delta - \kappa^2\right)\mathrm{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{2}$$

where $\kappa > 0$. The partial differential equation (2) models the heat conduction in a fin where u is the dimensionless local fin temperature, $\kappa^2 = h/\left(\widetilde{k}\ \delta_f\right)$, h is the surface heat transfer coefficient $[\mathrm{W}/\left(\mathrm{m}^2\mathrm{K}\right)]$, $\widetilde{k}$ is the thermal conductivity of the fin $[\mathrm{W}/\left(\mathrm{mK}\right)]$ and δ_f is the half-fin thickness [m].

We now let $\mathbf{n}(\mathbf{x}) = \left(\mathrm{n}_1(\mathbf{x}), \ldots, \mathrm{n}_d(\mathbf{x})\right)^{\mathsf{T}}$ be the outward unit normal vector at $\mathbf{x} \in \partial\Omega$ and $\mathscr{N}\mathrm{u}(\mathbf{x}) \equiv \mathrm{q}(\mathbf{x}) = \nabla\mathrm{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$, where $\mathscr{N}$ is the boundary-differential operator associated with the modified Helmholtz differential operator, $\mathscr{L} \equiv \Delta - \kappa^2$, and Neumann boundary conditions on $\partial\Omega$. In the direct problem formulation, the knowledge of the constant κ, the location, shape and size of the entire boundary $\partial\Omega$, the temperature and/or the normal heat flux on the entire boundary $\partial\Omega$ gives the corresponding Dirichlet, Neumann, or mixed boundary conditions, which enable one to determine the unknown boundary conditions, as well as the temperature distribution in the solution domain.

A different and more interesting situation arises when it is possible to measure both the temperature and the normal heat flux on a part of the boundary $\partial\Omega$, say Γ_0, and this leads to the mathematical formulation of the Cauchy problem consisting of the partial differential equation (2) and the boundary conditions

$$\mathrm{u}(\mathbf{x}) = \varphi(\mathbf{x}), \qquad \mathscr{N}\mathrm{u}(\mathbf{x}) \equiv \mathrm{q}(\mathbf{x}) = \psi(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_0, \tag{3}$$

where $\varphi \in \mathrm{H}^{1/2}(\Gamma_0)$ and $\psi \in \left(\mathrm{H}^{1/2}_{00}(\Gamma_0)\right)^*$ are prescribed temperature and normal heat flux, respectively. In the above formulation of the boundary conditions (3), it can be seen that the boundary Γ_0 is over-specified by prescribing both the temperature $\mathrm{u}\big|_{\Gamma_0} = \varphi$ and the normal heat flux $\mathrm{q}\big|_{\Gamma_0} = \psi$, while the boundary Γ_1 is under-specified since both the temperature $\mathrm{u}\big|_{\Gamma_1}$ and the normal heat flux $\mathrm{q}\big|_{\Gamma_1}$ are unknown and have to be determined. We also assume that data are chosen such that there exists a solution to this Cauchy problem. This solution is unique according to the so-called unique continuation properties for elliptic equations.

The Cauchy problem for the modified Helmholtz equation is considerably more difficult to solve than the direct problem, both analytically and numerically, since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well-known that this solution is unstable with respect to small perturbations in the data on Γ_1, see Hadamard (1923). Thus the problem is ill-posed and we cannot use a direct approach in order to solve the system of linear equations which arises from the discretization of the partial differential equation (2) and the boundary conditions (3).

3 Alternating iterative algorithms with relaxation

In this section we propose two alternating iterative algorithms with relaxation which aim to improve the computational time of the alternating iterative algorithm introduced by Kozlov, Maz′ya and Fomin (1991), at the same time maintaining the accuracy of the numerical results obtained with the latter.

Alternating iterative algorithm with relaxation I:

Step 1.1. If $k = 1$ then choose an arbitrary function $\xi^{(1)} \in \left(\mathrm{H}_{00}^{1/2}(\Gamma_1)\right)^*$.

Step 1.2. If $k \geq 2$ then solve the direct problem

$$\begin{cases} \mathscr{L}\mathrm{u}^{(2k-2)}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \mathrm{q}^{(2k-2)}(\mathbf{x}) \equiv \nabla \mathrm{u}^{(2k-2)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \psi(\mathbf{x}), & \mathbf{x} \in \Gamma_0, \\ \mathrm{u}^{(2k-2)}(\mathbf{x}) = \eta^{(k-1)}(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \end{cases} \tag{4}$$

where $\eta^{(k-1)}(\mathbf{x}) = \mathrm{u}^{(2k-3)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_1$, to obtain $\mathrm{u}^{(2k-2)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{q}^{(2k-2)}(\mathbf{x}) \equiv \nabla \mathrm{u}^{(2k-2)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \Gamma_1$.

Step 2. Provided that $k \geq 2$ update the unknown Neumann data on Γ_1 as:

$$\xi^{(k)}(\mathbf{x}) = \theta\, \mathrm{q}^{(2k-2)}(\mathbf{x}) + (1-\theta)\, \xi^{(k-1)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{5}$$

where the relaxation factor, $0 \leq \theta \leq 2$, is fixed. For $k \geq 1$ solve the direct problem

$$\begin{cases} \mathscr{L}\mathrm{u}^{(2k-1)}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \mathrm{u}^{(2k-1)}(\mathbf{x}) = \varphi(\mathbf{x}), & \mathbf{x} \in \Gamma_0, \\ \mathrm{q}^{(2k-1)}(\mathbf{x}) \equiv \nabla \mathrm{u}^{(2k-1)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \xi^{(k)}(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \end{cases} \tag{6}$$

to determine $\mathrm{u}^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{u}^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_1$.

Step 3. Set $k = k+1$ and repeat Steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 3.1 *The value* $\theta = 1$ *in Eqn.* (5) *corresponds to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Neumann data, whilst values* $\theta \in (0,1)$ *and* $\theta \in (1,2)$ *in Eqn.* (5) *correspond to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Neumann data and a constant under- and over-relaxation factor, respectively.*

Alternating iterative algorithm with relaxation II:

Step 1.1. If $k = 1$ then choose an arbitrary function $\eta^{(1)} \in \mathrm{H}^{1/2}(\Gamma_0)$.
Step 1.2. If $k \geq 2$ then solve the direct problem

$$\begin{cases} \mathscr{L}\mathrm{u}^{(2k-2)}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \mathrm{u}^{(2k-2)}(\mathbf{x}) = \varphi(\mathbf{x}), & \mathbf{x} \in \Gamma_0, \\ \mathrm{q}^{(2k-2)}(\mathbf{x}) \equiv \nabla \mathrm{u}^{(2k-2)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \xi^{(k-1)}(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \end{cases} \tag{7}$$

where $\xi^{(k-1)}(\mathbf{x}) = \mathrm{q}^{(2k-3)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_1$, to obtain $\mathrm{u}^{(2k-2)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{u}^{(2k-2)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_1$.
Step 2. Provided that $k \geq 2$ update the unknown Dirichlet data on Γ_1 as:

$$\eta^{(k)}(\mathbf{x}) = \theta\, \mathrm{u}^{(2k-2)}(\mathbf{x}) + (1-\theta)\eta^{(k-1)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{8}$$

where the relaxation factor, $0 \leq \theta \leq 2$, is fixed. For $k \geq 1$ solve the direct problem

$$\begin{cases} \mathscr{L}\mathrm{u}^{(2k-1)}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \mathrm{q}^{(2k-1)}(\mathbf{x}) \equiv \nabla \mathrm{u}^{(2k-1)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \psi(\mathbf{x}), & \mathbf{x} \in \Gamma_0, \\ \mathrm{u}^{(2k-1)}(\mathbf{x}) = \eta^{(k)}(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \end{cases} \tag{9}$$

to determine $\mathrm{u}^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{q}^{(2k-1)}(\mathbf{x}) \equiv \nabla \mathrm{u}^{(2k-1)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \Gamma_1$.

Step 3. Set $k = k+1$ and repeat Steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 3.2 *The value* $\theta = 1$ *in Eqn.* (8) *corresponds to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for* *the Dirichlet data, whilst values* $\theta \in (0,1)$ *and* $\theta \in (1,2)$ *in Eqn.* (8) *correspond to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Dirichlet data and a constant under- and over-relaxation factor, respectively.*

Remark 3.3 *In was reported in* Marin and Johansson (2010) *that, in the case of elasticity, both these relaxation versions numerically produced similar results and this turns out to be valid also in the case of the modified Helmholtz equation* (2)*. Thus, since it is easier and somewhat more natural to make a guess for function values on the boundary part* Γ_1 *than for normal derivatives on* Γ_1*, from now on, we shall mainly concentrate on producing theoretical and numerical results for the alternating iterative algorithm with relaxation II.*

4 Convergence of the alternating iterative algorithms with relaxation

Following the ideas of Jourhmane and Nachaoui (2002) we shall prove:

Theorem 4.1 *Let* $\varphi \in \mathrm{H}^{1/2}(\Gamma_0)$ *and* $\psi \in \left(\mathrm{H}^{1/2}_{00}(\Gamma_0)\right)^*$*. Assume that the Cauchy problem* (2) *and* (3) *has a solution* $\mathrm{u} \in \mathrm{H}^1(\Omega)$*. Let* $\mathrm{u}^{(k)}$ *be the k-th approximate solution in the alternating procedure II described above. Then there exists a number* $1 < b \leq 2$ *such that when the relaxation parameter* θ *is chosen with* $1 \leq \theta \leq b$*, then*

$$\lim_{k\to\infty} \left\|\mathrm{u} - \mathrm{u}^{(k)}\right\|_{\mathrm{H}^1(\Omega)} = 0 \tag{10}$$

for any initial data element $\eta^{(1)} \in \mathrm{H}^{1/2}(\Gamma_1)$.

To obtain a proof of this theorem we shall first rewrite the Cauchy problem (2) and for this we need to introduce an operator. To define this operator, let $\mathrm{u}^{(1)}$ be the solution to (9) with $k=1$ for given functions $\eta^{(1)} = \eta$ and $\psi = 0$. Let then $\mathrm{u}^{(2)}$ be the solution to (7) for $k=2$ with $\varphi = 0$ and $\xi^{(1)} = \mathrm{q}^{(2k-3)}$ on Γ_1. Define the linear operator $T_\theta : \mathrm{H}^{1/2}(\Gamma_1) \longrightarrow \mathrm{H}^{1/2}(\Gamma_1)$ for $\theta \geq 0$ by

$$T_\theta \eta = \theta\, \mathrm{u}^{(2)}(\eta)\big|_{\Gamma_1} + (1-\theta)\eta, \tag{11}$$

which is well-defined. In a similar way, let $\mathrm{v}^{(2)}$ be the element obtained from the second approximation in the alternating iterative algorithm with relaxation II, with the initial guess $\eta = 0$, and define the element $G_\theta(\varphi,\psi)$ by

$$G_\theta(\varphi,\psi) = \theta\, \mathrm{v}^{(2)}\big|_{\Gamma_1}. \tag{12}$$

The Cauchy problem (2) is equivalent to the fixed point equation

$$T_\theta \eta + G_\theta(\varphi, \psi) = \eta. \tag{13}$$

Thus, to show convergence of the procedure, we shall investigate the properties of the operator T_θ. We point out that to the authors' knowledge, in the previous works on alternating iterative methods, the operator T_θ has not been examined, and the reason for this is represented by the fact that, previously, mostly Neumann data have been the starting initial guess instead of the trace.

To find the properties of T_θ, we first introduce the bilinear form

$$a(\mathrm{u}, \mathrm{v}) = \int_\Omega \nabla \mathrm{u} \cdot \nabla \mathrm{v} \, d\mathbf{x} + \kappa^2 \int_\Omega \mathrm{u}\mathrm{v} \, d\mathbf{x}, \tag{14}$$

for $\mathrm{u}, \mathrm{v} \in \mathrm{H}^1(\Omega)$. We also define the following bilinear form in $\mathrm{H}^{1/2}(\Gamma_1)$

$$\big(\eta, \zeta\big) = a(\mathrm{u}, \mathrm{v}), \tag{15}$$

where u solves (9) with $\eta^{(k)} = \eta$ and $\psi = 0$, and similarly v solves (9) with $\eta^{(k)} = \zeta$ and $\psi = 0$, where η and ζ are in $\mathrm{H}^{1/2}(\Gamma_1)$. Since κ is a real number it is straightforward to check that $\big(\cdot, \cdot\big)$ is a well-defined inner product in $\mathrm{H}^{1/2}(\Gamma_1)$ and that the corresponding norm $\|\cdot\|$ is equivalent with the standard norm on $\mathrm{H}^{1/2}(\Gamma_1)$.

Lemma 4.1 *Let $T = T_1$, where T_1 is defined by* (11). *Then T is injective, self-adjoint, positive definite, non-expansive with respect to the inner product* (15)*, and one is not an eigenvalue.*

Proof: This follows using techniques from Kozlov and Maz′ya (1989) and Kozlov, Maz′ya and Fomin (1991). For the sake of completeness, we show that T is non-expansive; the other properties of T can be deduced in the similar way. Let $\mathrm{u}^{(k)}$ be generated from the alternating iterative algorithm II, with $\varphi = 0$ and $\psi = 0$. Since $\mathrm{u}^{(k)}|_{\Gamma_0} = 0$ or $\mathrm{q}^{(k)}|_{\Gamma_0} = 0$, for $k = 1, 2, 3$, we find using Green's formula

$$a\big(\mathrm{u}^{(k)}, \mathrm{u}^{(k)}\big) = \int_{\Gamma_1} \mathrm{u}^{(k)} \mathrm{q}^{(k)} \, d\Gamma(\mathbf{y}), \quad k = 1, 2, 3. \tag{16}$$

Similarly, since $\mathrm{q}^{(3)} = 0$ on Γ_0 and $\mathrm{u}^{(2)} = \mathrm{u}^{(3)}$ on Γ_1,

$$a\big(\mathrm{u}^{(2)}, \mathrm{u}^{(3)}\big) = \int_{\Gamma_1} \mathrm{u}^{(3)} \mathrm{q}^{(3)} \, d\Gamma(\mathbf{y}). \tag{17}$$

Using (16), we obtain, from Eqn. (17), $a\left(\mathrm{u}^{(2)},\mathrm{u}^{(3)}\right) = a\left(\mathrm{u}^{(3)},\mathrm{u}^{(3)}\right)$. From this,

$$a\left(\mathrm{u}^{(3)}-\mathrm{u}^{(2)},\mathrm{u}^{(3)}-\mathrm{u}^{(2)}\right) = a\left(\mathrm{u}^{(2)},\mathrm{u}^{(2)}\right) - a\left(\mathrm{u}^{(3)},\mathrm{u}^{(3)}\right). \tag{18}$$

In a similar way,

$$a\left(\mathrm{u}^{(2)}-\mathrm{u}^{(1)},\mathrm{u}^{(2)}-\mathrm{u}^{(1)}\right) = a\left(\mathrm{u}^{(1)},\mathrm{u}^{(1)}\right) - a\left(\mathrm{u}^{(2)},\mathrm{u}^{(2)}\right). \tag{19}$$

Since $\left(\eta,\eta\right) = a\left(\mathrm{u}^{(1)},\mathrm{u}^{(1)}\right)$ and $\left(T\eta,T\eta\right) = a\left(\mathrm{u}^{(3)},\mathrm{u}^{(3)}\right)$, where $\left(\cdot,\cdot\right)$ is defined by (15), combining (18) and (19) we have

$$\left(T\eta,T\eta\right) \leq \left\|\eta\right\|^2,$$

and thus T is non-expansive. Note that if the number one is an eigenvalue, then an eigenfunction u has $\mathrm{u} = 0$ and $\mathrm{q} = 0$ on Γ_0. Using the uniqueness of the Cauchy problem, we conclude that u is zero, hence one is not an eigenvalue, which implies that T has the norm less than unity.

We can now finish the proof of Theorem 4.1. It is sufficient to consider the case when $\varphi = 0$ and $\psi = 0$. Let then $\mathrm{u}^{(k)}$ be generated from the second alternating iterative algorithm with the initial guess η. Note that

$$T_\theta\eta = \theta T\eta + (1-\theta)\eta. \tag{20}$$

Clearly, using Lemma 4.1, we find that T_θ is self-adjoint for $\theta \geq 0$. Using Lemma 4.1 and the representation (20), following the steps in the proof of Theorem 4.1 in Marin and Johansson (2010), one can verify that T_θ is positive definite for $1 \leq \theta \leq b$, non-expansive with respect to the inner product and one is not an eigenvalue. One can then check that

$$a\left(\mathrm{u}^{(2k+1)},\mathrm{u}^{(2k+1)}\right) = \left(T_\theta^{(2k+1)}\eta, T_\theta^{(2k+1)}\eta\right). \tag{21}$$

Thus, from the properties of the operator T_θ, we conclude that the right-hand side tends to zero and therefore also $\lim_{k\to\infty} a\left(\mathrm{u}^{(2k+1)},\mathrm{u}^{(2k+1)}\right) = 0$. Now, $a\left(\mathrm{u}^{(2k+1)},\mathrm{u}^{(2k+1)}\right) \geq \min\{1,\kappa^2\}\left\|\mathrm{u}^{(2k+1)}\right\|_{\mathrm{H}^1(\Omega)}$, and we conclude that $\lim_{k\to\infty}\left\|\mathrm{u}^{(2k+1)}\right\|_{\mathrm{H}^1(\Omega)} = 0$. Finally, using the identity

$$a\left(\mathrm{u}^{(2k+2)}-\mathrm{u}^{(2k+1)},\mathrm{u}^{(2k+2)}-\mathrm{u}^{(2k+1)}\right) = a\left(\mathrm{u}^{(2k+1)},\mathrm{u}^{(2k+1)}\right) - a\left(\mathrm{u}^{(2k+2)},\mathrm{u}^{(2k+2)}\right), \tag{22}$$

we find that also $\lim_{k\to\infty}\left\|\mathrm{u}^{(2k)}\right\|_{\mathrm{H}^1(\Omega)} = 0$, which completes the proof.

Remark 4.1 *Let* v *be the solution to the Helmholtz equation* (2) *with Neumann conditions $q = \psi$ on Γ_0 and $\mathrm{q} = \xi_0$ on Γ_1, and put $\eta = \mathrm{v}\big|_{\Gamma_1}$. Starting the alternating iterative algorithm II with the initial guess $\eta_0 = \eta$, the approximations $\mathrm{u}^{(k)}$ obtained for $k \geq 2$ will be precisely those produced by the alternating iterative algorithm I with the initial guess $\xi = \xi_0$. Thus, from* Theorem 4.1*, convergence is settled also for the the alternating iterative algorithm I.*

5 Boundary element method

In the two-dimensional case, i.e. $d = 2$, the modified Helmholtz equation (2) can be formulated in integral form as, see e.g. Chen and Zhou (1992),

$$\begin{aligned} &\mathrm{c}(\mathbf{x})\,\mathrm{u}(\mathbf{x}) + \fint_{\partial\Omega} [\nabla_{\mathbf{y}}\mathscr{F}(\mathbf{x},\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})]\,\mathrm{u}(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}) = \int_{\partial\Omega} \mathscr{F}(\mathbf{x},\mathbf{y})\,\mathrm{q}(\mathbf{y})\,\mathrm{d}\Gamma(\mathbf{y}), \\ &\mathbf{x} \in \overline{\Omega}, \quad \mathbf{y} \in \partial\Omega, \end{aligned} \tag{23}$$

where the first integral is taken in the sense of the Cauchy principal value, $\mathrm{c}(\mathbf{x}) = 1$ for $\mathbf{x} \in \Omega$ and $\mathrm{c}(\mathbf{x}) = 1/2$ for $\mathbf{x} \in \partial\Omega$ (smooth). Here $\mathscr{F}$ is the fundamental solution for the two-dimensional modified Helmholtz equation given by

$$\mathscr{F}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\,\mathrm{K}_0\big(\kappa\|\mathbf{x}-\mathbf{y}\|\big), \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^2, \tag{24}$$

where K_0 is the modified Bessel function of the second kind of order zero.

A BEM with constant boundary elements is employed in order to discretise the integral equation (23), see Chen and Zhou (1992). If the boundaries Γ_0 and Γ_1 are discretised into N_0 and N_1 constant boundary elements, respectively, such that $\mathrm{N} = \mathrm{N}_0 + \mathrm{N}_1$, then on applying the boundary integral equation (23) at each node/collocation point, we arrive at the following system of linear algebraic equations

$$\mathbf{A}\,\mathbf{U} = \mathbf{B}\,\mathbf{Q}. \tag{25}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are matrices which depend solely on the geometry of the boundary $\partial\Omega$ and material properties, i.e. the so-called wave number, κ, and can be calculated analytically, while the vectors $\mathbf{U}$ and $\mathbf{Q}$ consist of the discretised values of the boundary temperatures and normal heat fluxes, respectively. The BEM system of linear algebraic equations (25) can be re-written as

$$\begin{bmatrix} \mathbf{A}^{(00)} & \mathbf{A}^{(01)} \\ \mathbf{A}^{(10)} & \mathbf{A}^{(11)} \end{bmatrix} \begin{pmatrix} \mathbf{U}^{(0)} \\ \mathbf{U}^{(1)} \end{pmatrix} = \begin{bmatrix} \mathbf{B}^{(00)} & \mathbf{B}^{(01)} \\ \mathbf{B}^{(10)} & \mathbf{B}^{(11)} \end{bmatrix} \begin{pmatrix} \mathbf{Q}^{(0)} \\ \mathbf{Q}^{(1)} \end{pmatrix}, \tag{26}$$

where the vectors $\mathbf{U}^{(0)} = \left(\mathrm{u}^{(1)}, \ldots, \mathrm{u}^{(\mathrm{N}_0)}\right)^{\mathsf{T}} \in \mathbb{R}^{\mathrm{N}_0}$ and $\mathbf{T}^{(0)} = \left(\mathrm{q}^{(1)}, \ldots, \mathrm{q}^{(\mathrm{N}_0)}\right)^{\mathsf{T}} \in \mathbb{R}^{\mathrm{N}_0}$ contain the values of the temperature and normal heat flux, respectively, at the nodes/collocation points on the under-specified boundary Γ_0, while the vectors $\mathbf{U}^{(1)} = \left(\mathrm{u}^{(\mathrm{N}_0+1)}, \ldots, \mathrm{u}^{(\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}} \in \mathbb{R}^{\mathrm{N}_1}$ and $\mathbf{T}^{(1)} = \left(\mathrm{q}^{(\mathrm{N}_0+1)}, \ldots, \mathrm{q}^{(\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}} \in \mathbb{R}^{\mathrm{N}_1}$ consist of the values of the temperature and normal heat flux, respectively, at the nodes/collocation points on the over-specified boundary Γ_1. Here the matrices $\mathbf{A}^{(ij)} \in \mathbb{R}^{\mathrm{N}_i \times \mathrm{N}_j}$ and $\mathbf{B}^{(ij)} \in \mathbb{R}^{\mathrm{N}_i \times \mathrm{N}_j}$, $i, j = 0, 1$, contain the elements of the BEM matrices $\mathbf{A}$ and $\mathbf{B}$, respectively, corresponding to the decomposition of the global vectors $\mathbf{U}$ and $\mathbf{Q}$, in the sense that the indices i and j denote the fact that the nodes/collocation points belong to the the boundary Γ_i, $i = 0, 1$, and the field points are located on the boundary Γ_j, $j = 0, 1$, respectively.

It should be mentioned that at each step of the two alternating iterative algorithms with relaxation presented in Section 3 two direct mixed well-posed problems are solved using the BEM. Consequently, the general form of the BEM system of linear algebraic equations associated with these direct problems may be recast as

$$\mathbf{C}\,\mathbf{X} = \mathbf{F}, \tag{27}$$

where

$$\begin{aligned} &\mathbf{C} = \begin{bmatrix} \mathbf{A}^{(00)} & -\mathbf{B}^{(01)} \\ \mathbf{A}^{(10)} & -\mathbf{B}^{(11)} \end{bmatrix}, \qquad \mathbf{X} = \begin{pmatrix} \mathbf{U}^{(2k-2)} \\ \mathbf{Q}^{(2k-2)} \end{pmatrix}, \\ &\mathbf{F} = \begin{bmatrix} \mathbf{B}^{(00)} & -\mathbf{A}^{(01)} \\ \mathbf{B}^{(10)} & -\mathbf{A}^{(11)} \end{bmatrix} \begin{pmatrix} \Psi \\ \mathbf{E}^{(k-1)} \end{pmatrix}, \end{aligned} \tag{28.1}$$

$$\begin{aligned} &\mathbf{U}^{(2k-2)} = \left(\mathrm{u}^{(2k-2;1)}, \ldots, \mathrm{u}^{(2k-2;\mathrm{N}_0)}\right)^{\mathsf{T}}, \\ &\mathbf{Q}^{(2k-2)} = \left(\mathrm{q}^{(2k-2;\mathrm{N}_0+1)}, \ldots, \mathrm{q}^{(2k-2;\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}}, \end{aligned} \tag{28.2}$$

$$\Psi = \left(\psi^{(1)}, \ldots, \psi^{(\mathrm{N}_0)}\right)^{\mathsf{T}}, \quad \mathbf{E}^{(k-1)} = \left(\eta^{(k-1;\mathrm{N}_0+1)}, \ldots, \eta^{(k-1;\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}}, \tag{28.3}$$

and

$$\begin{aligned} &\mathbf{C} = \begin{bmatrix} -\mathbf{B}^{(00)} & \mathbf{A}^{(01)} \\ -\mathbf{B}^{(10)} & \mathbf{A}^{(11)} \end{bmatrix}, \qquad \mathbf{X} = \begin{pmatrix} \mathbf{Q}^{(2k-1)} \\ \mathbf{U}^{(2k-1)} \end{pmatrix}, \\ &\mathbf{F} = \begin{bmatrix} -\mathbf{A}^{(00)} & \mathbf{B}^{(01)} \\ -\mathbf{A}^{(10)} & \mathbf{B}^{(11)} \end{bmatrix} \begin{pmatrix} \Phi \\ \Xi^{(k)} \end{pmatrix}, \end{aligned} \tag{29.1}$$

$$\begin{aligned} &\mathbf{Q}^{(2k-1)} = \left(\mathrm{q}^{(2k-1;1)}, \ldots, \mathrm{q}^{(2k-1;\mathrm{N}_0)}\right)^{\mathsf{T}}, \\ &\mathbf{U}^{(2k-1)} = \left(\mathrm{u}^{(2k-1;\mathrm{N}_0+1)}, \ldots, \mathrm{u}^{(2k-1;\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}}, \end{aligned} \tag{29.2}$$

$$\Phi = \left(\varphi^{(1)}, \ldots, \varphi^{(\mathrm{N}_0)}\right)^{\mathsf{T}}, \quad \Xi^{(k)} = \left(\xi^{(k;\mathrm{N}_0+1)}, \ldots, \xi^{(k;\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}}, \tag{29.3}$$

for the direct mixed well-posed problems (4) and (6), respectively, in the case of the alternating iterative algorithm with relaxation I. Formulae (28) and (29) are also valid for the direct mixed well-posed problems (9) and (7), respectively, corresponding to the alternating iterative algorithm with relaxation II, with the mention that the pairs of indices $(2k-2, k-1)$ and $(2k-1, k)$ are interchanged.

6 Numerical results and discussion

Due to Remark 3.3, we shall mainly produce numerical results for the alternating iterative algorithm with relaxation II. Therefore, it is the purpose of this section to present the numerical implementation of this method using the BEM presented in Section 5, for the Cauchy problem associated with the two-dimensional modified Helmholtz equation and analyse the numerical convergence and stability of this procedure, as well as the influence of the constant relaxation parameter, θ.

6.1 Examples

We consider three typical examples in both simply and doubly connected smooth geometries in which the two-dimensional modified Helmholtz equation is satisfied and we solve the Cauchy problem (2) – (3) for:

Example 1. (Doubly connected, smooth geometry) We consider the following analytical solutions for the temperature and normal heat flux on the boundary $\partial\Omega$:

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x}) = \exp\left(\mathrm{a}_1\mathrm{x}_1 + \mathrm{a}_2\mathrm{x}_2\right), \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \overline{\Omega}, \tag{30.1}$$

and

$$\mathrm{q}^{(\mathrm{an})}(\mathbf{x}) = \exp\left(\mathrm{a}_1\mathrm{x}_1 + \mathrm{a}_2\mathrm{x}_2\right)\left[\mathrm{a}_1\,\mathrm{n}_1(\mathbf{x}) + \mathrm{a}_2\,\mathrm{n}_2(\mathbf{x})\right], \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \partial\Omega, \tag{30.2}$$

respectively, where $\kappa = 2.0$, $\mathrm{a}_1 = 1.0$, $\mathrm{a}_2 = -\sqrt{\kappa^2 - \mathrm{a}_1^2} = -\sqrt{3}$, $\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 \mid r_{\mathrm{int}} < \rho(\mathbf{x}) < r_{\mathrm{out}}\right\}$, $\rho(\mathbf{x}) = \sqrt{\mathrm{x}_1^2 + \mathrm{x}_2^2}$ is the radial polar coordinate of $\mathbf{x}$, $r_{\mathrm{int}} = 0.5$ and $r_{\mathrm{out}} = 1.0$. Here $\Gamma_0 = \Gamma_{\mathrm{out}} = \left\{\mathbf{x} \in \partial\Omega \mid \rho(\mathbf{x}) = r_{\mathrm{out}}\right\}$ and $\Gamma_1 = \Gamma_{\mathrm{int}} = \left\{\mathbf{x} \in \partial\Omega \mid \rho(\mathbf{x}) = r_{\mathrm{int}}\right\}$.

Example 2. (Doubly connected, smooth geometry) We consider the same geometry and analytical solutions for the temperature and normal heat flux on the boundary $\partial\Omega$ as those corresponding to Example 1, and take $\Gamma_0 = \Gamma_{\mathrm{int}} = \left\{\mathbf{x} \in \partial\Omega \mid \rho(\mathbf{x}) = r_{\mathrm{int}}\right\}$ and $\Gamma_1 = \Gamma_{\mathrm{out}} = \left\{\mathbf{x} \in \partial\Omega \mid \rho(\mathbf{x}) = r_{\mathrm{out}}\right\}$.

Example 3. (Simply connected, smooth geometry) We consider the following analytical solutions for the temperature and normal heat flux on the boundary $\partial\Omega$:

$$u^{(an)}(\mathbf{x}) = \exp\left(a_1 x_1 + a_2 x_2\right), \quad \mathbf{x} = (x_1, x_2) \in \overline{\Omega}, \tag{31.1}$$

and

$$q^{(an)}(\mathbf{x}) = \exp\left(a_1 x_1 + a_2 x_2\right)\left[a_1 n_1(\mathbf{x}) + a_2 n_2(\mathbf{x})\right], \quad \mathbf{x} = (x_1, x_2) \in \partial\Omega, \tag{31.2}$$

respectively, where $\kappa = 1.0$, $a_1 = 0.5$, $a_2 = \sqrt{\kappa^2 - a_1^2} = \sqrt{3}/2$, $\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 \,\middle|\, \rho(\mathbf{x}) < r\right\}$ and $r = 1.0$. Here $\Gamma_0 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, \pi/2 < \theta(\mathbf{x}) < 2\pi\right\}$ and $\Gamma_1 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, 0 < \theta(\mathbf{x}) < \pi/2\right\}$, with $\theta(\mathbf{x})$ the angular polar coordinate of $\mathbf{x}$.

For the inverse problems analysed, the BEM system of linear algebraic equations (25) or (26) has been solved for each of the well-posed, direct, mixed boundary value problems that occur at each iteration, k, of the algorithms presented in Section 3 to provide simultaneously the unspecified boundary temperature and normal heat flux on Γ_1. In this study, the numbers of constant boundary elements used for discretising the over- and under-specified boundaries Γ_0 and Γ_1, respectively, were taken as follows:

(i) $N_0 = 40$ and $N_1 = 80$ elements for Example 1;

(ii) $N_0 = N_1 = 80$ elements in the case of Example 2;

(iii) $N_0 = 20$ and $N_1 = 60$ elements for Example 3.

It is also important to mention that for the inverse problems investigated in this paper, as well as the alternating iterative algorithm II, the initial guess $\eta^{(1)}$ for the temperature $u\big|_{\Gamma_1}$, was taken to be

$$\eta^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_1. \tag{32}$$

Moreover, all numerical computations have been performed in `FORTRAN 90` in double precision on a 3.00 GHz Intel Pentium 4 machine.

6.2 Results obtained with exact data: Convergence of the algorithm

In order to analyse the accuracy, convergence and stability of the proposed alternating iterative algorithm with relaxation II, for $k \geq 1$ we introduce the following errors

$$e_u(k) = \left\| u^{(2k)} - u^{(an)} \right\|_{L^2(\Gamma_1)^d} \tag{33.1}$$

and

$$e_{\mathrm{q}}(k) = \left\|\mathrm{q}^{(2k-1)} - \mathrm{q}^{(\mathrm{an})}\right\|_{\mathrm{L}^2(\Gamma_1)^d}. \tag{33.2}$$

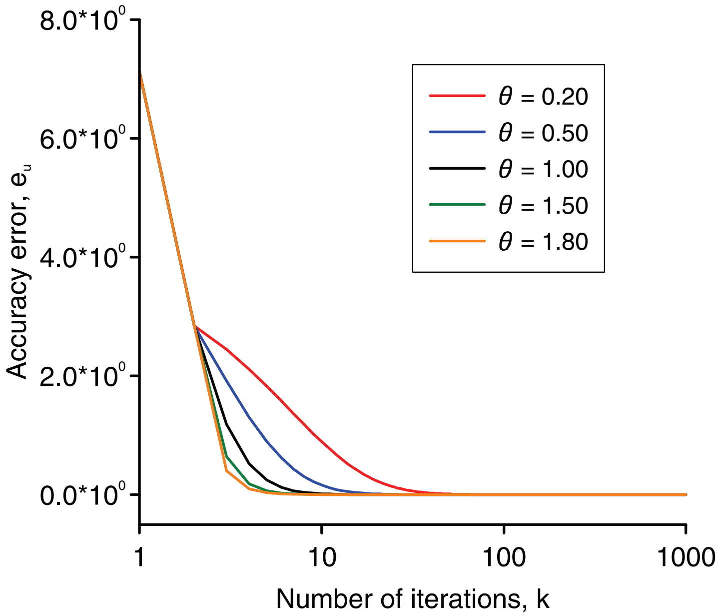
Here $\mathrm{u}^{(2k)}$ and $\mathrm{q}^{(2k-1)}$ are the temperature and normal heat flux retrieved on the under-specified boundary Γ_1 after k iterations using the alternating iterative algorithm with relaxation II, with the mention that each iteration consists of solving two direct mixed well-posed problems, namely equations (7) and (9).

Figs. $1(a)$ and $1(b)$ show, on a logarithmic scale, the accuracy errors e_{u} and e_{q}, as functions of the number of iterations, k, obtained using the alternating iterative algorithm II, exact Cauchy data and various values of the relaxation parameter θ, in the case of Example 1. It can be seen from these figures that, for all values of the relaxation parameter used in this paper, both errors e_{u} and e_{q} keep decreasing until a specific number of iterations, after which the convergence rate of the aforementioned accuracy errors becomes very slow so that they reach a plateau. As expected, for each value of the relaxation parameter employed, $e_{\mathrm{u}}(k) < e_{\mathrm{q}}(k)$ for all $k \geq 1$, i.e. temperatures are more accurate than normal heat fluxes; also, the larger the parameter θ, the lower the number of iterations and, consequently, CPU time required for obtaining accurate numerical results for both the temperature and normal heat flux on Γ_1. Therefore, choosing $\theta \in (1,2)$ in the alternating iterative algorithms I and II results in a significant reduction of the number of iterations as compared with the corresponding original alternating iterative algorithms proposed by Kozlov, Maz′ya and Fomin (1991), i.e. for $\theta = 1$.

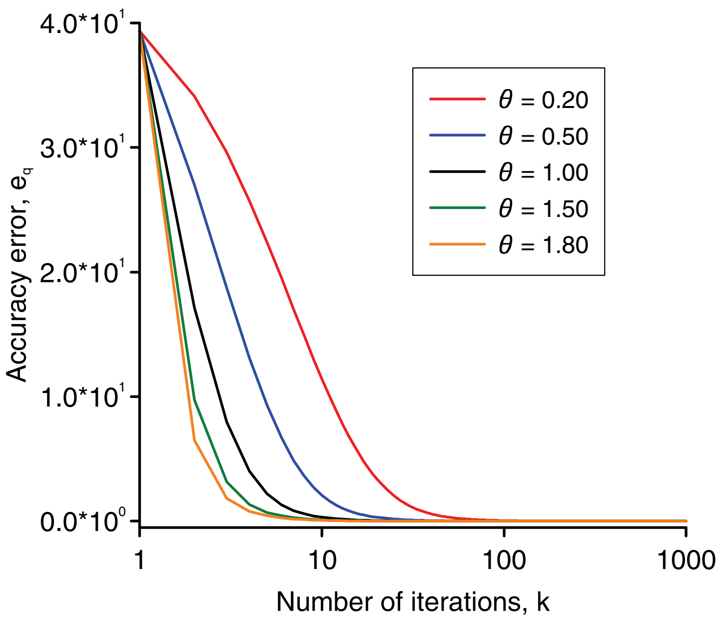
The same conclusions can be drawn from Fig. $2(a)$, which illustrates the analytical and numerical temperature $\mathrm{u}\big|_{\Gamma_1}$ obtained with $\theta = 1.80$ after $k = 1000$ iterations, and Fig. $2(b)$, which presents graphically the corresponding analytical and numerical values for the numerical heat flux $\mathrm{q}\big|_{\Gamma_1}$. From Figs. 1 and 2, it can be concluded that the alternating iterative algorithm with relaxation II described in Section 3 provides excellent approximations for the unknown Dirichlet and Neumann data on Γ_1 and is convergent with respect to increasing the number of iterations, k, if exact Cauchy data are prescribed on the over-specified boundary Γ_0. Although not presented, it is reported that similar results have been obtained for Examples 2 and 3, and all admissible values of the relaxation parameter, as well as the alternating iterative algorithm with relaxation I applied to Examples $1-3$.

6.3 Regularizing stopping criterion

Once the convergence of the numerical solution to the exact solution with respect to number of iterations performed, k, has been established, we investigate the stability of the numerical solution for the examples considered. To do so and also simulate

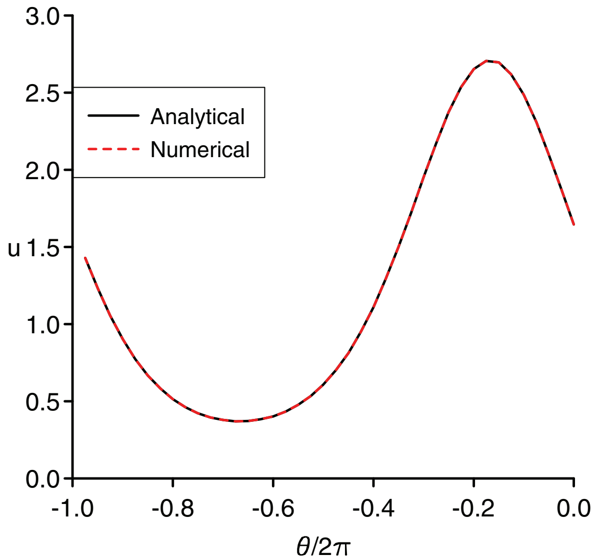


(a) Example 1: e_{u}, $p_{\mathrm{u}} = p_{\mathrm{q}} = 0\%$, algorithm II

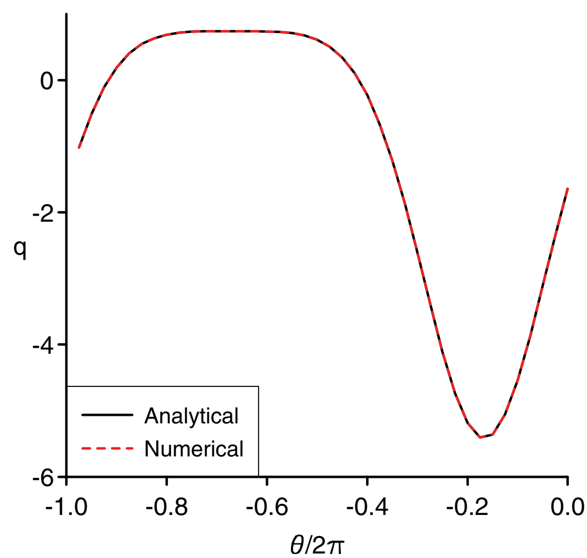


(b) Example 1: e_{q}, $p_{\mathrm{u}} = p_{\mathrm{q}} = 0\%$, algorithm II

Figure 1: The accuracy errors (a) e_{u} and (b) e_{q}, as functions of the number of iterations, k, obtained using the alternating iterative algorithm II, exact Cauchy data and several values of the relaxation parameter θ, namely $\theta \in \{0.20, 0.50, 1.00, 1.50, 1.80\}$, for the Cauchy problem given by Example 1.



(a) Example 1: $u|_{\Gamma_1}$, $p_u = p_q = 0\%$, $\theta = 1.80$, algorithm II



(b) Example 1: $q|_{\Gamma_1}$, $p_u = p_q = 0\%$, $\theta = 1.80$, algorithm II

Figure 2: The analytical and numerical temperatures (a) $u|_{\Gamma_1}$, and fluxes (b) $q|_{\Gamma_1}$, obtained using the alternating iterative algorithm II, the discrepancy principle, $\theta = 1.80$ and exact Cauchy data, i.e. $p_u = p_q = 0\%$, for the Cauchy problem given by Example 1.

the inherent inaccuracies in the measured data on Γ_0, we assume that various levels of Gaussian random noise, p_{u} and p_{q}, have been added into the exact temperature $\mathrm{u}\big|_{\Gamma_0} = \varphi$ and normal heat flux $\mathrm{q}\big|_{\Gamma_0} = \psi$ data, respectively, so that the following perturbed temperature and normal heat flux are available:

$$\varphi^{\delta} \in \mathrm{L}^2(\Gamma_0): \quad \big\|\mathrm{u}^{(\mathrm{an})}\big|_{\Gamma_0} - \varphi^{\delta}\big\|_{\mathrm{L}^2(\Gamma_0)} = \delta, \tag{34.1}$$

and

$$\psi^{\delta} \in \mathrm{L}^2(\Gamma_0): \quad \big\|\mathrm{q}^{(\mathrm{an})}\big|_{\Gamma_0} - \psi^{\delta}\big\|_{\mathrm{L}^2(\Gamma_0)} = \delta. \tag{34.2}$$

Figs. 3(a) and 3(b) present, on a logarithmic scale, the accuracy errors e_{u} and e_{q}, respectively, as functions of the number of iterations, k, obtained using the alternating iterative algorithm II, $\theta = 1.80$ and $p_{\mathrm{u}} \in \{1\%, 2\%, 3\%\}$, for the Cauchy problem given by Example 1. From these figures it can be seen that, for each fixed value of p_{u}, the errors in predicting the temperature and normal heat flux on the under-specified boundary Γ_1 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion has to be used in order to cease the iterative process at the point where the errors in the numerical solutions start increasing.

To define the stopping criterion required for regularizing/stabilizing the iterative methods analysed in this paper, for $k \geq 1$, the following convergence error is introduced:

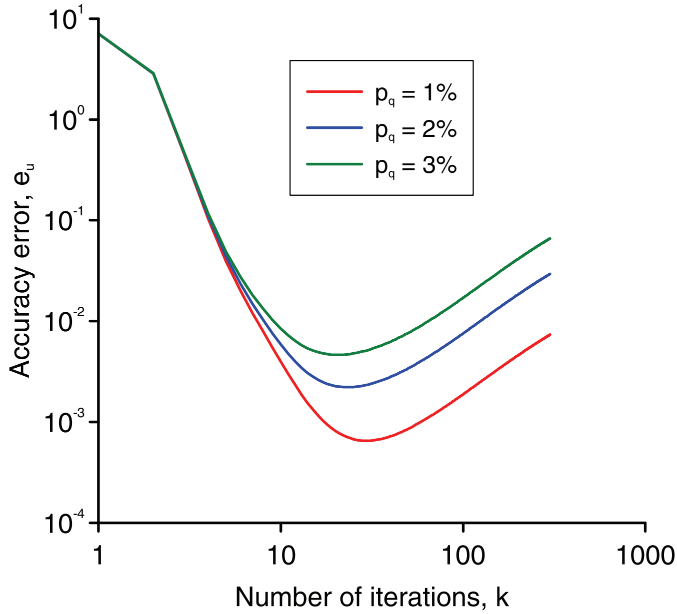
$$E(k) = \big\|\mathbf{A}\,\mathbf{U}^{(k)} - \mathbf{B}\,\mathbf{Q}^{(k)}\big\|, \tag{35}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the BEM matrices. Here the vectors $\mathbf{U}^{(k)}$ and $\mathbf{Q}^{(k)}$ are given as follows:

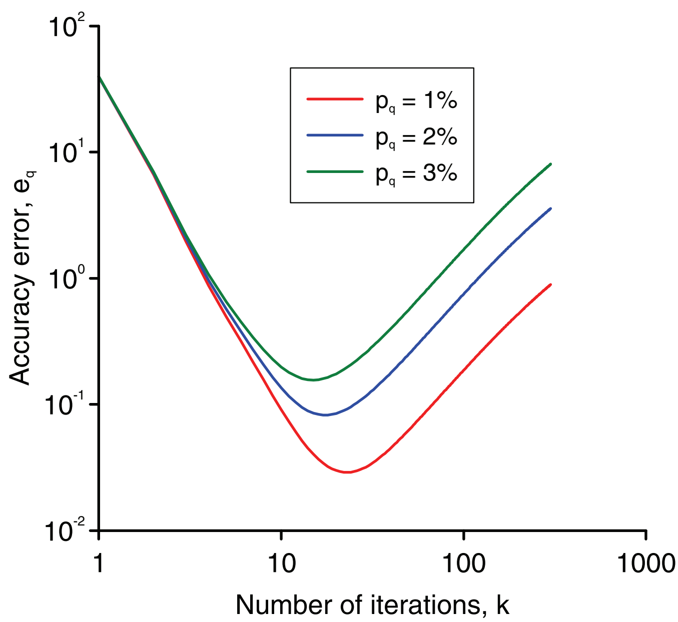
(i) For the alternating iterative algorithm with relaxation I

$$\mathbf{U}^{(k)} = \begin{pmatrix} \mathbf{\Phi} \\ \mathbf{U}^{(2k-1)} \end{pmatrix}, \quad \begin{aligned} &\mathbf{\Phi}^{\delta} = \big(\varphi^{(\delta;1)}, \ldots, \varphi^{(\delta;\mathrm{N}_0)}\big)^{\mathsf{T}}, \\ &\mathbf{U}^{(2k-1)} = \big(\mathrm{u}^{(2k-1;1)}, \ldots, \mathrm{u}^{(2k-1;\mathrm{N}_0)}\big)^{\mathsf{T}}, \end{aligned} \tag{36.1}$$

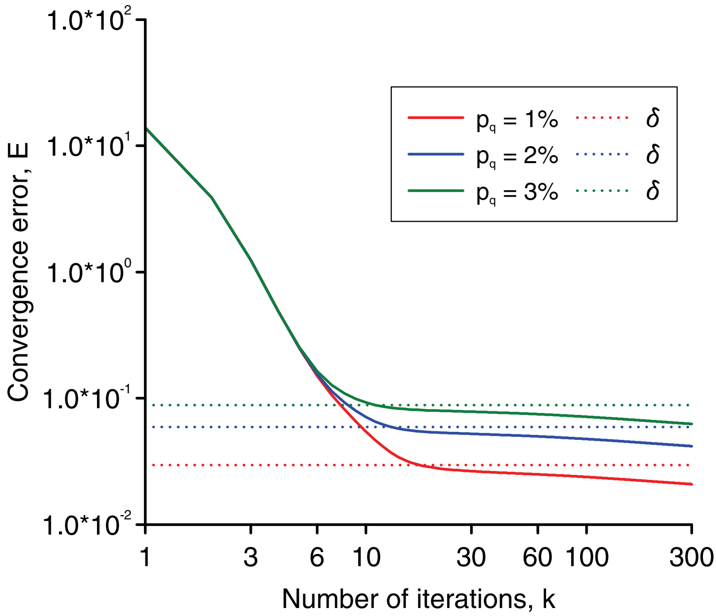
$$\mathbf{Q}^{(k)} = \begin{pmatrix} \mathbf{\Psi} \\ \mathbf{Q}^{(2k)} \end{pmatrix}, \quad \begin{aligned} &\mathbf{\Psi}^{\delta} = \big(\psi^{(\delta;1)}, \ldots, \psi^{(\delta;\mathrm{N}_0)}\big)^{\mathsf{T}}, \\ &\mathbf{Q}^{(2k)} = \big(\mathrm{q}^{(2k;\mathrm{N}_0+1)}, \ldots, \mathrm{q}^{(2k;\mathrm{N}_0+\mathrm{N}_1)}\big)^{\mathsf{T}}; \end{aligned} \tag{36.2}$$



(a) Example 1: e_{u}, $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, algorithm II



(b) Example 1: e_{q}, $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, algorithm II



(c) Example 1: E, $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, algorithm II

Figure 3: The accuracy errors (a) e_{u} and (b) e_{q}, and the convergence error (c) E, as functions of the number of iterations, k, obtained using the alternating iterative algorithm II, $\theta = 1.80$ and various amounts of noise added into $\mathrm{q}\big|_{\Gamma_0}$, i.e. $p_{\mathrm{u}} = 0\%$ and $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, for the Cauchy problem given by Example 1.

(ii) For the alternating iterative algorithm with relaxation II

$$\mathbf{U}^{(k)} = \begin{pmatrix} \mathbf{\Phi} \\ \mathbf{U}^{(2k)} \end{pmatrix}, \quad \begin{aligned} &\mathbf{\Phi}^{\delta} = \left(\varphi^{(\delta;1)}, \ldots, \varphi^{(\delta;\mathrm{N}_0)}\right)^{\mathsf{T}}, \\ &\mathbf{U}^{(2k)} = \left(\mathrm{u}^{(2k;1)}, \ldots, \mathrm{u}^{(2k;\mathrm{N}_0)}\right)^{\mathsf{T}}, \end{aligned} \tag{37.1}$$

$$\mathbf{T}^{(k)} = \begin{pmatrix} \mathbf{\Psi} \\ \mathbf{Q}^{(2k-1)} \end{pmatrix}, \quad \begin{aligned} &\mathbf{\Psi}^{\delta} = \left(\psi^{(\delta;1)}, \ldots, \psi^{(\delta;\mathrm{N}_0)}\right)^{\mathsf{T}}, \\ &\mathbf{Q}^{(2k-1)} = \left(\mathrm{q}^{(2k-1;\mathrm{N}_0+1)}, \ldots, \mathrm{q}^{(2k-1;\mathrm{N}_0+\mathrm{N}_1)}\right)^{\mathsf{T}}. \end{aligned} \tag{37.2}$$

The alternating iterative algorithms I and II described in Section 3 are ceased according to the discrepancy principle of Morozov (1966), see also Marin, Elliott, Heggs, Ingham, Lesnic and Wen (2003a) and Marin and Johansson (2010), namely at the optimal iteration number, k_{opt}, which is the smallest integer with

$$E(k) \approx \mathrm{O}(\delta). \tag{38}$$

Fig. $3(c)$ presents the evolution of the convergence error E with respect to the number of iterations performed, k, using the alternating iterative algorithm II, $\theta = 1.80$ and $p_{\text{u}} \in \{1\%, 2\%, 3\%\}$, for the Cauchy problem given by Example 1. By comparing Figs. $3(a) - 3(c)$, it can be seen that selecting the optimal iteration number, k_{opt}, according to the stopping rule (38) captures very well the minimum values for the accuracy errors e_{u} and e_{q}. Therefore, Eqn. (38) represents a stabilizing stopping criterion for the alternating iterative algorithm with relaxation II, at the same time being consistent with the findings of Marin, Elliott, Heggs, Ingham, Lesnic and Wen (2003a), who analysed a particular case of the aforementioned algorithm, namely $\theta = 1$. Although not illustrated, it is important to mention that similar results and conclusions have been obtained for the other examples considered and $\theta \in (0, 2)$. The same conclusions can be drawn if the alternating iterative algorithm with relaxation I is applied to solving the Cauchy problem given by Example 1, using $\theta = 1.80$ and $p_{\text{q}} \in \{1\%, 2\%, 3\%\}$.

As mentioned in Section 6.2, for exact data the iterative process is convergent with respect to increasing the number of iterations, k, since the accuracy errors e_{u} and e_{q} keep decreasing even after a large number of iterations, see Fig. 3. It should be noted that, in this case, a stopping criterion is not necessary since the numerical solution is convergent with respect to increasing the number of iterations. However, even for exact Cauchy data on Γ_0 the errors E, e_{u} and e_{q} have a similar behaviour and the error E may be used to stop the iterative process at the point where the rate of convergence is very small and no substantial improvement in the numerical solution is obtained if the iterative process is continued. Hence it can be concluded that the regularizing stopping criterion proposed for the alternating iterative algorithms with relaxation I and II is very efficient in locating the point where the errors start increasing and the iterative process should be ceased.

6.4 Results obtained with noisy data: Stability of the algorithms

Based on the stopping criterion (38) described in Section 6.3, the analytical and numerical values retrieved for the temperature and normal heat flux on the under-specified boundary Γ_1, using the alternating iterative algorithm II, $\theta = 1.80$ and $p_{\text{u}} \in \{1\%, 2\%, 3\%\}$, in the case of Example 1, are presented in Figs. $4(a)$ and $4(b)$, respectively. It can be seen from these figures that the numerical solution for both the temperature and normal heat flux is a stable approximation to its corresponding exact solution, free of unbounded and rapid oscillations, and it converges to its exact counterpart as p_{u} decreases.

θ	p_u	p_q	$e_u(k_{opt})$	$e_q(k_{opt})$	k_{opt}	CPU time [sec]
0.20	1%	0%	0.16413×10^{-2}	0.49761×10^{-1}	134	16.97
	2%	0%	0.48845×10^{-2}	0.11603×10^{0}	102	15.77
	3%	0%	0.94039×10^{-2}	0.19476×10^{0}	87	15.06
0.50	1%	0%	0.16490×10^{-2}	0.49610×10^{-1}	54	13.72
	2%	0%	0.49227×10^{-2}	0.11594×10^{0}	41	13.14
	3%	0%	0.94820×10^{-2}	0.19517×10^{0}	35	12.86
1.00	1%	0%	0.16683×10^{-2}	0.49446×10^{-1}	27	12.47
	2%	0%	0.49938×10^{-2}	0.11611×10^{0}	21	12.27
	3%	0%	0.96999×10^{-2}	0.19738×10^{0}	18	12.23
1.50	1%	0%	0.16560×10^{-2}	0.49066×10^{-1}	19	12.25
	2%	0%	0.50573×10^{-2}	0.11797×10^{0}	15	11.98
	3%	0%	0.10042×10^{-1}	0.20484×10^{0}	13	11.94
1.80	1%	0%	0.16599×10^{-2}	0.48958×10^{-1}	16	12.06
	2%	0%	0.50894×10^{-2}	0.11593×10^{0}	12	12.00
	3%	0%	0.10127×10^{-1}	0.20579×10^{0}	11	12.00

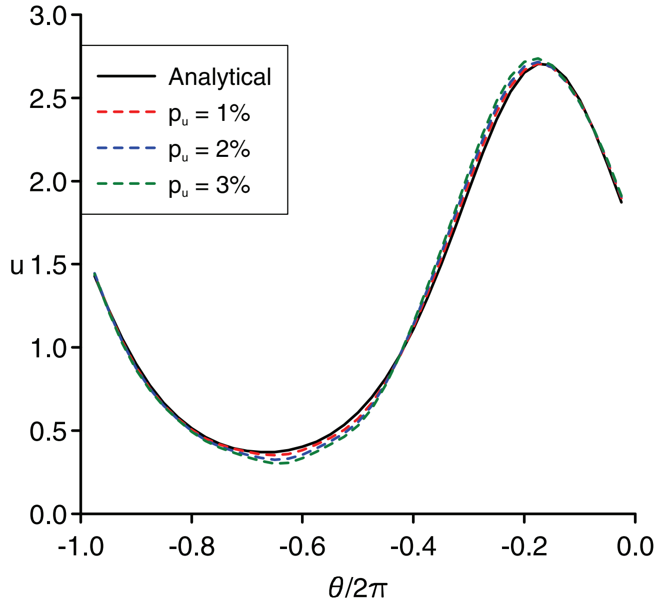
Table 1: The values of the accuracy errors, $e_u(k_{opt})$ and $e_q(k_{opt})$, the optimal iteration number, k_{opt}, and the CPU time, obtained using the alternating iterative algorithm II, the discrepancy principle, various amounts of noise added into $u\big|_{\Gamma_0}$, i.e. $p_u \in \{1\%, 2\%, 3\%\}$ and $p_q = 0\%$, and various values for the relaxation parameter, θ, for the Cauchy problem given by Example 1.

The values of the accuracy errors, $e_{\mathrm{u}}(k_{\mathrm{opt}})$ and $e_{\mathrm{q}}(k_{\mathrm{opt}})$, the corresponding optimal iteration number, k_{opt}, and the CPU time, obtained using the alternating iterative algorithm II, the stopping criterion (38), various levels of noise added into the Dirichlet data on Γ_0 and various values of the relaxation parameter, $\theta \in (0,2)$, for the Cauchy problem given by Example 1, are presented in Table 1. The following major conclusions can be drawn from Table 1:

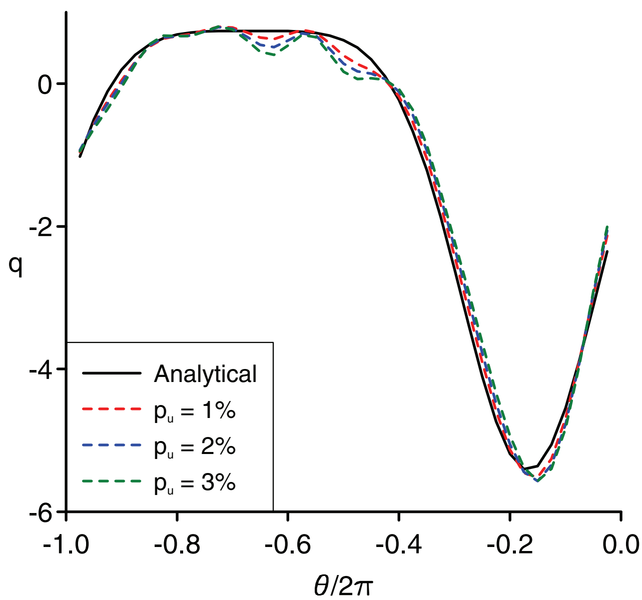
(i) For all fixed values of the relaxation parameter $\theta \in (0,2)$, both accuracy errors $e_{\mathrm{u}}(k_{\mathrm{opt}})$ and $e_{\mathrm{q}}(k_{\mathrm{opt}})$ decrease as p_{u} decreases (i.e. the algorithm II is stable with respect to decreasing the level of noise added into the Dirichlet data on Γ_0), while the optimal number of iterations k_{opt} and, consequently, the CPU time required for the alternating iterative algorithm II to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 increase as p_{u} decreases;

(ii) For all fixed amounts of noise added into the temperature on the over-specified boundary Γ_0, the accuracy errors $e_{\mathrm{u}}(k_{\mathrm{opt}})$ and $e_{\mathrm{q}}(k_{\mathrm{opt}})$, the optimal number of iterations, k_{opt} and the CPU time required for the alternating iterative algorithm II to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 decrease as $\theta \longrightarrow 2$, i.e. as more over-relaxation is introduced in the algorithm II. However, it should be stressed out that the differences, in terms of accuracy, between the numerical results for both $\mathrm{u}\big|_{\Gamma_1}$ and $\mathrm{q}\big|_{\Gamma_1}$, obtained for various values of the relaxation parameter, θ, are not very significant.

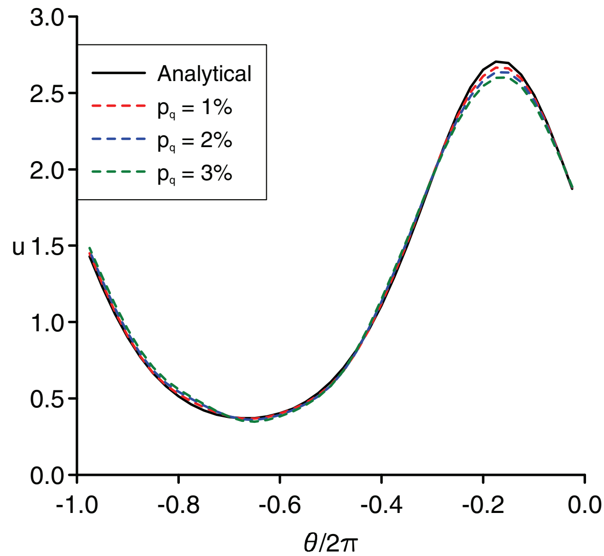
Next, we exemplify the performance of the alternating iterative algorithm II with under-, no and over-relaxation by considering Example 1 with $p_{\mathrm{u}} = 1\%$. In this case, the CPU times needed for the alternating iterative algorithm I with $\theta = 0.20$ (under-relaxation), $\theta = 1.00$ (no relaxation) and $\theta = 1.80$ (over-relaxation) to reach the numerical solutions for the temperature and normal heat flux on Γ_1 were found to be 16.97, 12.47 and 12.06 s, respectively, while the corresponding values for the optimal number of iterations required, k_{opt}, were found to be 134, 27 and 16, respectively. This means that, to attain the numerical solutions for the unknown Dirichlet and Neumann data on Γ_1, the alternating iterative algorithm II with over-relaxation ($\theta = 1.80$) requires a reduction in the number of iterations performed by approximately 29% and 88% with respect to those corresponding to the standard iterative algorithm II as proposed by Kozlov, Maz′ya and Fomin (1991), i.e. without relaxation ($\theta = 1.00$), and the alternating iterative algorithm II with under-relaxation ($\theta = 0.20$), respectively.



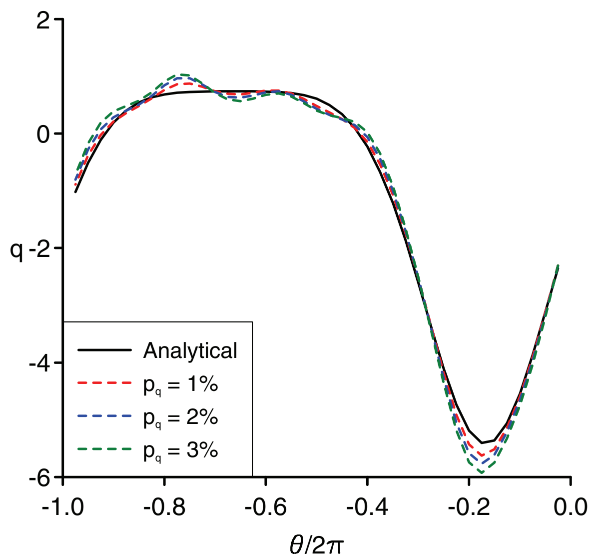
(a) Example 1: $\mathrm{u}\big|_{\Gamma_1}$, $p_{\mathrm{u}} \in \{1\%, 2\%, 3\%\}$, algorithm II



(b) Example 1: $\mathrm{q}\big|_{\Gamma_1}$, $p_{\mathrm{u}} \in \{1\%, 2\%, 3\%\}$, algorithm II



(c) Example 1: $u\big|_{\Gamma_1}$, $p_q \in \{1\%, 2\%, 3\%\}$, algorithm II



(d) Example 1: $q\big|_{\Gamma_1}$, $p_q \in \{1\%, 2\%, 3\%\}$, algorithm II

Figure 4: The analytical and numerical temperatures (a) and (c) $u\big|_{\Gamma_1}$, and fluxes (b) and (d) $q\big|_{\Gamma_1}$, obtained using the alternating iterative algorithm II, the discrepancy principle, $\theta = 1.80$ and various amounts of noise added into the Cauchy data, for Example 1.

Similar conclusions to those obtained from Figs. $4(a)$ and $4(b)$ can be drawn from Figs. $4(c)$ and $4(d)$, which present the numerical values for the temperature and normal heat flux on the under-specified boundary Γ_1, in comparison with their analytical counterparts, obtained using the alternating iterative algorithm II, the regularizing stopping criterion (38), $\theta = 1.80$ and $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, for Example 1. From Figs. $4(a) - 4(d)$, it can be observed that the alternating iterative algorithm II applied to Example 1 is more sensitive to noise added into the temperature $\mathrm{u}|_{\Gamma_0}$ than to perturbations of the normal heat flux $\mathrm{u}|_{\Gamma_0}$.

θ	p_{u}	p_{q}	$e_{\mathrm{u}}(k_{\mathrm{opt}})$	$e_{\mathrm{q}}(k_{\mathrm{opt}})$	k_{opt}	CPU time [sec]
0.20	0%	1%	0.97549×10^{-3}	0.35063×10^{-1}	154	18.16
	0%	2%	0.34161×10^{-2}	$0.10249 \times 10^{\ 0}$	113	16.31
	0%	3%	0.72020×10^{-2}	$0.19710 \times 10^{\ 0}$	94	15.34
0.50	0%	1%	0.97334×10^{-3}	0.34608×10^{-1}	62	14.12
	0%	2%	0.33653×10^{-2}	0.99853×10^{-1}	46	13.41
	0%	3%	0.71921×10^{-2}	$0.19350 \times 10^{\ 0}$	38	13.02
1.00	0%	1%	0.98644×10^{-3}	0.34254×10^{-1}	31	12.80
	0%	2%	0.34281×10^{-2}	0.98518×10^{-1}	23	12.45
	0%	3%	0.73623×10^{-2}	$0.19095 \times 10^{\ 0}$	19	12.19
1.50	0%	1%	0.97438×10^{-3}	0.33350×10^{-1}	21	12.48
	0%	2%	0.32708×10^{-2}	0.93299×10^{-1}	16	12.08
	0%	3%	0.72582×10^{-2}	$0.18368 \times 10^{\ 0}$	13	12.00
1.80	0%	1%	0.93884×10^{-3}	0.32267×10^{-1}	18	12.33
	0%	2%	0.30741×10^{-2}	0.88923×10^{-1}	14	13.32
	0%	3%	0.71987×10^{-2}	$0.17959 \times 10^{\ 0}$	11	11.89

Table 2: The values of the accuracy errors, $e_{\mathrm{u}}(k_{\mathrm{opt}})$ and $e_{\mathrm{q}}(k_{\mathrm{opt}})$, the optimal iteration number, k_{opt}, and the CPU time, obtained using the alternating iterative algorithm II, the discrepancy principle, various amounts of noise added into $\mathrm{q}|_{\Gamma_0}$, i.e. $p_{\mathrm{u}} = 0\%$ and $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, and various values for the relaxation parameter, θ, for the Cauchy problem given by Example 1.

Table 2 tabulates the values of the corresponding accuracy errors given by equations (33.1) and (33.2), the optimal iteration number, k_{opt}, and the CPU time, obtained using the alternating iterative algorithm II, the discrepancy principle (38), various levels of noise added into the Neumann data on Γ_0 and various values of the relaxation parameter, $\theta \in (0, 2)$, for the Cauchy problem given by Example 1. From Tables 1 and 2 it can be noticed that the sensitivity of the alternating iterative algorithm II with respect to noisy Dirichlet and Neumann data on Γ_0, for Example 1, results in the following:

(i) More inaccurate numerical results for both $u\big|_{\Gamma_1}$ and $q\big|_{\Gamma_1}$ are obtained for perturbed temperature on Γ_0 than for noisy normal heat flux on Γ_0;

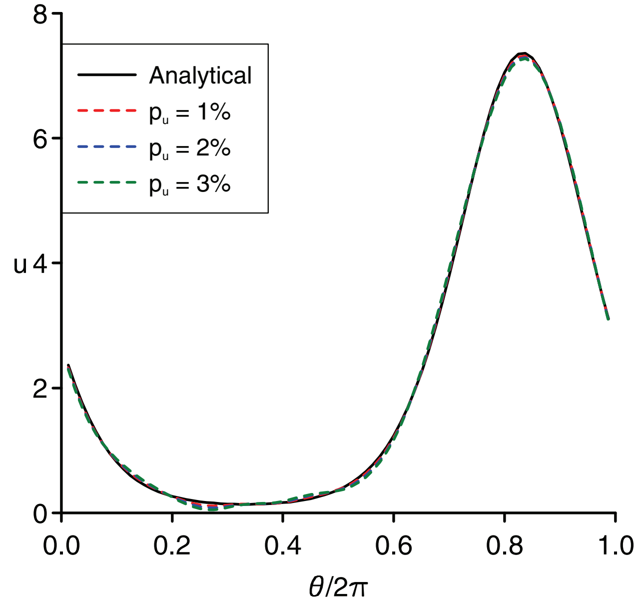
(ii) The optimal number of iterations, k_{opt}, required for the alternating iterative algorithm II to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 in the case of perturbed temperature on Γ_0 is larger than that corresponding to noisy normal heat flux on Γ_0.

Although not presented herein, it is important to report that accurate, convergent and stable numerical results for the unknown temperature and normal heat flux on Γ_1 have also been obtained, in the case of the Cauchy problem associated with Example 1, when using the alternating iterative algorithm I, $\theta = 1.80$ and various amounts of noise added into the temperature $u\big|_{\Gamma_0}$. We point out that it has been noticed that the alternating iterative algorithm II provides slightly more inaccurate numerical results for $u\big|_{\Gamma_1}$ and $q\big|_{\Gamma_1}$ for noisy temperature on Γ_0 than for perturbed normal heat flux on Γ_0.

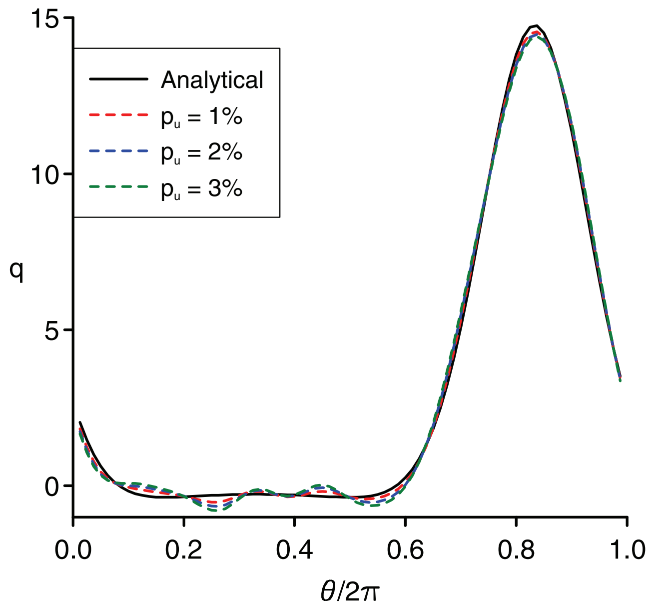
Again, accurate, stable and convergent numerical solutions for $u\big|_{\Gamma_1}$ and $q\big|_{\Gamma_1}$ have been obtained for the Cauchy problem given by Example 2. Fig. 5 presents the analytical and numerical results for the unknown temperature and normal heat flux on Γ_1, obtained using $\theta = 1.80$, various levels of noise added into the Dirichlet or Neumann data on Γ_0 and the alternating iterative algorithm II, for Example 2.

6.5 Limitations of the algorithms

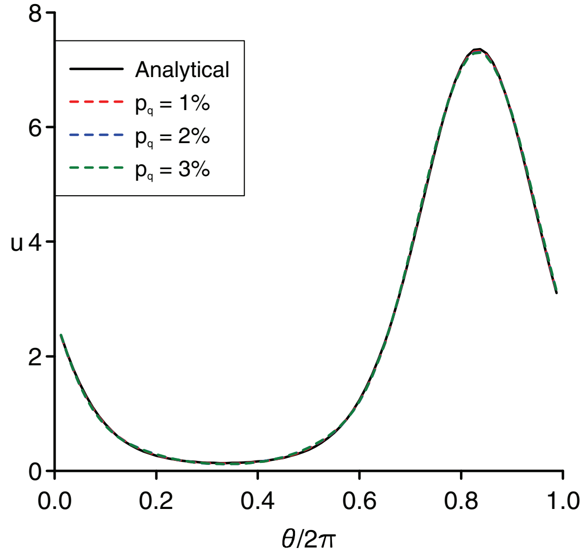
The analytical and numerical values for the temperature $u\big|_{\Gamma_1}$ and normal heat flux $q\big|_{\Gamma_1}$, obtained using the alternating iterative algorithm II, the discrepancy principle, $\theta = 1.50$ and various amounts of noise added into the temperature or normal flux data on Γ_0, in the case of the Cauchy problem given by Example 3, are presented in Fig. 6. Although the numerically retrieved temperatures on Γ_1 are reasonable approximations for their analytical values, see Figs. $6(a)$ and $6(c)$, and the errors in the numerical results obtained for both the temperature and the normal heat flux on the under-specified boundary Γ_1 decrease with respect to decreasing the level of noise added into the Cauchy data on Γ_0, it can be seen from Figs. $6(b)$ and $6(d)$ that the numerical normal heat fluxes on Γ_1 still remain inaccurate approximations for their corresponding analytical counterparts. In addition, the numerical normal heat fluxes on Γ_1 are very inaccurate approximations for their analytical counterparts, as well as highly oscillatory, at the endpoints of Γ_1. Figs. $6(a)-6(d)$ clearly



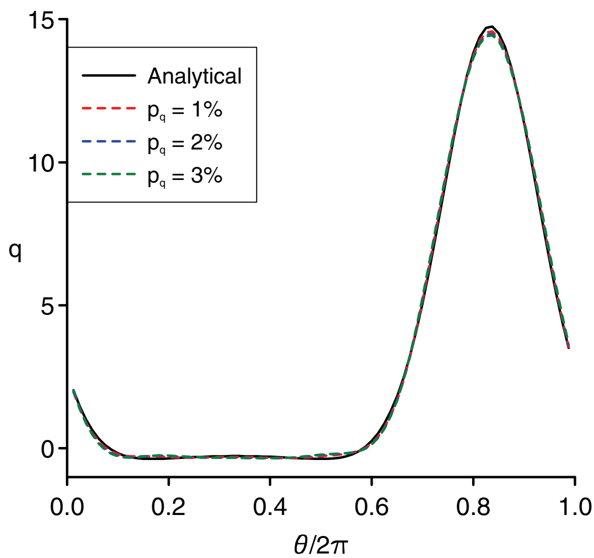
(a) Example 2: $u\big|_{\Gamma_1}$, $p_u \in \{1\%, 2\%, 3\%\}$, algorithm II



(b) Example 2: $q\big|_{\Gamma_1}$, $p_u \in \{1\%, 2\%, 3\%\}$, algorithm II

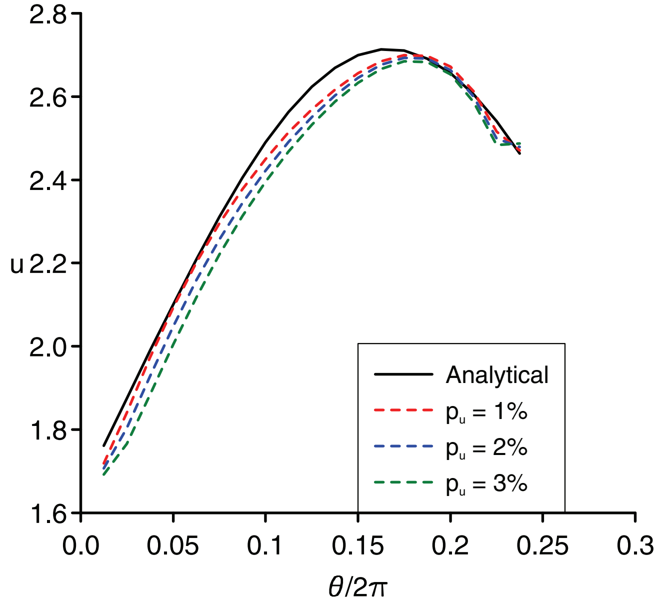


(c) Example 2: $u\big|_{\Gamma_1}$, $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, algorithm II

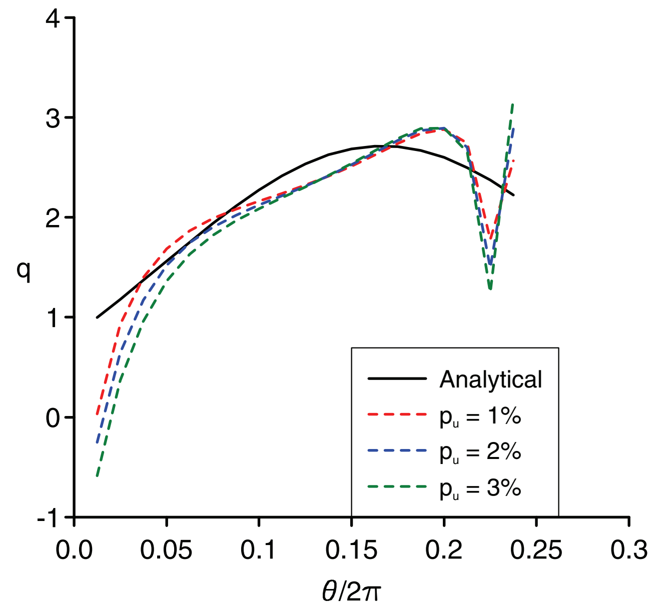


(d) Example 2: $q\big|_{\Gamma_1}$, $p_{\mathrm{q}} \in \{1\%, 2\%, 3\%\}$, algorithm II

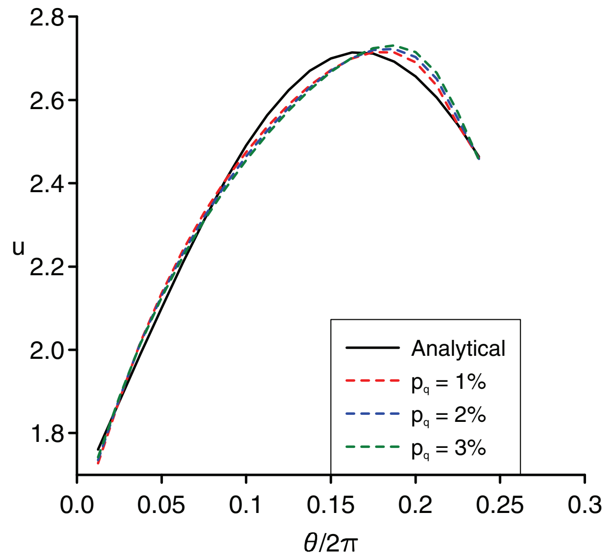
Figure 5: The analytical and numerical temperatures (a) and (c) $u\big|_{\Gamma_1}$, and fluxes (b) and (d) $q\big|_{\Gamma_1}$, obtained using the alternating iterative algorithm II, the discrepancy principle, $\theta = 1.80$ and various amounts of noise added into the Cauchy data, for Example 2.



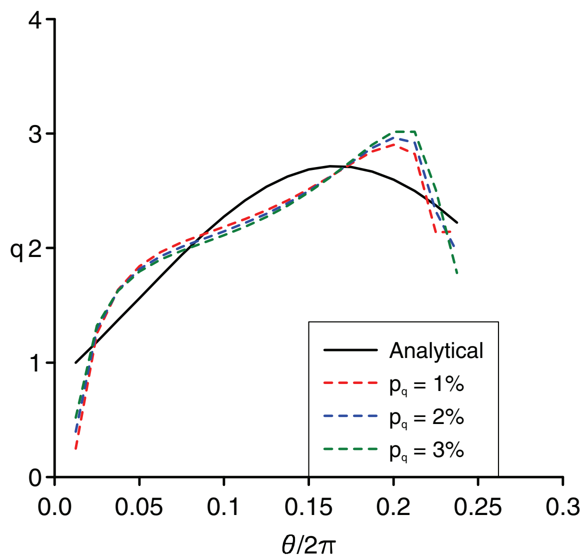
(a) Example 3: $u|_{\Gamma_1}$, $p_u \in \{1\%, 2\%, 3\%\}$, algorithm II



(b) Example 3: $q|_{\Gamma_1}$, $p_u \in \{1\%, 2\%, 3\%\}$, algorithm II



(c) Example 3: $u|_{\Gamma_1}$, $p_q \in \{1\%, 2\%, 3\%\}$, algorithm II



(d) Example 3: $q|_{\Gamma_1}$, $p_q \in \{1\%, 2\%, 3\%\}$, algorithm II

Figure 6: The analytical and numerical temperatures (a) and (c) $u|_{\Gamma_1}$, and fluxes (b) and (d) $q|_{\Gamma_1}$, obtained using the alternating iterative algorithm II, the discrepancy principle, $\theta = 1.50$ and various amounts of noise added into the Cauchy data, for Example 3.

show the difficulty of the alternating iterative algorithm II in reconstructing the unknown temperature and normal heat flux on the under-specified boundary Γ_1 from noisy Cauchy measurements on the remaining boundary Γ_0 in the case of a simply connected domain and hence the limitation of the proposed numerical procedure for such geometries. For this type of problems, special treatment is required for the temperature at the common endpoints of the over- and under-specified boundaries, i.e. points belonging to $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$. One may use weight functions at each iteration of the algorithm in order to cancel the singularities, see e.g. Johansson and Marin (2007), but this will be investigated in a future work. Although not presented, it is reported that similar results have been obtained when solving the Cauchy problem given by Example 3 using the alternating iterative algorithm I, as well as Cauchy problems for the modified Helmholtz equation in piecewise smooth, simply connected geometries, such as rectangular or square domains.

7 Conclusions

In this paper, we proposed two algorithms involving the relaxation of either the given Dirichlet data (temperature) or the prescribed Neumann data (normal heat flux) on the over-specified boundary in the case of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991) applied to Cauchy problems for the modified Helmholtz equation. A convergence proof of these relaxation methods was given, as well as a regularizing stopping criterion. The aforementioned alternating iterative algorithms with relaxation were implemented, for Cauchy problems governed by the two-dimensional modified Helmholtz equation, by employing constant boundary elements. The numerical results obtained using these procedures, in the case of doubly connected domains, i.e. domains whose over- and under-specified boundaries have no common points, showed the numerical stability, convergence, accuracy, consistency and computational efficiency of the proposed method. More specifically, both alternating iterative algorithms with constant over-relaxation of either the temperature or the prescribed normal heat flux on the over-specified boundary significantly reduced the number of iterations performed in order to achieve the numerical solutions for the temperature and normal heat flux on the under-specified boundary, as well as the CPU time allocated for this purpose.

The limitation of the proposed algorithm is related to Cauchy problems in two-dimensional simply connected domains, i.e. geometries for which the over- and under-specified boundaries have common endpoints. In such situations, the present alternating iterative methods fail to produce accurate approximations for the temperature and normal heat flux on the under-specified boundary from measured Cauchy data available on the remaining boundary. Future work is related to adapting the present procedures to Cauchy problems for the modified Helmholtz equation in domains with corners by employing weight functions.

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