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Slow Rotation of an Axisymmetric Slip Particle about Its Axis of Revolution

Yi W. Wan[1] **and Huan J. Keh**[2]

Abstract: The problem of the rotation of a rigid particle of revolution about its axis in a viscous fluid is studied theoretically in the steady limit of low Reynolds number. The fluid is allowed to slip at the surface of the particle. A singularity method based on the principle of distribution of a set of spherical singularities along the axis of revolution within a prolate particle or on the fundamental plane within an oblate particle is used to find the general solution for the fluid velocity field that satisfies the boundary condition at infinity. The slip condition on the surface of the rotating particle is then satisfied by applying a boundary collocation technique to this general solution to numerically determine the unknown coefficients. The torque exerted on the particle by the fluid is evaluated with good convergence behavior for various cases. For the rotation of a slip sphere and of a no-slip prolate or oblate spheroid about its axis of symmetry, our torque results agree excellently with the corresponding exact solutions. For the rotation of a slip spheroid, the agreement between our results and the available approximate analytical solution is also very good. It is found that the hydrodynamic torque on the rotating spheroid with a given equatorial radius is a monotonically increasing function of the axial-to-radial aspect ratio of the spheroid for a no-slip or finite-slip spheroid and vanishes for a perfectly slip spheroid. For the rotation of a spheroid with a fixed aspect ratio, the hydrodynamic torque decreases monotonically with an increase in the slip coefficient of the particle.

Keywords: Creeping flow, Particle of revolution, Prolate and oblate spheroids, Singularity method, Boundary collocation technique, Slip-flow surface, Hydrodynamic torque

[1] Department of Chemical Engineering, National Taiwan University, Taipei, 10617, Taiwan, ROC
[2] Corresponding Author: Department of Chemical Engineering, National Taiwan University, Taipei, 10617, Taiwan, ROC. E-mail: huan@ntu.edu.tw

1 Introduction

The translational and rotational motions of small particles in a continuous medium at low Reynolds numbers are of much fundamental and practical interest in the areas of chemical, biomedical, and environmental engineering and science. The theoretical treatment of this subject has grown out of the classic work of Stokes (1951) for a rigid sphere moving in an unbounded viscous fluid. Oberbeck (1876) and Jeffery (1915) extended this result to the translation of an ellipsoid and rotation of a particle of revolution, respectively. More recently, the creeping flow caused by the motion of a particle of more general shape has been treated in the literature by the symbolic operator method [Brenner (1966)], boundary collocation method [Gluckman, Weinbaum, and Pfeffer (1972); Hsu and Ganatos (1989)], singularity method [Chwang and Wu (1975)], and boundary integral (boundary element) method [Youngren and Acrivos (1975); Tsai, Young, and Cheng (2002); Diaz and Barthès-Biesel (2002); Staben, Zinchenko, and Davis (2003); Sellier (2008)].

When one tries to solve the Stokes problems, it is usually assumed that no slippage arises at the solid-fluid interfaces. Actually, this is an idealization of the transport processes involved. The phenomena that the adjacent fluid can slip frictionally over a solid surface occur for cases such as the rarified gas flow around an aerosol particle [Kennard (1938); Keh and Shiau (2000)], the water flow near a hydrophobic surface [Tretheway and Meinhart (2002); Willmott (2008)], the micropolar fluid flow past a rigid particle [Sherif, Faltas, and Saad (2008)], and the viscous fluid flow over the surface of a porous medium [Beavers and Joseph (1967); Saffman (1971); Nir (1976)] or a small particle of molecular size [Hu and Zwanzig (1974)]. Presumably, any such slipping would be proportional to the local shear stress of the fluid next to the solid surface [Happel and Brenner (1983); Keh and Chen (1996)], known as the Navier slip [see Eq. (3a)], at least as the velocity gradient is small. The constant of proportionality, β^{-1}, is termed the slip coefficient of the solid surface. The quantity η/β (where η is the fluid viscosity) is a length, which can be pictured by noting that the fluid motion is the same as if the solid surface was displaced inward by a distance η/β with the velocity gradient extending uniformly right up to no-slip velocity at the surface.

Basset (1961) has found that the hydrodynamic torque acting on a rotating rigid sphere of radius b with a slip-flow boundary condition at its surface is

$$T = 8\pi\eta b^3\Omega\frac{\beta b}{\beta b+3\eta}, \tag{1}$$

where Ω is the angular velocity of the particle. When $\eta/\beta b = 0$, there is no slip at the particle surface and Eq. (1) degenerates to the well-known Stokes result. In the limiting case of $\eta/\beta b \to \infty$, there is a perfect slip at the particle surface and the torque vanishes.

The problem of rotation of nonspherical particles with frictionally slip surfaces is obviously a matter of analytical difficulty. The friction coefficients for the uniform rotation of a perfect-slip spheroid in a viscous fluid were computed numerically by fitting the slip condition approximately with a general solution of the Stokes equations in the form of an infinite series of spheroidal harmonics [Hu and Zwanzig (1974)]. Recently, the low-Reynolds-number rotation of a rigid particle which departs but little in shape from a sphere with the slip boundary condition was analyzed, and an explicit expression for the hydrodynamic torque experienced by it was obtained to the second order in the small parameter characterizing the deformation [Senchenko and Keh (2006); Chang and Keh (2009)]. However, the problem of the slow rotation of nonspherical particles with the general slipping condition in a viscous fluid has not been exactly solved yet, mainly due to the fact that, if frictional slip is included, a separation-of-variable solution may not be feasible for most orthogonal curvilinear coordinate systems, such as the prolate and oblate spheroidal coordinate systems [Leong (1984); Williams (1986)].

In this paper we use a method of distributed internal singularities incorporated with a boundary collocation technique to analyze the creeping flow generated by a slip particle of revolution rotating about its axis of symmetry; the particle can be either prolate or oblate. The analytical-numerical approach is a meshless method applicable to boundary value problems closely related to the method of fundamental solutions, which consists of approximating the solutions of the problem by a linear combination of fundamental solutions with respect to some singularities and determining the unknown coefficients of the fundamental solutions and the coordinates of the singularities by requiring the approximation to satisfy the boundary conditions [Marin (2008)], to the multipole Trefftz method, in which a truncated finite system of simultaneous linear algebraic equations is derived and solved for the unknown coefficients of the infinite sum of the multipole representation for the interested field with the features of excellent accuracy, fast rate of convergence, and high computational efficiency [Lee and Chen (2009)], and to the boundary-collocation Trefftz method [Liu, Yeih, and Atluri (2009)]. With this approach the torque exerted on the particle by the fluid as a function of the slip coefficient of the particle is numerically calculated for various cases. For the special cases of a slip sphere and of a no-slip spheroid, our torque results show excellent agreement with the exact solutions. For the case of a slip spheroid whose shape deviates slightly from that of a sphere, our results also agree quite well with the available approximate analytical solution.

2 Mathematical description of the problem

We consider the steady rotational motion of a general axisymmetric particle in an incompressible, Newtonian fluid about its axis of revolution at the steady state. The fluid may slip frictionally at the surface of the particle and is at rest at infinity. The circular cylindrical coordinates (ρ,ϕ,z) and spherical coordinates (r,θ,ϕ) are utilized and the particle center is chosen to be the origin of the coordinates, as shown in Fig. 1. The angular velocity of the particle is $\Omega\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in the positive z direction. The Reynolds number is assumed to be sufficiently small so that the inertial terms in the fluid momentum equation can be neglected, in comparison with the viscous terms.

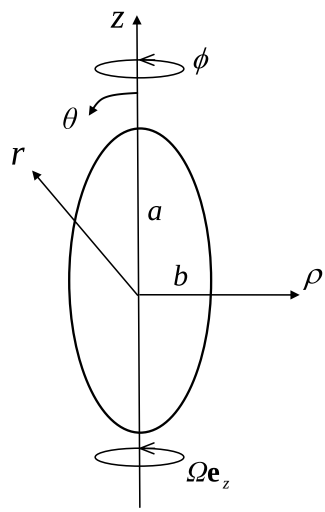

Figure 1: Geometrical sketch for the rotation of an arbitrary axisymmetric particle about its axis of revolution

The fluid flow is governed by the Stokes equations,

$$\eta\nabla^2\mathbf{v}-\nabla p=\mathbf{0},$$

$$\nabla\cdot\mathbf{v}=0, \tag{2}$$

where $\mathbf{v}$ is the fluid velocity field and p is the dynamic pressure distribution. Since the relative tangential velocity of the fluid at the particle surface is proportional to the local tangential stress [Happel and Brenner (1983)] and the fluid is motionless far away from the particle, the boundary conditions are

$$\mathbf{v}=\Omega\mathbf{e}_z\times\mathbf{r}+\frac{1}{\beta}(\mathbf{I}-\mathbf{nn})\mathbf{n}:\tau \text{ on } S_p, \tag{3a}$$

$$\mathbf{v} \to \mathbf{0} \text{ as } r \to \infty. \tag{3b}$$

Here, $\mathbf{r}$ is a position vector from the particle center, $\tau(= \eta[\nabla\mathbf{v} + (\nabla\mathbf{v})^T])$ is the viscous stress tensor, $1/\beta$ is the frictional slip coefficient about the particle surface which is taken to be a constant, $\mathbf{I}$ is the unit dyadic, and $\mathbf{n}$ is the unit normal vector at the particle surface S_p pointing into the fluid.

Evidently, for the axisymmetric rotational Stokes flow considered here, the dynamic pressure keeps constant everywhere and the only nonzero velocity component is v_ϕ in the ϕ direction. Thus, the torque $T\mathbf{e}_z$ exerted by the fluid on the surface of the particle can be determined from

$$T\mathbf{e}_z = \iint_{S_p} \mathbf{r} \times (\mathbf{n} \cdot \tau) dS. \tag{4}$$

To solve Eqs. (2) and (3) for v_ϕ, a set of fundamental spherical singularities is chosen and distributed along the axis of revolution within a prolate particle or on the fundamental plane within an oblate particle [Keh and Huang (2004)]. The flow field surrounding the particle is approximated by the superposition of the set of the spherical singularities and the boundary condition (3a) on the particle surface can be satisfied by making use of the multipole collocation method. For the special case of a spherical particle, only a single singularity which is placed at the particle center is needed.

The velocity component v_ϕ for the fluid motion caused by a spherical singularity at the point $\rho = 0$ and $z = h$ is [Jeffery (1915)]

$$v_\phi = \sum_{n=2}^{\infty} B_n A_n(\rho, z-h), \tag{5}$$

where A_n are functions of position defined by Eq. (A1) in Appendix A, and B_n are unknown constant coefficients. Note that boundary condition (3b) is immediately satisfied by a solution in the form of Eq. (5).

In cylindrical coordinates, boundary condition (3a) on the particle surface for the fluid flow can be expressed as

$$v_\phi = \Omega\rho + \frac{1}{\beta}(n_\rho \tau_{\rho\phi} + n_z \tau_{z\phi}) \text{ on } S_p, \tag{6}$$

where n_ρ and n_z are the local ρ and z components of the unit normal vector $\mathbf{n}$. From Eq. (5), the components of the viscous stress tensor in Eq. (6) can be obtained as

$$\tau_{\rho\phi} = \eta \sum_{n=2}^{\infty} B_n \alpha_n(\rho, z-h),$$

$$\tau_{z\phi} = \eta \sum_{n=2}^{\infty} B_n \beta_n(\rho, z-h), \tag{7}$$

where α_n and β_n are functions of position defined by Eqs. (A2) and (A3).

Equations (5) and (7) for the fluid velocity and stress fields caused by a spherical singularity and boundary condition (6) on the particle surface will be utilized in the following sections to solve for the hydrodynamic torque induced by the rotation of an axisymmetric particle about it axis of revolution.

3 Solution for the rotation of a spherical particle

In this section a singularity described in the previous section is used to obtain the solution for the rotation of a slip spherical particle of radius b. The torque results will be compared with the exact solution given by Eq. (1).

The flow field generated by the rotation of a sphere can be represented by a singularity placed at its center which is the origin of the coordinate frame. Thus, the velocity and stress components for the fluid motion caused by the rotating sphere are given by Eqs. (5) and (7) with h=0. To determine the unknown constants B_n, one can apply the boundary condition (6) at the particle surface to these velocity and stress components to yield

$$\sum_{n=2}^{\infty} B_n A_n^*(\rho, z) = \Omega\rho \text{ at } r = b, \tag{8}$$

where

$$A_n^*(\rho, z) = A_n(\rho, z) - \frac{\eta}{\beta}[n_\rho \alpha_n(\rho, z) + n_z \beta_n(\rho, z)]. \tag{9}$$

Substituting Eq. (7) into Eq. (4) and applying orthogonality properties of the Gegenbauer polynomials, one can obtain a simple formula for the torque exerted by the fluid on the particle,

$$T = 8\pi\eta B_2. \tag{10}$$

That is, only the first multipole of the singularity contributes to the hydrodynamic torque on the particle.

To satisfy the boundary condition (8) exactly along the entire semicircular generating arc of the sphere in a meridian plane would require the solution of the entire infinite array of unknown constants B_n. However, the boundary collocation technique [Gluckman, Pfeffer, and Weinbaum (1971); Keh and Chang (2008)] enforces the boundary condition at a finite number of discrete points on the particle's generating arc and truncates the infinite series in Eqs. (5), (7), and (8) into finite ones. The unknown constant in each term of the series permit one to satisfy the exact boundary condition at one discrete point on the particle surface. Thus, if the boundary is approximated by satisfying condition (8) at N discrete points, then the infinite series is truncated after N terms, resulting in a system of N simultaneous linear algebraic equations in the truncated form of Eq. (8). This matrix equation can be solved by any of the standard matrix-reduction techniques to yield the N unknown constants B_n required in the truncated equations for the velocity and stress fields. The accuracy of the truncation technique can be improved to any degree by taking a sufficiently large value of N. Naturally, the truncation error vanishes as $N \to \infty$.

When specifying the points along the semicircular generating arc of the sphere where the boundary condition is exactly satisfied, the first point that should be chosen is $\theta = \pi/2$, since this point defines the projected area of the particle normal to its axis of rotation. In addition, the points $\theta = 0$ and $\theta = \pi$ are also important. However, an examination of the system of linear algebraic equations for the unknown constants B_n shows that the coefficient matrix becomes singular if these points are used. To overcome the difficulty of singularity and to preserve the geometric symmetry of the spherical boundary about the equatorial plane $\theta = \pi/2$, points at $\theta = \alpha$, $\pi/2 - \alpha$, $\pi/2 + \alpha$ and $\pi - \alpha$ are taken to be four basic collocation points. Additional points along the boundary are selected as mirror-image pairs about the plane $\theta = \pi/2$ to divide the θ coordinate into equal parts. The optimum value of α in this work is found to be 0.01°, with which the numerical results of the hydrodynamic torque on the particle can converge to at least four significant figures. In principle, as long as the number of the collocation points is sufficiently large and the distribution of the collocation points is adequate, the solution of the torque will converge and the shape of the particle can be well approximated, irrespective of the particle shape or boundary condition.

In Table 1, a number of numerical solutions of the dimensionless hydrodynamic torque $T/8\pi\eta b^3\Omega$ for the rotation of a sphere are presented for various values of the dimensionless slip parameter $\eta/\beta b$ using the boundary collocation technique. All of the results were obtained by increasing the number of collocation points N until the convergence of four significant digits is achieved. The exact solutions for $T/8\pi\eta b^3\Omega$ calculated using Eq. (1) are also listed in the bottom row of Table 1 for comparison. It can be seen that the results from the collocation method agree very well with the exact results for the desired accuracy and the rate of convergence is very rapid.

Table 1: Numerical results of the dimensionless torque for a rotating sphere with various values of the slip parameter

N		$T/8\pi\eta b^3\Omega$			
		$\eta/\beta b=0$	$\eta/\beta b=0.1$	$\eta/\beta b=1$	$\eta/\beta b=10$
4		1.0000	0.7692	0.2500	0.0323
8		1.0000	0.7692	0.2500	0.0323
Exact solution		1.0000	0.7692	0.2500	0.0323

Exact solutions are given by Eq. (1).

4 Axisymmetric rotation of prolate particles

We consider in this section the fluid motion caused by a general prolate axisymmetric particle rotating about its axis of symmetry. The fluid may slip frictionally at the surface of the particle. A segment between points A$(\rho=0, z=-c_1)$ and B$(\rho=0, z=c_2)$ is taken along the axis of revolution inside the particle on which a set of spherical singularities are distributed (c_1 and c_2 are positive constants). If the nose and the tail of the particle are round, then their centers of curvature can be chosen as A and B. The general solution of the flow field outside the particle can be constructed by the superposition of the singularities distributed on the segment AB, and Eq. (5) is used to result in

$$v_\phi=\sum_{n=2}^{\infty}\int_{-c_1}^{c_2}B_n(t)A_n(\rho,z-t)dt. \tag{11}$$

The corresponding expressions for the components of the viscous stress tensor can be obtained using Eq. (7). Equation (11) provides an exact solution for Eq. (2) that satisfies Eq. (3b), and the unknown density distribution function for the singularities, $B_n(t)$, must be determined from the remaining boundary condition (3a) or (6) using the collocation technique. The torque exerted by the fluid on the particle is obtained by the substitution of Eq. (11) into Eq. (4), with a result similar to Eq. (10),

$$T=8\pi\eta\int_{-c_1}^{c_2}B_2(\mathrm{t})\mathrm{d}t. \tag{12}$$

In order to use the multipole collocation technique to satisfy the boundary condition at the particle surface, the integration encountered in Eqs. (11) and (12) must be treated numerically. Here, we use the M-point Gauss-Legendre quadrature formula [Hornbeck (1975)]

$$\int_{-\mathrm{c}_1}^{c_2}f(q)dq=\frac{1}{2}(c_1+c_2)\sum_{m=1}^{M}w_mf(q_m), \tag{13}$$

where $f(q)$ is any function of q, q_m are the quadrature zeros, and w_m are the corresponding quadrature weights.

Applying Eq. (13) to Eq. (11) and truncating the infinite series after N terms, we obtain

$$v_\phi = \sum_{n=2}^{N+1} \sum_{m=1}^{M} B_{nm} A_n(\rho, z - q_m), \tag{14}$$

where B_{nm} are the unknown density constants. Accordingly, the corresponding stress components are expressed using Eq. (7) as

$$\begin{bmatrix} \tau_{\rho\phi} \\ \tau_{z\phi} \end{bmatrix} = \eta \sum_{n=2}^{N+1} \sum_{m=1}^{M} B_{nm} \begin{bmatrix} \alpha_n(\rho, z - q_m) \\ \beta_n(\rho, z - q_m) \end{bmatrix}. \tag{15}$$

Application of the boundary condition (6) to Eqs. (14) and (15) yields

$$\sum_{n=2}^{N+1} \sum_{m=1}^{M} B_{nm} A_n^*(\rho, z - q_m) = \Omega\rho \text{ on } S_\mathrm{p}, \tag{16}$$

where the functions A_n^* is given by Eq. (9). The collocation method allows the particle's boundary to be approximated by satisfying Eq. (16) at MN discrete values of z (rings) on its surface and results in a set of MN simultaneous linear algebraic equations, which can be solved numerically to yield the MN density constants B_{nm} required in Eq. (14) for the fluid velocity field. Once these constants are determined, the torque acting on the particle by the fluid can be obtained from Eq. (12), with the result

$$T = 8\pi\eta \sum_{m=1}^{M} B_{2m}. \tag{17}$$

5 Solution for the rotation of a prolate spheroid

In this section the method presented in the previous section is used to obtain the solution for the rotation of a slip prolate spheroid about its axis of revolution. The surface of the prolate spheroid is represented in cylindrical coordinates by

$$z(\rho) = \pm a[1 - (\frac{\rho}{b})^2]^{1/2}, \tag{18}$$

where $a > b > 0$ and $0 \leq \rho \leq b$ (a and b are the major and minor semi-axes, respectively, of the prolate spheroid).

For a prolate spheroidal particle with a no-slip surface ($\eta/\beta b = 0$) rotating with an angular velocity Ω about its axis of revolution in an unbounded fluid, the exact solution for the torque exerted on the particle by the fluid is [Happel and Brenner (1983)]

$$T_\infty = 8\pi\eta b^3 \Omega t_\infty, \tag{19a}$$

$$t_\infty = \{\frac{3}{2}(\zeta^2-1)^{1/2}[\zeta-(\zeta^2-1)\coth^{-1}\zeta]\}^{-1}. \tag{19b}$$

In Eq. (19b), $\zeta = a/c$ and $c = (a^2-b^2)^{1/2}$, which is the half distance between the two foci of the prolate spheroid.

In Section 3, collocation solutions for the rotational motion of a spherical particle with a slip surface were presented and shown to be in perfect agreement with the exact solution. We now use the same collocation scheme incorporated with the method of distribution of spherical singularities to obtain the corresponding solution for a slip prolate spheroid. In Table 2, numerical results of the nondimensional torque $T/8\pi\eta b^3\Omega$ for the rotation of a prolate spheroid about its axis of revolution are presented for five representative cases of the aspect ratio a/b with various values of the slip parameter $\eta/\beta b$. The values of $T/8\pi\eta b^3\Omega$ are computed for different values of N (which shows convergence tests) with $M = 60$ in Eqs. (14)-(16). For each case, small values of N and M can achieve good convergence behavior for the calculation of T. The exact solution of t_∞ for the axisymmetric rotation of a no-slip prolate spheroid (with $\eta/\beta b = 0$) given by Eq. (19b) is also given in Table 2 for comparison. It can be seen that our collocation results from the method of distributed singularities agree excellently with the exact solution in this limit. In general, the convergence behavior of the method of singularities is very good.

Recently, Chang and Keh (2009) investigated the problem of slow flow of a viscous incompressible fluid past a slip particle whose shape deviates slightly from that of a sphere. Their analytical result corresponding to the hydrodynamic torque exerted on a spheroid rotating about its axis of revolution, which is correct to the second order in the small parameter ε characterizing the deformation, can be expressed as

$$\frac{T}{8\pi\eta b^3\Omega} = \gamma_3 - \varepsilon\frac{3\gamma_3^2}{5\gamma_4} + \varepsilon^2\frac{3\gamma_3^3\gamma_5}{175}(1+3\nu+36\nu^2+150\nu^3), \tag{20}$$

where $\varepsilon = 1-(a/b)$, $\gamma_i = \beta b/(\beta b + i\eta)$, and $\nu = \eta/\beta b$. The values of the dimensionless torque $T/8\pi\eta b^3\Omega$ calculated from this approximate formula are also listed in Table 2 for comparison. It can be found that the solution correct to the second order in ε given by Eq. (20) agrees quite well with our collocation solutions for small magnitudes of ε. The errors are less than 1.2% for particles with

$1 \leq a/b \leq 2$. However, the accuracy of this approximate solution begins to deteriorate, as expected, when the value of a/b becomes greater, especially for the case of an intensely slip spheroid.

Table 2: Numerical results of the dimensionless torque for the rotation of a prolate spheroid about its axis of revolution for various values of the aspect ratio and slip parameter of the spheroid (M=60 is used)

$\eta/\beta b$	N	$T/8\pi\eta b^3\Omega$ a/b=1.1	$a/b=2$	$a/b=5$	$a/b=10$	$a/b=20$
0	2	1.06017	1.61335	3.53040	6.80571	13.53307
	3	1.06017	1.61335	3.53040	6.80471	13.42393
	4	1.06017	1.61335	3.53040	6.80471	13.42393
	Exact solution	1.06017	1.61335	3.53040	6.80471	13.42393
0.1	4	0.81903	1.27414	2.83908	5.50313	11.00286
	5	0.81903	1.27414	2.83908	5.50200	10.87700
	6	0.81903	1.27414	2.83908	5.50200	10.87700
	Approximate solution	0.81903	1.27569	2.90804	6.00525	13.61199
1	4	0.26883	0.44262	1.03937	2.04808	4.13131
	5	0.26883	0.44262	1.03937	2.04709	4.07223
	6	0.26883	0.44262	1.03937	2.04709	4.07223
	Approximate solution	0.26883	0.44598	1.13571	2.62455	6.87455
3	4	0.10784	0.18080	0.43250	0.85748	1.73519
	5	0.10784	0.18080	0.43250	0.85699	1.70882
	6	0.10784	0.18080	0.43250	0.85699	1.70882
	Approximate solution	0.10785	0.18270	0.48715	1.18247	3.27767
10	4	0.03483	0.05889	0.14215	0.28250	0.57289
	5	0.03483	0.05889	0.14215	0.28250	0.56396
	6	0.03483	0.05889	0.14215	0.28250	0.56396
	Approximate solution	0.03484	0.05959	0.16239	0.40305	1.14440

Exact and approximate solutions are given by Eqs. (19) and (20), respectively.

The dimensionless torque $T/8\pi\eta b^3\Omega$ for the axisymmetric rotation of a prolate spheroid as a function of the aspect ratio a/b for several different values of the slip parameter $\eta/\beta b$ is plotted in Fig. 2. For a prolate spheroid with a no-slip surface or a slip surface having a finite value of $\eta/\beta b$, the value of $T/8\pi\eta b^3\Omega$ increases monotonically with an increase in the value of a/b because of the increase in the surface area with an increase in a/b for a given value of the equatorial radius b. For a perfectly slip spheroid (with $\eta/\beta b \to \infty$), $T/8\pi\eta b^3\Omega$ disappears irrespective of its aspect ratio. On the other hand, $T/8\pi\eta b^3\Omega$ is a monotonically decreasing function of $\eta/\beta b$ for a given shape of spheroids, and its dependence becomes quite sensitive when the value of a/b is large. It can be seen that the hydrodynamic torque on the spheroid can be quite large when a/b is large and $\eta/\beta b$ is small.

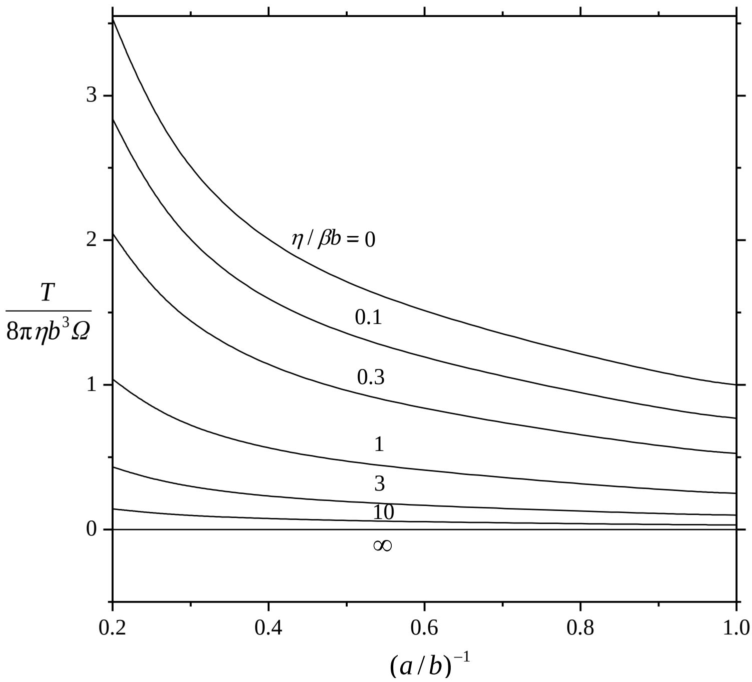

Figure 2: Plots of the dimensionless torque for the rotation of a prolate spheroid with a slip surface about its axis of revolution versus the aspect ratio of the spheroid for various values of the slip parameter $\eta/\beta b$

6 Axisymmetric rotation of oblate particles

The rotational motion of a general prolate axisymmetric particle with a slip surface about its axis of revolution was considered in Section 4 and a set of spherical singularities must be distributed on a segment along the axis inside the particle. In this section we consider the corresponding rotation of a general slip oblate particle and the singularities should be distributed on the fundamental plane within the particle.

Since the oblate particle and the fluid motion are axisymmetric, the fundamental plane should be a circular disk S_d normal to the z-axis and with its center at the origin of the coordinate frame (the center of the particle).

Let Q be an arbitrary point on S_d which is expressed with the circular polar coordinates $\rho = \hat{\rho}$, $\phi = \hat{\phi}$, and $z = 0$. Then the velocity disturbance at another point P($\rho = \rho$, $\phi = 0$, $z = z$) generated by the spherical singularity at Q can be obtained using Eq. (5),

$$\hat{v}_\phi = \frac{\rho - \hat{\rho}\cos\hat{\phi}}{\rho^*} \sum_{n=2}^{\infty} B_n A_n(\rho^*, z),$$

$$\hat{v}_\rho = \frac{\hat{\rho}\sin\hat{\phi}}{\rho^*} \sum_{n=2}^{\infty} B_n A_n(\rho^*, z), \tag{21}$$

where ρ^* is the distance from point Q to the projection of point P on the plane $z = 0$,

$$\rho^* = (\rho^2 + \hat{\rho}^2 - 2\rho\hat{\rho}\cos\hat{\phi})^{1/2}. \tag{22}$$

Due to the axisymmetry of the fluid motion, the singularities must be distributed uniformly on the circles in S_d with their centers at the origin of coordinates. Hence, the unknown density distribution coefficients B_n in Eq. (21) are functions of $\hat{\rho}$ only.

The total disturbance of the fluid velocity field produced by the rotation of the oblate particle can be approximated by the superposition of the individual disturbances in Eq. (21) induced by the whole set of singularities on the fundamental disk S_d. Thus, at an arbitrary location, we have

$$v_\phi = \sum_{n=2}^{\infty} \int_0^{2\pi} \int_0^R \left(\frac{\rho - \hat{\rho}\cos\hat{\phi}}{\rho^*}\right) B_n(\hat{\rho}) A_n(\rho^*, z) \hat{\rho}\, d\hat{\rho}\, d\hat{\phi}, \tag{23}$$

where R is the radius of the disk S_d and the corresponding integral of $\hat{v}_\rho$ vanishes. Equation (23) provides an exact solution for Eq. (2) that satisfies Eq. (3b), and the unknown density distribution functions $B_n(\hat{\rho})$ must be determined from the remaining boundary condition (3a) or (6) using the collocation method. In Eq. (6), the stress components can be calculated from Eq. (23) and expressed as

$$\tau_{\rho\phi} = \eta \sum_{n=2}^{\infty} \int_0^{2\pi} \int_0^R B_n(\hat{\rho}) \{ [\frac{\hat{\rho}^2 \sin^2\hat{\phi}}{(\rho^*)^3} - \frac{\rho - \hat{\rho}\cos\hat{\phi}}{\rho\rho^*}] A_n(\rho^*, z)$$

$$+ (\frac{\rho - \hat{\rho}\cos\hat{\phi}}{\rho^*})^2 \delta_n(\rho^*, z) \} \hat{\rho}\, d\hat{\rho}\, d\hat{\phi},$$

$$\tau_{z\phi} = \eta \sum_{n=2}^{\infty} \int_0^{2\pi} \int_0^R \left(\frac{\rho - \hat{\rho}\cos\hat{\phi}}{\rho^*}\right) B_n(\hat{\rho}) \beta_n(\rho^*, z) \hat{\rho} d\hat{\rho} d\hat{\phi}. \tag{24}$$

where δ_n are functions of position defined by Eq. (A4).

The torque exerted by the fluid on the oblate axisymmetric particle can be obtained by substituting Eq. (24) into Eq. (4) and applying the orthogonality properties of the Gegenbauer polynomials. The result is

$$T = 16\pi^2 \eta \int_0^R B_2(\hat{\rho}) \hat{\rho} d\hat{\rho}. \tag{25}$$

Similar to the case of the rotation of a prolate particle examined in Section 4, the integration in Eqs. (23) and (24) with respect to $\hat{\rho}$ can be approximated by the M-point Gauss-Legendre quadrature formula, as shown in Eq. (13), and the infinite series is truncated after N terms. With this arrangement, Eqs. (23) and (24) become

$$v_\phi = \sum_{n=2}^{N+1} \sum_{m=1}^{M} B'_{nm} A_{nm}(\rho, z), \tag{26}$$

$$\begin{bmatrix} \tau_{\rho\phi} \\ \tau_{z\phi} \end{bmatrix} = \eta \sum_{n=2}^{N+1} \sum_{m=1}^{M} B'_{nm} \begin{bmatrix} \alpha_{nm}(\rho, z) \\ \beta_{nm}(\rho, z) \end{bmatrix}, \tag{27}$$

where functions A_{nm}, α_{nm}, and β_{nm} are defined by Eqs. (A5)-(A7).

Application of the boundary condition (6) to Eqs. (26) and (27) yields

$$\sum_{n=2}^{N+1} \sum_{m=1}^{M} B'_{nm} A^*_{nm}(\rho, z) = \Omega\rho \text{ on } S_\text{p}, \tag{28}$$

where A^*_{nm} is given by Eq. (9) with the subscript n of its functions being replaced by nm. Thus, the collocation technique described in Section 4 can be used to satisfy the boundary condition (28) and to determine the MN density constants B'_{nm} required for the fluid velocity field. Once these constants are determined, the hydrodynamic torque exerted on the particle can be obtained from Eq. (25), with the result

$$T = 16\pi^2 \eta \sum_{m=1}^{M} B'_{2m}. \tag{29}$$

7 Solution for the rotation of an oblate spheroidal particle

In Section 5, numerical solutions of the hydrodynamic torque experienced by a prolate spheroid rotating about its axis of revolution were presented. In this section the similar singularity method and boundary collocation technique described in the previous section will be used to solve the corresponding rotation of an oblate spheroid. The shape of the oblate spheroid can still be represented by Eq. (18), but now with $b > a > 0$. In addition, the exact solution for the torque exerted on an oblate spheroid with a no-slip surface rotating with an angular velocity Ω about its axis of symmetry in an unbounded fluid can be expressed as Eq. (19a), with the coefficient t_∞ given by [Happel and Brenner (1983)]

$$t_\infty = \{\frac{3}{2}(\zeta^2+1)^{1/2}[(\zeta^2+1)\cot^{-1}\zeta - \zeta]\}^{-1}, \tag{30}$$

where $\zeta = a/c$ and $c = (b^2-a^2)^{1/2}$, which is the radius of the focal circle of the oblate spheroid.

The numerical solutions of the nondimensional torque $T/8\pi\eta b^3\Omega$ for the rotation of an oblate spheroid about its axis of revolution are presented in Table 3 for four representative cases of the aspect ratio a/b with various values of the slip parameter $\eta/\beta b$. The values of $T/8\pi\eta b^3\Omega$ are computed for different values of N with $M = 60$ in Eqs. (26)-(28). For a spheroid with its aspect ratio close to unity, small values of N and M can usually achieve good convergence behavior for the calculation of T. When the aspect ratio of the spheroid deviates further from unity (especially for the case with relatively large values of $\eta/\beta b$), greater values of N and M may be needed. The exact solution for the rotation of a no-slip oblate spheroid (with $\eta/\beta b = 0$) about its axis of revolution given by Eq. (30) and the approximate solution for the axisymmetric rotation of a slip spheroid whose shape deviates slightly from that of a sphere given by Eq. (20) are also listed in Table 3 for comparison. Analogous to the case of a prolate spheroid considered in Section 5, the convergence behavior of the method of spherical singularities in general is quite satisfactory. The agreement between our results and the exact and approximate solutions is very good. The errors of Eq. (20) are less than about 1.1% for particles with $0.5 \le a/b \le 1$; but as expected, the accuracy of this approximate solution begins to deteriorate when the value of a/b becomes smaller.

In Fig. 3, the dimensionless torque $T/8\pi\eta b^3\Omega$ for the axisymmetric rotation of an oblate spheroid as a function of the aspect ratio a/b for several different values of the slip parameter $\eta/\beta b$ is plotted. Similarly to the rotation of a prolate spheroid discussed in Section 5, the value of $T/8\pi\eta b^3\Omega$ decreases monotonically (and almost linearly) with a decrease in a/b for an oblate spheroid with a no-slip surface or a slip surface having a finite value of $\eta/\beta b$. Again, $T/8\pi\eta b^3\Omega$ is a monotonically decreasing function of $\eta/\beta b$ for spheroids with a fixed value of a/b andvanishes for a perfectly slip spheroid (with $\eta/\beta b \to \infty$).

Table 3: Numerical results of the dimensionless torque for the rotation of an oblate spheroid about its axis of revolution for various values of the aspect ratio and slip parameter of the spheroid (M=60 is used)

$\eta/\beta b$	N	$T/8\pi\eta b^3\Omega$			
		a/b=0.9	a/b=0.5	a/b=0.2	a/b=0.1
0	2	0.94018	0.70502	0.53437	0.80324
	3	0.94018	0.70502	0.53437	0.47894
	4	0.94018	0.70502	0.53437	0.47894
	Exact solution	0.94018	0.70502	0.53437	0.47894
0.1	4	0.71962	0.52335	0.38014	0.33536
	5	0.71962	0.52335	0.38014	0.33545
	6	0.71962	0.52335	0.38014	0.33545
	Approximate solution	0.71962	0.52306	0.37762	0.32952
1	4	0.23134	0.15953	0.11284	0.08594
	5	0.23134	0.15953	0.11284	0.10091
	6	0.23134	0.15953	0.11284	0.10091
	Approximate solution	0.23133	0.15837	0.10543	0.08812
3	4	0.09225	0.06279	0.04431	0.03919
	5	0.09225	0.06279	0.04431	0.03986
	6	0.09225	0.06279	0.04431	0.03986
	Approximate solution	0.09225	0.06217	0.04061	0.03360
10	4	0.02972	0.02011	0.01418	0.01278
	5	0.02972	0.02011	0.01418	0.01280
	6	0.02972	0.02011	0.01418	0.01280
	Approximate solution	0.02972	0.01989	0.01289	0.01062

Exact and approximate solutions are given by Eqs. (30) and (20), respectively.

8 Concluding remarks

The slow rotation of a general axisymmetric particle with a slip surface in a viscous fluid about its axis of revolution has been investigated by the use of the method of internal singularity distributions combined with the boundary collocation technique. For the case of the axisymmetric rotation of a prolate particle, a truncated set of spherical singularities is distributed along the axis inside the particle, whereas for the case of an oblate particle, the singularities are placed on the fundamental disk

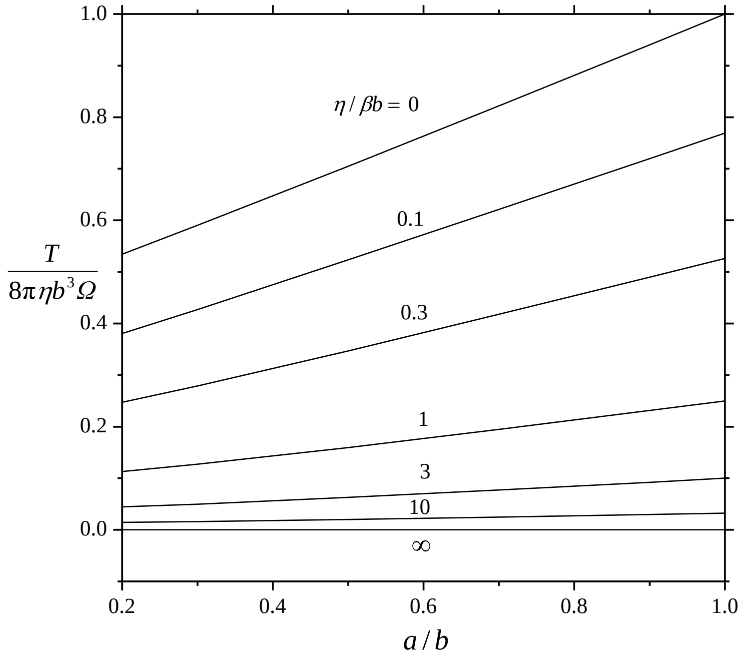

Figure 3: Plots of the dimensionless torque for the rotation of an oblate spheroid with a slip surface about its axis of revolution versus the aspect ratio of the spheroid for various values of the slip parameter $\eta/\beta b$

of the particle. The numerical results for the torque exerted on the particle by the fluid indicate that the solution procedure converges rapidly and accurate solutions can be obtained for various cases of the particle shape. Although the numerical solutions were presented in the previous sections only for the rotation of a sphere, a prolate spheroid, and an oblate spheroid, the combined analytical and numerical technique utilized in this work can easily provide the hydrodynamic calculations for the rotation of an axisymmetric particle of other shapes, such as a prolate or oblate Cassini oval [Keh and Tseng (1994)].

For an axisymmetric prolate particle whose nose and tail are not round, for example, a long cylinder, or for an axisymmetric oblate particle without a smooth surface, for example, a short cylinder or a relatively thin disk, the problem of its rotation will be difficult to be solved by any analytical or numerical solution technique, and it is certainly not the main target of the solution technique considered in this paper. In principle, however, the present numerical solution technique might be better than other ones for solving this kind of problems. The spherical singularities can still be distributed on a segment along the axis of revolution of the prolate particle or on a fundamental disk within the oblate particle. Obviously, the choices of the end points of the segment and the radius of the fundamental disk are somewhat indefinite for this case. One would expect that slightly different but reasonable choices of the end points of the segment or the radius of the fundamental disk may lead to almost the same convergent solution for a given problem.

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Appendix A: Definitions of functions in Sections 2, 4, and 6

For conciseness the definitions of some functions in Sections 2, 4, and 6 are listed in this appendix. The functions appearing in Eqs. (5), (7), and (24) are defined as

$$A_n(\rho,z) = n\rho^{-1}(\rho^2+z^2)^{-(n-1)/2}G_n^{-1/2}(\mu) \tag{A1}$$

$$\alpha_n(\rho,z) = -n\rho^{-2}(\rho^2+z^2)^{-(n+2)/2}\{(\rho^2+z^2)^{1/2}[2z^2+(1+n)\rho^2]G_n^{-1/2}(\mu)-z\rho^2P_{n-1}(\mu)\} \tag{A2}$$

$$\beta_n(\rho,z) = -n\rho^{-1}(\rho^2+z^2)^{-(n+2)/2}[(n-1)z(\rho^2+z^2)^{1/2}G_n^{-1/2}(\mu)+\rho^2P_{n-1}(\mu)], \tag{A3}$$

$$\delta_n(\rho,z) = -n\rho^{-2}(\rho^2+z^2)^{-(n+1)/2}[(n\rho^2+z^2)G_n^{-1/2}(\mu)-\rho^2\mu P_{n-1}(\mu)], \tag{A4}$$

where $\mu = z(\rho^2+z^2)^{-1/2}$, $G_n^{-1/2}$ is the Gegenbauer polynomial of the first kind of order n and degree –1/2, and P_n is the Legendre polynomial of order n.

The following are the definitions of some functions used in Eqs. (26) and (27) in Section 6:

$$A_{nm}(\rho,z) = \int_0^{2\pi} (\frac{\rho - q_m\cos\hat{\phi}}{\rho^*})A_n(\rho^*,z)d\hat{\phi}, \tag{A5}$$

$$\alpha_{nm}(\rho,z) = \int_0^{2\pi} \{[\frac{q_m^2\sin^2\hat{\phi}}{(\rho^*)^3} - \frac{\rho - q_m\cos\hat{\phi}}{\rho\rho^*}]A_n(\rho^*,z) + (\frac{\rho - q_m\cos\hat{\phi}}{\rho^*})^2\delta_n(\rho^*,z)\}d\hat{\phi}, \tag{A6}$$

$$\beta_{nm}(\rho,z) = \int_0^{2\pi} (\frac{\rho - q_m\cos\hat{\phi}}{\rho^*})\beta_n(\rho^*,z)d\hat{\phi}, \tag{A7}$$

where $\rho^* = (\rho^2+q_m^2-2\rho q_m\cos\hat{\phi})^{1/2}$ and q_m are the quadrature zeros referred to Eq. (13). The integrations in Eqs. (A5)-(A7) can be performed numerically after the substitution of Eqs. (A1), (A3), and (A4).

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