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Particular Solutions of Chebyshev Polynomials for Polyharmonic and Poly-Helmholtz Equations

Chia-Cheng Tsai[1]

Abstract: In this paper we develop analytical particular solutions for the polyharmonic and the products of Helmholtz-type partial differential operators with Chebyshev polynomials at right-hand side. Our solutions can be written explicitly in terms of either monomial or Chebyshev bases. By using these formulas, we can obtain the approximate particular solution when the right-hand side has been represented by a truncated series of Chebyshev polynomials. These formulas are further implemented to solve inhomogeneous partial differential equations (PDEs) in which the homogeneous solutions are complementarily solved by the method of fundamental solutions (MFS). Numerical experiments, which include eighth order PDEs and three-dimensional cases, are carried out. Due to the exponential convergence of the Chebyshev interpolation and the MFS, our numerical results are extremely accurate.

Keyword: Particular solution, Chebyshev polynomials, polyharmonic equation, product of Helmholtz equation, boundary element method, method of fundamental solutions, Trefftz method, radial basis function

1 Introduction

There are a number of numerical methods that can be classified as boundary-type numerical methods [Cheng and Cheng (2005)] because the numerical discretization is performed either on the solution boundary, or on a boundary-like geometry, which resides in a lower spatial dimension than the solution domain. Examples include the boundary element method (BEM), the MFS [Kupradze and Aleksidze (1964); Bogomolny (1985)], the Trefftz Methods (TM) [Cozzano, Rodríguez (2007); Liu (2007A); Liu (2007B)], the meshless local boundary integral equation method [Zhu, Zhang, and Atluri (1998)], and *et al.* Boundary-type numerical methods generally require the governing equation to be homogeneous hence a special algorithm is needed for inhomogeneous PDEs.

One way to eliminate the inhomogeneous term is by the method of particular solutions (MPS) addressed by Golberg and Chen (1999). Consider an inhomogeneous linear PDE:

$$\Lambda u = f(\mathbf{x}); \quad \mathbf{x} \in \Omega \tag{1}$$

which is subject to the boundary conditions

$$\mathbf{B}(u) = \mathbf{g}(\mathbf{x}); \quad \mathbf{x} \in \Gamma \tag{2}$$

In the above, Λ and $\mathbf{B}$ are partial differential operators, Ω is the solution domain, and Γ is its boundary. We can decompose the solution into two parts, a particular solution part u_p and a homogeneous part u_h, such that

$$u = u_p + u_h \tag{3}$$

We require the particular solution to satisfy Eq. (1) as

$$\Lambda u_p = f(\mathbf{x}); \quad \mathbf{x} \in \Omega \tag{4}$$

but not the boundary condition Eq. (2), such that the solution is easier to find. The solutions of Eq. (4) can be found by the MPS which will be described later. Then, it is easy to show that the homogeneous solution must satisfy

$$\Lambda u_h = 0; \quad \mathbf{x} \in \Omega \tag{5}$$

and is subject to the modified boundary condition

$$\mathbf{B}u_h = \mathbf{g}(\mathbf{x}) - \mathbf{B}u_p; \quad \mathbf{x} \in \Gamma \tag{6}$$

[1] Department of Information Technology, Toko University, Chia-Yi County, 61363, Taiwan. E-mail: tsaichiacheng@ntu.edu.tw

Eqs. (5) and (6) can then be formally solved by boundary-type numerical methods. The original idea of this formulation stemmed form the dual reciprocity boundary element method (DRBEM) innovated by Nardini and Brebbia (1982). Actually, the dual reciprocity procedure is equivalent to the MPS.

From the above description, it is clear that the applicability of previous formulation depends on the availability of the particular solution u_p associated with right hand side function f and the operator Λ. It is obvious that analytical expressions for u_p are rare. Hence, in the same spirit of numerical solutions, approximate expressions of the particular solution are sought. This can be accomplished by approximating the right hand side function as a summation of basis (trial) functions,

$$f(\mathbf{x}) \approx \sum_i \alpha_i \varphi_i(\mathbf{x}) \tag{7}$$

where φ_i are basis functions, and α_i are constants to be determined by solving a linear system resulting from collocations or least squares. Once the basis functions in Eq. (7) are selected, the problem of finding particular solution is reduced to

$$\Lambda \psi_i = \varphi_i(\mathbf{x}); \quad \mathbf{x} \in \Omega \tag{8}$$

where ψ_i is the particular solution corresponding to φ_i. Once ψ_i is found, the approximate particular solution sought for becomes

$$u_p \approx \sum_{i=1}^{n} \alpha_i \psi_i(\mathbf{x}) \tag{9}$$

Therefore, the success of the MPS depends on the availability of the exact expression of ψ_i associated to the basis function φ_i and operators Λ.

In the last few decades, significant progress has been made in obtaining analytical particular solutions for various basis functions. Among these are the radial basis functions [Cheng, Lafe, and Grilli (1994); Golberg (1995); Golberg, Chen, and Karur (1996); Golberg and Chen (1999); Muleshkov, Golberg, and Chen (1999); Cheng (2000); Muleshkov and Golberg (2007)], the trigonometric functions [Atkinson (1985); Li and, Chen (2004)], the monomials [Janssen and, Lambert (1992); Cheng, Lafe, and Grilli (1994); Cheng, Chen, Golberg, Rashed (2001); Golberg, Muleshkov, Chen, and Cheng (2003)], the Chebyshev polynomials [Golberg, Muleshkov, Chen, and Cheng (2003); Reutskiy and Chen (2006); Karageorghis, and Kyza (2007); Ding and Chen (2007)] and others. In this study, we consider the analytical particular solution corresponding to the Chebyshev polynomials.

In the original study of Chebyshev interpolation, Golberg, Muleshkov, Chen, and Cheng (2003) utilized symbolic software *Mathematica* to connect monomials and Chebyshev polynomials and used their derived particular solution as follows

$$\Lambda \sum_{i,j} \alpha_{ij} x^i y^j = x^m y^n \tag{10}$$

to implement floating number computing. However some book keepings are required in their study. Reutskiy and Chen (2006) remedied the tedious book keeping by using two-stage approximations of trigonometric functions and Chebyshev polynomials. On the other hand, Karageorghis, and Kyza (2007) studied the same issue by directly considering

$$\Lambda \sum_{i,j} \beta_{ij} T_i(x) T_j(y) = T_m(x) T_n(y) \tag{11}$$

where $T_m(x)$ is the Chebyshev polynomial of degree m. However matrix inverses are conducted to their final formulas. Thus they have to face the issue of ill-conditioning. Recently, Ding and Chen (2007) discovered a recursive formulation free from book keepings and matrix inverses. Thus their formulation can be implemented by floating number computing.

It is well known that systems involving the coupling of a set of second order elliptic equations are encountered in some engineering problems, such as a multilayered aquifer system [Cheng and Morohunfola (1993A); Cheng and Morohunfola (1993B)], or a multiple porosity system [Cheng (2000)]. These coupled systems can be reduced to a single partial differential equation by using the Hörmander operator decomposition technique [Hörmander (1963)]. The resultant partial differential equations usually involve the polyharmonic or the products of Helmholtz-type operators. This motivated us to generalize the situation by considering analytical particular solutions of

$$\Lambda \sum_{i,j,k} \alpha_{ijk} x^i y^j z^k = x^m y^n z^l \tag{12}$$

where the partial differential operator Λ is in a very general form

$$\begin{aligned}\Lambda &= \prod_{i=1}^{\sigma} (\Delta - \lambda_i)^{A_i} \\ &= (\Delta - \lambda_1)^{A_1} (\Delta - \lambda_2)^{A_2} \dots (\Delta - \lambda_\sigma)^{A_\sigma}\end{aligned} \tag{13}$$

with $A_1, A_2, \dots, A_\sigma \in$ N, $\lambda_1, \lambda_2, ..., \lambda_\sigma \in$ C, and Δ is the Laplacian. We notice that when $\prod_{i=1}^{\sigma} \lambda_i \neq 0$ in Eq. (13), the operator is the product of the Helmholtz-type operators, and when $\sigma = 1$, Eq. (13) becomes the poly-Hemholtz or polyharmonic operator.

In our implementation, we use the explicit formulas between monomials and Chebyshev polynomials [Mason and Handscomb (2003)] as well as the explicit formulas in Eq. (12), which can be easily coded by multiple loops. Our implementation is free from book keepings and matrix inverses. Compared to the recursive formulation [Ding and Chen (2007)], our formulation is easier and more suitable for higher order PDEs and three dimensions. Furthermore, we solve the homogeneous solution by the MFS in our implementation [Kupradze and Aleksidze (1964); Bogomolny (1985); Tsai, Young, Cheng (2002); Smyrlis, Karageorghis (2003); Chen, Fan, Young, Murugesan, Tsai (2005); Young, Ruan (2005); Young, Chen, Chen, Kao (2007); Hu, Young, Fan (2008)]. This two-stage procedure forms a boundary-type meshless numerical method, which is opposite to domain-type meshless numerical methods solving PDEs in one stage, such as Kansa's method [Emdadi, Kansa, Libre, Rahimian, Shekarchi (2008); Kosec, Šarler (2008); Libre, Emdadi, Kansa, Rahimian, Shekarchi (2008)] and the meshless local Petrov-Galerkin method [Gao, Liu, Liu (2006); Han, Liu, Rajendran, Atluri (2006); Sladek, Sladek, Wen, Aliabadi (2006); Sladek, Sladek, Zhang, Solek (2007); Wu, Shen, Tao (2007)] and *et al*. Our numerical results are very accurate even for eighth order PDEs and three dimensions due to the exponential convergence of the Chebyshev interpolation and the MFS.

In order to complete the mathematical consideration, we also illustrate by an example for obtaining explicit particular solutions with Chebyshev polynomials at right hand side

$$\Lambda \sum_{i,j,k} \beta_{ijk} T_i(x) T_j(y) T_k(z) = T_m(x) T_n(y) T_l(z) \tag{14}$$

$$\Lambda \sum_{i,j,k} \gamma_{ijk} x^i y^j z^k = T_m(x) T_n(y) T_l(z) \tag{15}$$

by using the explicit formulas between monomials and Chebyshev polynomials [Mason and Handscomb (2003)]. We do not suggest direct implementations of Eqs. (14) and (15) because they are less efficient.

2 Chebyshev interpolation

We begin with the trivariate Chebyshev polynomial interpolation $\tilde{f}(x,y,z)$ for a function $f(x,y,z)$, in which lower dimensional situations are included. The Chebyshev interpolant using Gauss-Lobatto nodes for cubic domain $[x_a, x_b] \times [y_a, y_b] \times [z_a, z_b]$ takes the form:

$$\begin{aligned}\tilde{f}(x,y,z) = \sum_{i}^{l} \sum_{j}^{m} \sum_{k}^{n} a_{ijk} T_i \left(\frac{2x - x_b - x_a}{x_b - x_a} \right) \\ T_j \left(\frac{2y - y_b - y_a}{y_b - y_a} \right) T_k \left(\frac{2z - z_b - z_a}{z_b - z_a} \right)\end{aligned} \tag{16}$$

where

$$\begin{aligned}a_{ijk} = \frac{8}{lmn \overline{c}_{l,i} \overline{c}_{m,j} \overline{c}_{n,k}} \sum_{\overline{i}}^{l} \sum_{\overline{j}}^{m} \sum_{\overline{k}}^{n} \frac{f(x_{\overline{i}}, x_{\overline{j}}, x_{\overline{k}})}{\overline{c}_{l,\overline{i}} \overline{c}_{m,\overline{j}} \overline{c}_{n,\overline{k}}} \\ \cos \frac{i\pi\overline{i}}{l} \cos \frac{j\pi\overline{j}}{m} \cos \frac{k\pi\overline{k}}{n}\end{aligned} \tag{17}$$

with

$$\overline{c}_{l,0} = \overline{c}_{l,l} = 2, \; \overline{c}_{l,i} = 1, \; 1 \leq i \leq l-1 \tag{18a}$$

$$x_{\overline{i}} = \frac{(x_b - x_a)}{2} \cos \frac{\overline{i}\pi}{l} + \frac{x_b + x_a}{2} \tag{18b}$$

$$y_{\overline{j}} = \frac{(y_b - y_a)}{2} \cos \frac{\overline{j}\pi}{m} + \frac{y_b + y_a}{2} \tag{18c}$$

$$z_{\overline{k}} = \frac{(z_b - z_a)}{2}\cos\frac{\overline{k}\pi}{n} + \frac{z_b + z_a}{2} \tag{18d}$$

Note that l, m and n are the numbers of Gauss-Lobatto nodes in the x, y and z directions, respectively. Also, it is well known that the use of Gauss-Lobatto nodes ensure the spectral convergence for Chebyshev interpolation. Details of Chebyshev interpolation can be found in a recent excellent review book of Mason and Handscomb (2003).

In the application of MPS, the right hand side $f(x,y,z)$ in Eq. (4) is first approximated by the Chebyshev interpolation. In the following derivations, we assume $x_a = y_a = z_a = -1$, $x_b = y_b = z_b = 1$, and $[-1,\ 1]\times[-1,\ 1]\times[-1,\ 1]$ big enough to enclose Ω. These assumptions do not lose the generality. Then, Eq. (16) reduces to the following form:

$$\tilde{f}(x,y,z) = \sum_i^l \sum_j^m \sum_k^n a_{ijk} T_i T_j T_k \tag{19}$$

Eq. (19) can also be rewritten in terms of monomial basis by using [Mason and Handscomb (2003)]

$$T_n(x) = \sum_{k=0}^{[n/2]} c_k^{(n)} x^{n-2k} \tag{20}$$

with

$$c_k^{(n)} = (-1)^k 2^{n-2k-1} \frac{n(n-k-1)!}{k!(n-2k)!}, \quad n > 2k \tag{21a}$$

$$c_k^{(2k)} = (-1)^k, \quad k \geq 0 \tag{21b}$$

In Eq. (20), the brackets [ ] in the summation limits indicate taking the integer part of the argument. Using Eqs. (19) and (20) it is enough to obtain

$$\tilde{f}(x,y,z) = \sum_i^l \sum_j^m \sum_k^n b_{ijk} x^i y^j z^k \tag{22}$$

It should be noted that, we can actually write b_{ijk} explicitly, however that is not the best way for implementations. We will return to this issue in Section 6.

3 Particular solutions of monomials – poly-Helmholtz equation

Since we have the approximated right hand side in terms of monomials, we should find the particular solutions defined by Eq. (12) in order to apply the MPS. In this section, we consider $\Lambda = (\Delta - \lambda)^L$ first. In other words, we are going to seek $P_\lambda^{(L,l,m,n)}(x,y,z)$ so that

$$(\Delta - \lambda)^L P_\lambda^{(L,l,m,n)}(x,y,z) = x^l y^m z^n \tag{23}$$

Golberg, Muleshkov, Chen, and Cheng (2003) had found the solution for $L = 1$, $\lambda \neq 0$

$$P_\lambda^{(1,l,m,n)}(x,y,z) = \sum_{i=0}^{\left[\frac{l}{2}\right]} \sum_{j=0}^{\left[\frac{m}{2}\right]} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{-(i+j+k)!\,l!\,m!\,n!\,x^{l-2i} y^{m-2j} z^{n-2k}}{\lambda^{i+j+k+1} i!\,j!\,k!\,(l-2i)!\,(m-2j)!\,(n-2k)!} \tag{24}$$

or

$$\Delta P_\lambda^{(1,l,m,n)} - \lambda P_\lambda^{(1,l,m,n)} = x^l y^m z^n \tag{25}$$

Taking partial derivative with respect λ, we have

$$\begin{aligned} &\Delta \frac{\partial P_\lambda^{(1,l,m,n)}}{\partial \lambda} - P_\lambda^{(1,l,m,n)} - \lambda \frac{\partial P_\lambda^{(1,l,m,n)}}{\partial \lambda} = 0 \\ &\Rightarrow (\Delta - \lambda) \frac{\partial P_\lambda^{(1,l,m,n)}}{\partial \lambda} = P_\lambda^{(1,l,m,n)} \\ &\Rightarrow (\Delta - \lambda)^2 \frac{\partial P_\lambda^{(1,l,m,n)}}{\partial \lambda} = x^l y^m z^n \end{aligned} \tag{26}$$

From Eq. (26), it is clear that

$$P_\lambda^{(2,l,m,n)} = \frac{\partial P_\lambda^{(1,l,m,n)}}{\partial \lambda} \tag{27}$$

Repeating the above derivations we can obtain

$$P_\lambda^{(L,l,m,n)} = \frac{1}{(L-1)!} \frac{\partial^{L-1} P_\lambda^{(1,l,m,n)}}{\partial \lambda^{L-1}} \tag{28}$$

Using Eqs. (24) and (28) we can have the desired particular solution for $\lambda \neq 0$ as follows

$$P_\lambda^{(L,l,m,n)}(x_1,x_2,x_3) = \sum_{i=0}^{\left[\frac{l}{2}\right]}\sum_{j=0}^{\left[\frac{m}{2}\right]}\sum_{k=0}^{\left[\frac{n}{2}\right]} \left[(-1)^L(i+j+k+L-1)!l!m!n!x^{l-2i}y^{m-2j}z^{n-2k}\right] \Big/ \left[(L-1)!\lambda^{i+j+k+L}i!j!k!(l-2i)!(m-2j)!(n-2k)!\right] \quad (29)$$

It should be noticed that Eq. (29) reduce to two-dimensional solutions by setting $n=0$. Also, Eq. (29) can be implemented very easily by multiple loops. If we compare our formula with the recursive formulation introduced by Ding and Chen (2007), it is clear that our formula is easier for implementation for higher order PDEs and three dimensions which is often occurred in engineering applications.

4 Particular solutions of monomials – polyharmonic equation

Then, we consider the particular solutions corresponding to polyharmonic operators as follows:

$$\Delta^L P_0^{(L,l,m,n)}(x,y,z) = x^l y^m z^n \quad (30)$$

In order to find the solutions, we consider Eq. (29) with $l=0$

$$P_\lambda^{(L,0,m,n)}(x_2,x_3) = \sum_{j=0}^{\left[\frac{m}{2}\right]}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^L(j+k+L-1)!m!n!y^{m-2j}z^{n-2k}}{(L-1)!\lambda^{j+k+L}j!k!(m-2j)!(n-2k)!} \quad (31)$$

Then substitute $\lambda = -\frac{\partial^2}{\partial x^2}$ to Eq. (31) and multiply x^l, we can obtain

$$\Delta^L \sum_{j=0}^{\left[\frac{m}{2}\right]}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{j+k}(j+k+L-1)!m!n!x^l y^{m-2j}z^{n-2k}}{(L-1)!\left(\frac{\partial^2}{\partial x^2}\right)^{j+k+L} j!k!(m-2j)!(n-2k)!} = x^l y^m z^n \quad (32)$$

It is easy to show that

$$\left(\frac{\partial^2}{\partial x^2}\right)^{j+k+L}\left[\frac{l!}{(l+2j+2k+2L)!}x^{l+2j+2k+2L}\right] = x^l \quad (33)$$

Using Eq. (33) to eliminate the $\left(\frac{\partial^2}{\partial x^2}\right)^{j+k+L}$ operator in Eq. (32), we obtain the particular solution for polyharmonic operator given as

$$P_0^{(L,l,m,n)}(x,y,z) = \sum_{j=0}^{\left[\frac{m}{2}\right]}\sum_{k=0}^{\left[\frac{n}{2}\right]} \left[(-1)^{j+k}(j+k+L-1)!\, l!m!n!x^{l+2j+2k+2L}y^{m-2j}z^{n-2k}\right] \Big/ \left[(L-1)!\, j!k!(l+2j+2k+2L)!(m-2j)!(n-2k)!\right] \quad (34)$$

For $L=1$, the result obtained in Eq. (34) appears to be simpler as compared to those found in Golberg, Muleshkov, Chen, and Cheng (2003) for 3D Laplacian operator with monomial right hand side.

5 Particular solutions of monomials – $\Lambda = \prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}$

As we have mentioned that many problems in engineering and science involve the product of Helmholtz-type and harmonic operators. Therefore we are going to derive particular solutions in Eq. (12) with $\Lambda = \prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}$.

Banerjee (1994) introduced the difference trick to find the fundamental solution for the product of Helmholtz and harmonic operators. Here we find the difference trick can be linked to the partial fraction [O'Neil (2002)]. Therefore we use the particular solutions of poly-Helmholtz and polyharmonic operators, Eqs. (29) & (34), as well as the partial fraction to derive the desired particular solution. Consider we have the following partial fraction

$$\prod_{i=1}^{\sigma}\frac{1}{(\xi-\lambda_i)^{A_i}} = \sum_{j=1}^{\sigma}\sum_{k=0}^{A_j-1}\frac{\Theta_{jk}}{(\xi-\lambda_j)^{A_j-k}} \quad (35)$$

where the coefficients Θ_{jk} can be find formally by the decomposition of partial fraction [O'Neil

(2002)]. Then, the particular solution corresponding to

$$\left[\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}\right]P^{(l,m,n)}(x,y,z)=x^l y^m z^n \tag{36}$$

is

$$P^{(l,m,n)}(x,y,z)=\sum_{i=1}^{\sigma}\sum_{j=0}^{A_i-1}\Theta_{ij}P_{\lambda_i}^{(A_i-j,l,m,n)} \tag{37}$$

where $P_{\lambda_i}^{(A_i-j,l,m,n)}$ are given in Eqs. (29) and (34). To complete the derivation we first compare Eqs. (23), (30), and (36) to obtain

$$\begin{aligned}&\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}P^{(l,m,n)}(x,y,z)\\&=(\Delta-\lambda)^L P_{\lambda}^{(L,l,m,n)}(x,y,z)\\&\Rightarrow\frac{\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}P^{(l,m,n)}(x,y,z)}{(\Delta-\lambda)^L}=P_{\lambda}^{(L,l,m,n)}(x,y,z)\end{aligned} \tag{38}$$

Then, we substitute $\xi=\Delta$ to Eq. (35), multiply both side by $\left[\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}\right]P^{(l,m,n)}(x,y,z)$, and use Eq. (38) to obtain

$$\begin{aligned}&\prod_{i=1}^{\sigma}\frac{1}{(\Delta-\lambda_i)^{A_i}}=\sum_{j=0}^{\sigma}\sum_{k=0}^{A_j-1}\frac{\Theta_{jk}}{(\Delta-\lambda_j)^{A_j-k}}\\&\Rightarrow P^{(l,m,n)}(x,y,z)\\&\quad=\sum_{j=0}^{\sigma}\sum_{k=0}^{A_j-1}\frac{\Theta_{jk}\left[\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}\right]}{(\Delta-\lambda_j)^{A_j-k}}P^{(l,m,n)}(x,y,z)\\&\Rightarrow P^{(l,m,n)}(x,y,z)\\&\quad=\sum_{j=0}^{\sigma}\sum_{k=0}^{A_j-1}\Theta_{jk}P_{\lambda_j}^{(A_j-k,l,m,n)}(x,y,z)\end{aligned} \tag{39}$$

These finish our derivations.

Since we have the particular solution in Eq. (12) we are ready to apply the MPS. We can use Eqs. (29), (34), and (37) with Eq. (22) to get the approximated particular solution corresponding to

$$\left[\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}\right]\tilde{F}(x,y,z)=\tilde{f}(x,y,z) \tag{40}$$

as follows

$$\tilde{F}(x,y,z)=\sum_{i}^{l}\sum_{j}^{m}\sum_{k}^{n}b_{ijk}P^{(i,j,k)}(x,y,z) \tag{41}$$

These complete the MPS formulations.

Example 1: Let's find the particular solutions of

$$\Delta^2(\Delta-4)(\Delta+9)^2\overline{P}(x,y,z)=x^l y^m z^n \tag{42}$$

We first consider the following partial fraction by the method of undetermined coefficient [O'Neil (2002)]:

$$\begin{aligned}\frac{1}{\xi^2(\xi-4)(\xi+9)^2}&=-\frac{1}{324\xi^2}-\frac{1}{11664\xi}\\&+\frac{1}{2704(\xi-4)}-\frac{1}{1053(\xi+9)^2}\\&-\frac{35}{123201(\xi+9)}\end{aligned} \tag{43}$$

Then we have the particular solution as follows

$$\begin{aligned}\overline{P}=&-\frac{P_0^{(2,l,m,n)}}{324}-\frac{P_0^{(1,l,m,n)}}{11644}+\frac{P_4^{(1,l,m,n)}}{2704}\\&-\frac{P_{-9}^{(2,l,m,n)}}{1053}-\frac{35P_{-9}^{(1,l,m,n)}}{123201}\end{aligned} \tag{44}$$

where $P_0^{(2,l,m,n)}$ and $P_0^{(1,l,m,n)}$ are given in Eq. (34) as well as $P_4^{(1,l,m,n)}$, $P_{-9}^{(2,l,m,n)}$, and $P_{-9}^{(1,l,m,n)}$ are addressed in Eq. (29).

Example 2: Then we try to find the particular solutions of

$$(\Delta^2+1)\hat{P}(x,y,z)=x^l y^m z^n \tag{45}$$

We first consider the following partial fraction:

$$\frac{1}{\xi^2+1}=\frac{1}{(\xi+\mathbf{i})(\xi-\mathbf{i})}=\frac{1}{2\mathbf{i}(\xi-\mathbf{i})}-\frac{1}{2\mathbf{i}(\xi+\mathbf{i})} \tag{46}$$

Then we have the particular solution as follows

$$\hat{P}=\frac{P_{\mathbf{i}}^{(1,l,m,n)}}{2\mathbf{i}}-\frac{P_{-\mathbf{i}}^{(1,l,m,n)}}{2\mathbf{i}} \tag{47}$$

where $P_{\mathbf{i}}^{(1,l,m,n)}$ and $P_{-\mathbf{i}}^{(1,l,m,n)}$ are given in Eq. (29). It is interesting to note that $\hat{P}$ is a real valued function although we are working in complex numbers. This is another advantage over Ding and Chen (2007).

6 Particular solutions of Chebyshev polynomials

In order to complete the mathematical consideration, we also illustrate by an example about how to obtain explicit particular solutions with Chebyshev polynomials at right hand side as defined in Eqs. (14) and (15). Consider particular solutions of two-dimensional polyharmonic operator as

$$\Delta^L \tilde{P}(x,y) = T_h(x)T_i(y) \tag{48}$$

By using Eq. (20) we have

$$T_h(x)T_i(y) = \sum_{j=0}^{[h/2]}\sum_{k=0}^{[i/2]} c_j^{(h)} c_k^{(i)} x^{h-2j} y^{i-2k} \tag{49}$$

Then we can utilize Eq. (34) with $n=0$ to have

$$\Delta^L P_0^{(L,l,m,0)}(x,y,z) = x^l y^m \tag{50}$$

with

$$P_0^{(L,l,m,0)}(x,y,z) = \sum_{j=0}^{\left[\frac{m}{2}\right]} e_j^{L,l,m} x^{l+2j+2L} y^{m-2j} \tag{51a}$$

$$e_j^{(L,l,m)} = \frac{(-1)^j (j+L-1)!l!m!}{(L-1)!j!(l+2j+2L)!(m-2j)!} \tag{51b}$$

Combining Eqs. (48)–(50) we can obtain

$$\tilde{P}(x,y) = \sum_{j=0}^{\left[\frac{h}{2}\right]}\sum_{k=0}^{\left[\frac{i}{2}\right]}\sum_{l=0}^{\left[\frac{i}{2}\right]-k} c_j^{(h)} c_k^{(i)} e_l^{L,h-2j,i-2k} x^{h-2j+2l+2L} y^{i-2k-2l} \tag{52}$$

Eq. (52) is just the desired particular solution in terms of monomials. The particular solution can also be rewritten in terms of Chebyshev polynomials. We first introduce the explicit formula to express monomials in terms of Chebyshev polynomials introduced by Mason and Handscomb (2003) as follows:

$$x^n = \sum_{k=0}^{[n/2]} d_k^{(n)} T_{n-2k}(x) \tag{53}$$

with

$$d_k^{(n)} = 2^{1-n}\frac{n!}{k!(n-k)!} \tag{54}$$

Substituting Eq. (53) to Eq. (52) we have

$$\tilde{P}(x,y) = \sum_{j=0}^{\left[\frac{h}{2}\right]}\sum_{k=0}^{\left[\frac{i}{2}\right]}\sum_{l=0}^{\left[\frac{i}{2}\right]-k}\sum_{m=0}^{\left[\frac{h}{2}\right]-j+2l+2L}\sum_{n=0}^{\left[\frac{i}{2}\right]-k-l} c_j^{(h)} c_k^{(i)} e_l^{L,h-2j,i-2k} d_m^{h-2j+2l+2L} d_n^{i-2k-2l} T_{h-2j+2l+2L-2m}(x) T_{i-2k-2l-2n}(x) \tag{55}$$

Eq. (55) is just the explicit particular solutions in terms of Chebyshev polynomials. These derivations can be extended to poly-Helmholtz equations and three dimensions. Also, the particular solutions can be considered as an alternative to the formulations derived by Karageorghis, and Kyza (2007), in which they had to solve systems of linear equations.

Here we provide these formulas in order to show the mathematical possibility of obtaining explicit particular solutions with Chebyshev polynomials at right hand side. In this paper, we suggest to implement the MPS by using Eqs. (16), (22), and (37) in three stages instead of direct utilizing Eqs. (16) & (52) or Eq. (16) & (55) due to the computational efficiency. We can state the reason clearer that Eqs. (16) and (22) have to be executed once for a given $f(x,y,z)$ but the modified boundary condition in Eq. (6) should be calculated at all the discrete points by using Eq. (37). Thus the three-stage implementation is more efficient.

7 Numerical results

Once we find an approximate particular solution, we can solve the homogeneous problem (5) with given modified boundary data (6) by the MFS. Consider

$$\left[\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}\right] u_h = 0 \tag{56}$$

Then we can approximate $u_h(\mathbf{x})$ by the MFS as

$$u_h(\mathbf{x}) \cong \sum_{i=1}^{\sigma}\sum_{j=0}^{A_i-1}\sum_{k=1}^{K} \gamma_{ijk} G_{\lambda_i}^{(A_i-j)}(\mathbf{x}-\mathbf{s}_k) \tag{57}$$

where $G_\lambda^L(\mathbf{x})$ is the fundamental solution defined by

$$(\Delta-\lambda)^L G_\lambda^L(\mathbf{x}) = \delta(\mathbf{x}) \tag{58}$$

which can be found in Cheng, Antes, and Ortner (1994). And $\mathbf{s}_k$ are K prescribed source points outside the computation domains. The $K\sum_{i=1}^{\sigma} A_i$ unknown coefficient γ_{ijk} can be found by enforcing the boundary conditions (6) at K boundary field points. It is clear that the order of $\prod_{i=1}^{\sigma}(\Delta-\lambda_i)^{A_i}$ is $2\sum_{i=1}^{\sigma} A_i$. Therefore we need $\sum_{i=1}^{\sigma} A_i$ boundary conditions which are just the ranks of the vectored valued partial differential operator $\mathbf{B}$ in Eq. (6). In other words, the enforcement of boundary conditions at K boundary field points results in $K\sum_{i=1}^{\sigma} A_i$ linear equations, which can be used to solve the $K\sum_{i=1}^{\sigma} A_i$ unknown coefficients γ_{ijk}. Details of the MFS can be found in theoretical work of Bogomolny (1985). Also Tsai, Lin, Young, and Atluri (2006) discussed the locations of source and boundary field points.

Example 3: Let's solve two-dimensional modified Helmholtz equation

$$(\Delta-900)u = -899(e^{-x}+e^{-y}) \tag{59}$$

in $[-1,\ 1]\times[-1,\ 1]$, Dirichlet boundary condition is set up corresponding to exact solution

$$u = e^{-x}+e^{-y} \tag{60}$$

40 source points are selected in the MFS. Table I gives the root mean square errors (RMSEs) for different l and m in which excellent accuracy can be observed. Here l and m are the numbers of Gauss-Lobatto nodes in the x and y directions, respectively.

Table I: The RMSEs for Example 3

	$l=m=4$	$l=m=8$	$l=m=12$	$l=m=16$	$l=m=20$
RMSEs	2.16E-02	1.38E-06	4.18E-12	3.07E-13	2.41E-12

Example 4: Then we solve two-dimensional Poisson's equation

$$\Delta u = e^{-x}+e^{-y} \tag{61}$$

also in $[-1,\ 1]\times[-1,\ 1]$ with Dirichlet boundary condition. The exact solution is also

$$u = e^{-x}+e^{-y} \tag{62}$$

Table II gives the RMSEs. The accuracy is also great.

Table II: The RMSEs for Example 4

	$l=m=4$	$l=m=8$	$l=m=12$	$l=m=16$	$l=m=20$
RMSEs	4.77E-05	2.94E-10	1.92E-10	1.92E-10	1.76E-10

Example 5: Our explicit particular solutions enable us to solve three-dimensional problems very easily. Consider three-dimensional modified Helmholtz equation

$$(\Delta-900)u = -899(e^{-x}+e^{-y}+e^{-z}) \tag{63}$$

in $[-1,\ 1]\times[-1,\ 1]\times[-1,\ 1]$. Dirichlet boundary condition is set up by using the exact solution

$$u = e^{-x}+e^{-y}+e^{-z} \tag{64}$$

386 source points are selected in the MFS for this three-dimensional case. Table III gives the RMSEs for different l, m and n. The results also perform well. Here l, m and n are the numbers of Gauss-Lobatto nodes in the x, y and z directions, respectively.

Table III: The RMSEs for Example 5

	$l=m=n=4$	$l=m=n=8$	$l=m=n=12$	$l=m=n=16$	$l=m=n=20$
RMSEs	1.48E-01	5.45E-06	8.33E-12	4.15E-12	1.77E-11

Example 6: Also, we can solve three-dimensional Poisson's equation

$$\Delta u = e^{-x}+e^{-y}+e^{-z} \tag{65}$$

with Dirichlet boundary condition. The exact solution is also

$$u = e^{-x}+e^{-y}+e^{-z} \tag{66}$$

Table IV gives the RMSEs in which excellent results can also be observed.

Table IV: The RMSEs for Example 6

	$l=m=n=4$	$l=m=n=8$	$l=m=n=12$	$l=m=n=16$	$l=m=n=20$
RMSEs	4.18E-05	2.65E-10	4.17E-11	4.15E-11	2.89E-10

Example 7: As we have mentioned that our formulation can be extended to higher order PDEs. We consider polyharmonic equation in two-dimensional domain $[-1,\ 1]\times[-1,\ 1]$ as

$$\Delta^4 u = e^{-x} + e^{-y} \tag{67}$$

The exact solution of the problem is set up as

$$u = e^{-x} + e^{-y} \tag{68}$$

In this problem, we have to set up four boundary conditions in Eq. (6) as

$$\mathbf{B} = \begin{bmatrix} 1 & \frac{\partial}{\partial n} & \frac{\partial^2}{\partial n^2} & \frac{\partial^3}{\partial n^3} \end{bmatrix}^T \tag{69}$$

The MFS equation in Eq. (57) in this case reads

$$u_h(\mathbf{x}) \cong \sum_{k=1}^{K} \left(\gamma_{1k} r_k^6 \ln r_k + \gamma_{2k} r_k^4 \ln r_k + \gamma_{3k} r_k^2 \ln r_k + \gamma_{4k} \ln r_k\right) \tag{70}$$

where $r_k = |\mathbf{x} - \mathbf{s}_k|$.

Table V gives the RMSEs where nice results are also found.

Table V: The RMSEs for Example 7

	$l=m=4$	$l=m=8$	$l=m=12$	$l=m=16$	$l=m=20$
RMSEs	4.75E-10	2.98E-12	2.98E-12	2.98E-12	2.98E-12

Example 8: Then let's consider a two-dimensional problem in $[-1,\ 1]\times[-1,\ 1]$ as

$$\Delta^2(\Delta-900)(\Delta-100)u = 89001(e^{-x}+e^{-y}) \tag{71}$$

The same four types of boundary conditions are set up corresponding to exact solution

$$u = e^{-x} + e^{-y} \tag{72}$$

In this case, the MFS equation becomes

$$u_h(\mathbf{x}) \cong \sum_{k=1}^{K} \left(\gamma_{1k} K_0(30 r_k) + \gamma_{2k} K_0(10 r_k) + \gamma_{3k} r_k^2 \ln r_k + \gamma_{4k} \ln r_k\right) \tag{73}$$

where $K_0()$ is the zero order modified Bessel function of second kind.

We also address the nice RMSEs in Table VI.

Table VI: The RMSEs for Example 8

	$l=m=4$	$l=m=8$	$l=m=12$	$l=m=16$	$l=m=20$
RMSEs	7.29E-06	2.61E-10	3.29E-10	3.29E-10	3.23E-10

8 Discussions

Many problems in engineering and science are governed by a system of coupled linear partial differential equations. Through Hörmander linear partial differential operator theory and algebraic factorization, they can be reduced to a single equation involving the products of Helmholtz-type and polyharmonic operators. In this paper we derived explicit closed-form particular solutions for these operators with Chebyshev polynomials at right-hand side. With these particular solutions we can transform the inhomogeneous PDEs to homogeneous ones which can then be solved by the MFS. Numerical experiments including eighth order PDEs and three-dimensional cases are carried out. Our numerical results are extremely accurate due to the exponential convergence of both the Chebyshev interpolation and the MFS. Applications of these formulations to problems governed by a system of coupled linear partial differential equations are currently under investigations.

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