Generalized Extrapolation for Computation of Hypersingular Integrals in Boundary Element Methods

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Abstract: The trapezoidal rule for the computation of Hadamard finite-part integrals in boundary element methods is discussed, and the asymptotic expansion of error function is obtained. A series to approach the singular point is constructed and the convergence rate is proved. Based on the asymptotic expansion of the error functional, algorithm with theoretical analysis of the generalized extrapolation are given. Some examples show that the numerical results coincide with the theoretic analysis very well.

Keywords: Hypersingular integrals, trapezoidal rule, asymptotic error expansion, extrapolation algorithm

1 Introduction

Consider the following integral

$$I(f;s) = \displaystyle{=}\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-s)^2}dt \quad = \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon} \frac{f(t)}{(t-s)^2}dt + \int_{s+\varepsilon}^b \frac{f(t)}{(t-s)^2}dt - \frac{2f(s)}{\varepsilon}\right\}, \quad s\in(a,b), \tag{1}$$

where $=\!\!\!\!\!\!\int_a^b$ denotes a Hadamard finite-part integral and s is the singular point.

Integrals of the form Eq. 1 usually appear in boundary element methods (BEMs) which can be found in [Yu (1985); Yu (1993); Yu (2002); Yu and Huang (2008); Liu (2007); Liu and Yu (2008); Young, Chen, Chen, and Kao (2007); Albuquerque, Aliabadi (2008); He, Lim, and Lim (2008); Owatsiriwong, Phansri, and Park (2008)] and many engineering problems [Hui and Shia (1999); Brunner (2004); Hasegawa

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(2004); Sanz, Solis, and Dominguez (2007); Zhou, Li, and Yu (2008)]. The efficiency of BEMs and many engineering computational problems often depend on the efficiency of numerical evaluation of such finite-part integrals. Numerous work has been devoted in developing efficient quadrature formulas, such as the Gaussian method [Ioakimidis (1985); Hui and Shia (1999)], the transformation method [Elliott and Venturino (1997); Hasegawa (2004)] and some other methods [Yu (1992); Monehato (1994); Wu and Yu (1999); Du (2001); Kim and Jin (2003)].

In recent years, much attention has been paid to the composite Newton-Cotes method for Eq. 1 and the related superconvergence phenomenon. It's obvious that the accuracy of the (composite) Newton-Cotes rules for Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. Due to the hypersingularity of the kernel, the Newton-Cotes rules for Hadamard finite-part integrals can only achieve $O(h^k)$ in general case and $O(h^{k+1})$ [Wu and Sun (2005); Wu and Sun (2008)] at the superconvergence points. In the other word, one should take the higher order quadrature method if the higher precision is expected. But in the actual computation, the so-called 'Richardson extrapolation' can be used to to accelerating convergence for numerical integrals.

For the quadrature rules of Newton-Cotes type, the composite trapezoidal rule is among the simplest ones for implementation. The composite trapezoidal rule for evaluating Eq. 1 was first studied by [Linz (1985)], and the estimate is

$$|E_n(f)| \leq C\gamma^{-2}(h,s)h, \tag{2}$$

where $E_n(f)$ is the error functional and

$$\gamma(h,s) = \min_{0\leq j\leq n} \frac{|s-t_j|}{h}, \tag{3}$$

$t_j(0 \leq j \leq n)$ are the mesh points. Due to this estimate, one has to select the mesh very carefully such that the singular point is located near the center of a subinterval. This is the so-called mesh selection problem of the Newton-Cotes method. A more precise estimate is given [Wu and Sun (2008)] and takes the form of

$$|E_n(f)| \leq C|\ln\gamma(h,s)|h, \tag{4}$$

which makes the mesh selection problem less serious. From Eq. 2 and Eq. 4, we see that the accuracy of trapezoidal rule is still not very satisfactory. Thus, it is natural for us to construct more efficient algorithm to evaluate Eq. 1 based on the trapezoidal rule, and fortunately, we find that the extrapolation method can be applied for trapezoidal rule to get the arbitrary accuracy if the density function $f(t)$ is smooth enough.

It is well known that extrapolation as an accelerating convergence technique has been applied to many fields in computational mathematics [Liem, Lü,and Shih (1995); Sidi (2003)]. The classic extrapolation method based on polynomial and rational function has been well studied. The most famous one is Richardson extrapolation with the error function as

$$T(h) - a_0 = a_1 h^2 + a_2 h^4 + a_3 h^6 + \cdots,$$

where $T(0) = a_0$ and a_j are constant independent of h. Richardson eliminated the term h^2 by combining the approximations with two different meshes. In each case he used pairs of approximation to eliminate h^2, a process he named "h^2-extrapolation" to improve numerical solutions of integral equation and ordinary differential equation. In the paper of Lü and Huang (2006) based on a generalization of discrete Gronwall inequality, a new quadrature method for solving nonlinear weakly singular Volterra integral equations of the second kind is presented and the asymptotic expansion of the error is proved. However, to our knowledge, no attempt has been made to apply extrapolation technique to accelerate convergence for the computation of hypersingular integral.

In this paper we focus on the asymptotic error expansion of the trapezoidal rule for the computation of Hadamard finite-part integrals. The asymptotic error expansion takes the form of

$$E_n(f) = \sum_{i=1}^{l-1} \frac{h^i}{2^i} f^{(i+1)}(s) a_i(\tau) + O(h^{l-1}), \tag{5}$$

where $a_i(\tau)$ are functions independent of h, and τ the local coordinate of the singular point. Based on this asymptotic expansion, we suggest an extrapolation algorithm. For a given τ, a series of s_j is selected to approximate the singular point s accompanied by the refinement of the meshes. Moreover, by means of the extrapolation technique we not only obtain an approximation with higher order accuracy, but also get a posteriori error estimate.

The rest of this paper is organized as follows. In Sect.2, after introducing some basic formulas of the trapezoidal rule, we present the asymptotic error expansion. In Sect.3 we perform the proof. In Sect.4, extrapolation algorithm is proposed and a posteriori asymptotic error estimation to compute Hadamard finite-part integral is obtained. In the last section, several numerical examples are provided to validate our analysis.

2 Main result

Let $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a uniform partition of the interval $[a,b]$ with mesh size $h = (b-a)/n$. Define by $f_L(t)$ the piecewise linear interpolant for $f(t)$:

$$f_L(t) = \frac{t - t_{j-1}}{h} f(t_j) + \frac{t_j - t}{h} f(t_{j-1}), \;\; t \in [t_{j-1}, t_j], \;\; 1 \le j \le n. \tag{6}$$

Introduce a linear transformation

$$t = \hat{t}_j(\tau) := (\tau+1)h/2 + t_{j-1}, \;\; \tau \in [-1,1], \tag{7}$$

maps the reference element $[-1,1]$ onto the subinterval $[t_{j-1}, t_j]$. Replacing $f(t)$ in Eq. 1 with $f_L(t)$ gives the composite trapezoidal rule:

$$I_n(f;s) = \text{=}\!\!\!\!\!\!\int_a^b \frac{f_L(t)}{(t-s)^2} dt = \sum_{j=0}^{n} \omega_j(s) f(t_j) = I(f;s) - E_n(f), \tag{8}$$

where $\omega_j(s)$ denote the Cotes coefficients, see [Linz (1985); Wu and Sun (2005)], and $E_n(f) = I(f;s) - I_n(f;s)$.
We also define

$$F_i(\tau) = (\tau-1)(\tau+1)[(\tau+1)^i - (\tau-1)^i] \tag{9}$$

and its second function, defined by

$$\phi_{i,i+1}(t) = \begin{cases} -\dfrac{1}{2} \displaystyle -\!\!\!\!\!\!\int_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| < 1, \\ -\dfrac{1}{2} \displaystyle \int_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| > 1. \end{cases} \tag{10}$$

It is known that if $F_i(\tau)$ is replaced by the Legendre polynomial, $\phi_{i,i+1}$ defines the Legendre function of the second kind [Andrews (2002)]. Let

$$\phi_{ii}(t) = \begin{cases} -\dfrac{1}{2} \displaystyle =\!\!\!\!\!\!\int_{-1}^{1} \frac{F_i(\tau)}{(\tau - t)^2} d\tau, & |t| < 1, \\ -\dfrac{1}{2} \displaystyle \int_{-1}^{1} \frac{F_i(\tau)}{(\tau - t)^2} d\tau, & |t| > 1 \end{cases} \tag{11}$$

and

$$\phi_{ik}(t) = -\frac{1}{2} \int_{-1}^{1} \frac{F_i(\tau)(\tau-t)^{k-i-2}}{(k-i)!} d\tau, \;\; k > i+1. \tag{12}$$

We present our main theorem below. The proof will be given in the next section.

Theorem 2.1 Assume $f(t) \in C^l[a,b], \;\; l \geq 2.$ For the trapezoidal rule $I_n(f;s)$ defined in Eq. 8, there exist a positive constant C, independent of h and s and functions $a_i(\tau)$, independent of h, such that

$$E_n(f) = \sum_{i=1}^{l-1} \frac{h^i}{2^i} f^{(i+1)}(s) a_i(\tau) + R_n(s), \tag{13}$$

where $s = t_{m-1} + (1+\tau)h/2$, and

$$|R_n(s)| \leq C(\gamma^{-1}(h,s) + |\ln h|)h^{l-1}.$$

3 Proof of the main result

Before stating our idea, some notations and results are introduced. In the following analysis, C will denote a generic positive constant which is independent of h and s but which may depend on k and bounds of the derivatives of $f(t)$. Let P_l and Q_l denote the Legendre polynomial of degree l and the associated Legendre function of the second kind, respectively.

Define

$$\begin{aligned} M_{ik}^j(t,s) &= (t-t_{j-1})(t-t_j)(t-s)^{k-i}[(t-t_{j-1})^i - (t-t_j)^i] \\ &= F_i^j(t)(t-s)^{k-i}, \;\; k \geq i, \end{aligned} \tag{14}$$

where

$$F_i^j(t) = (t-t_{j-1})(t-t_j)[(t-t_{j-1})^i - (t-t_j)^i]. \tag{15}$$

By Eq. 7, we have

$$\begin{aligned} M_{ik}^j(t,s) &= \frac{h^{k+2}}{2^{k+2}}(\tau^2-1)(\tau-c_j)^{k-i}[(\tau+1)^i - (\tau-1)^i] \\ &= \frac{h^{k+2}}{2^{k+2}} M_{ik}(\tau,c_j) \\ &= \frac{h^{k+2}}{2^{k+2}} F_i(\tau)(\tau-c_j)^{k-i}, \end{aligned} \tag{16}$$

where $F_i(\tau)$ is defined by Eq. 9 and

$$M_{ik}(\tau,c_j) = (\tau^2-1)(\tau-c_j)^{k-i}[(\tau+1)^i - (\tau-1)^i], \;\; c_j = \frac{2}{h}(s-t_{j-1}) - 1. \tag{17}$$

Moreover, define

$$T_{ik}^{n,m}(\tau) := \phi_{ik}(\tau) + \sum_{j=1}^{n-m-1} \phi_{ik}(2j+\tau) + \sum_{j=1}^{m} \phi_{ik}(-2j+\tau), \ \ k = i, i+1 \tag{18}$$

and

$$b_i^{n,m}(\tau) = \begin{cases} \dfrac{(-1)^{i-1}}{i!} T_{i-1,i}^{n,m}(\tau) + \dfrac{(-1)^i}{(i+1)!} T_{ii}^{n,m}(\tau), & i > 1, \\ -\dfrac{1}{2} T_{11}^{n,m}(\tau), & i = 1. \end{cases} \tag{19}$$

Lemma 3.1 Let $\phi_{i,i+1}(t)$ and $\phi_{ii}(t)$ be defined in Eq. 10 and Eq. 11, respectively. Then

$$\phi_{i,i+1}(t) = \begin{cases} \displaystyle\sum_{j=1}^{i_1+1} \omega_{2j-1} Q_{2j-1}(t), & i = 2i_1, \\ \displaystyle\sum_{j=0}^{i_1} \omega_{2j} Q_{2j}(t), & i = 2i_1 - 1 \end{cases} \tag{20}$$

and

$$\phi_{ii}(t) = \begin{cases} \displaystyle\sum_{j=1}^{i_1} a_j Q_{2j}(t), & i = 2i_1, \\ \displaystyle\sum_{j=1}^{i_1} b_j Q_{2j-1}(t), & i = 2i_1 - 1, \end{cases} \tag{21}$$

where

$$\omega_j = \frac{2j+1}{2} \int_{-1}^{1} F_i(\tau) P_j(\tau) d\tau \tag{22}$$

and

$$a_j = -(4j+1) \sum_{k=1}^{j} \omega_{2k-1},$$

$$b_j = -(4j-1) \sum_{k=1}^{j} \omega_{2k-2}.$$

Proof: For $i = 2i_1$,

$$F_i(\tau) = (\tau^2 - 1)[(\tau+1)^i - (\tau-1)^i] = 2\tau(\tau^2-1)\sum_{j=1}^{i_1} C_{2i_1}^{2j-1}\tau^{2j-2}$$

and the polynomial $F_i(\tau)$ is an odd function. In terms of Legendre polynomials,

$$F_i(\tau) = \sum_{j=1}^{i_1+1} \omega_{2j-1}P_{2j-1}(\tau), \tag{23}$$

where ω_{2j-1} is defined in Eq. 22. The first part of Eq. 20 follows immediately from the definition of $\phi_{i,i+1}(\tau)$. Since

$$\sum_{j=1}^{i_1+1} \omega_{2j-1} = \sum_{j=1}^{i_1+1} \omega_{2j-1}P_{2j-1}(1) = F_i(1) = 0,$$

we can rewrite the first part of Eq. 20 by

$$\phi_{i,i+1}(t) = \sum_{j=1}^{i_1} \frac{a_j}{4j+1}[Q_{2j+1}(t) - Q_{2j-1}(t)]$$

with $a_j = -(4j+1)\sum_{k=1}^{j}\omega_{2k-1}$, which leads to the first part of Eq. 21 by using the recurrence relation [Andrews(2002)].

$$Q'_{l+1}(t) - Q'_{l-1}(t) = (2l+1)Q_l(t), \;\; l = 1,2,\cdots. \tag{24}$$

The proof for the second parts of Eq. 20 and Eq. 21 is similar.

Lemma 3.2 For $\tau \in (-1,1)$, there exists a function $c_{ik}(\tau)$, independent of h, such that

$$\lim_{n\to\infty} T_{ik}^{n,m}(\tau) = c_{ik}(\tau), \;\; k = i, i+1. \tag{25}$$

Proof: Consider the summation as follow:

$$\sum_{j=0}^{n-m-1} Q_l(2j+\tau) + \sum_{j=1}^{m} Q_l(-2j+\tau). \tag{26}$$

By the classical identity

$$Q_l(t) = \frac{1}{2^{l+1}}\int_{-1}^{1}\frac{(1-\tau^2)^l}{(t-\tau)^{l+1}}d\tau, \quad |t|>1, \;\; l = 1,2,\cdots \tag{27}$$

we get

$$|Q_l(t)| \le \frac{C}{(|t|-1)^{l+1}}, \quad |t|>1.$$

Since

$$Q_0(t)=\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|,$$

then we have

$$\sum_{j=0}^{n-m-1} Q_0(2j+\tau)+\sum_{j=1}^{m} Q_0(-2j+\tau)=\frac{1}{2}\log\frac{2(n-m)-1+\tau}{2m+1-\tau}=\frac{1}{2}\log\frac{b-s}{s-a}, \ n\to\infty,$$

where $m=\frac{s-a}{b-a}n+\frac{1-\tau}{2}$, $s\neq a,b$. By Eq. 20 and Eq. 21,we have

$$|\phi_{ik}(t)| \le \frac{C}{(|t|-1)^{2+[1+(-1)^i]/2}}, \quad |t|>2.$$

By Eq. 18 we get Eq. 25. The proof is complete.

Lemma 3.3 Assume $s\in(t_{m-1},t_m)$ for some m and let $c_j(1\le j\le n)$ be given by Eq. 17. Then, for $i=1,\cdots,l-1$ and $k=i$, we have

$$\phi_{ik}(c_j)=\begin{cases} -\dfrac{2^k}{h^{k+1}}\displaystyle\neq\!\!\!\!\!\!\int_{t_{j-1}}^{t_j}\frac{M_{ik}^j(t,s)}{(t-s)^2}dt, & j=m,\\ -\dfrac{2^k}{h^{k+1}}\displaystyle\int_{t_{j-1}}^{t_j}\frac{M_{ik}^j(t,s)}{(t-s)^2}dt, & j\neq m,\end{cases} \tag{28}$$

and

$$\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{F_i(\tau)}{(\tau-c_m)^2}d\tau=\begin{cases} 2\displaystyle\sum_{j=1}^{k_1}C_{2k_1}^{2j-1}\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{\tau^{2j-1}(\tau^2-1)}{(\tau-c_m)^2}d\tau, & i=2k_1,\\ 2\displaystyle\sum_{j=0}^{k_1}C_{2k_1+1}^{2j}\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{\tau^{2j}(\tau^2-1)}{(\tau-c_m)^2}d\tau, & i=2k_1+1.\end{cases} \tag{29}$$

Proof: If $j = m$, by the definition of Eq. 1 and noting $k = i$, we have

$$\begin{aligned}
\fint_{t_{j-1}}^{t_j} \frac{M_{ik}^j(t,s)}{(t-s)^2} dt &= \fint_{t_{j-1}}^{t_j} \frac{F_i^j(t)}{(t-s)^2} dt \\
&= \lim_{\varepsilon \to 0} \left\{ \left(\int_{t_{j-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{t_j} \right) \frac{F_i^j(t)}{(t-s)^2} dt - \frac{2F_i^j(s)}{\varepsilon} \right\} \\
&= \frac{h^{k+1}}{2^{k+1}} \lim_{\varepsilon \to 0} \left\{ \left(\int_{-1}^{c_m - \frac{2\varepsilon}{h}} + \int_{c_m + \frac{2\varepsilon}{h}}^{1} \right) \frac{F_i(\tau)}{(\tau - c_m)^2} d\tau - \frac{hF_i(c_m)}{\varepsilon} \right\} \\
&= \frac{h^{k+1}}{2^{k+1}} \fint_{-1}^{1} \frac{F_i(\tau)}{(\tau - c_m)^2} d\tau \\
&= -\frac{h^{k+1}}{2^k} \phi_{ii}(c_m).
\end{aligned}$$

The case $j \neq m$ can be proved by applying the same approach to the correspondent Riemann integral.

Now we consider Eq. 29, if $i = 2k_1$, then

$$\begin{aligned}
&\fint_{-1}^{1} \frac{F_i(\tau)}{(\tau - c_m)^2} d\tau \\
&= \lim_{\varepsilon \to 0} \left\{ \left(\int_{-1}^{c_m - \varepsilon} + \int_{c_m + \varepsilon}^{1} \right) \frac{F_i(\tau)}{(\tau - c_m)^2} d\tau - \frac{2F_i(c_m)}{\varepsilon} \right\} \\
&= 2 \sum_{j=1}^{k_1} C_{2k_1}^{2j-1} \lim_{\varepsilon \to 0} \left\{ \left(\int_{-1}^{c_m - \varepsilon} + \int_{c_m + \varepsilon}^{1} \right) \frac{\tau^{2j-1}(\tau^2 - 1)}{(\tau - c_m)^2} d\tau - \frac{2c_m^{2j-1}(c_m^2 - 1)}{\varepsilon} \right\} \\
&= 2 \sum_{j=1}^{k_1} C_{2k_1}^{2j-1} \fint_{-1}^{1} \frac{\tau^{2j-1}(\tau^2 - 1)}{(\tau - c_m)^2} d\tau,
\end{aligned}$$

where the following identity

$$(\tau + 1)^{2k_1} - (\tau - 1)^{2k_1} = 2 \sum_{j=1}^{k_1} C_{2k_1}^{2j-1} \tau^{2j-1}$$

has been used. The case for $i = 2k_1 + 1$ can be obtained similarly.

Lemma 3.4 Under the assumption of Lemma 3.3 and for $k = i + 1$, there holds that

$$\phi_{ik}(c_j) = \begin{cases} -\dfrac{2^k}{h^{k+1}} \displaystyle\fint_{t_{j-1}}^{t_j} \frac{M_{ik}^j(t,s)}{(t-s)^2} dt, & j = m, \\[2ex] -\dfrac{2^k}{h^{k+1}} \displaystyle\int_{t_{j-1}}^{t_j} \frac{M_{ik}^j(t,s)}{(t-s)^2} dt, & j \neq m \end{cases} \tag{30}$$

and

$$\fint_{-1}^{1}\frac{F_i(\tau)}{\tau-c_m}d\tau=\begin{cases} 2\sum_{j=1}^{k_1}C_{2k_1}^{2j-1}\fint_{-1}^{1}\frac{\tau^{2j-1}(\tau^2-1)}{\tau-c_m}d\tau, & i=2k_1,\\ 2\sum_{j=0}^{k_1}C_{2k_1+1}^{2j}\fint_{-1}^{1}\frac{\tau^{2j}(\tau^2-1)}{\tau-c_m}d\tau, & i=2k_1+1.\end{cases} \tag{31}$$

Proof: If $j=m$, by the definition of Cauchy principal value integral, we have

$$\begin{aligned}\mathop{=\!\!\!\!\!\!\int}_{t_{j-1}}^{t_j}\frac{M_{ik}^j(t,s)}{(t-s)^2}dt &= \fint_{t_{j-1}}^{t_j}\frac{F_i^j(t)}{t-s}dt\\ &=\lim_{\varepsilon\to 0}\left\{\left(\int_{t_{j-1}}^{s-\varepsilon}+\int_{s+\varepsilon}^{t_j}\right)\frac{F_i^j(t)}{t-s}dt\right\}\\ &=\frac{h^{k+1}}{2^{k+1}}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{c_m-\frac{2\varepsilon}{h}}+\int_{c_m+\frac{2\varepsilon}{h}}^{1}\right)\frac{F_i(\tau)}{\tau-c_m}d\tau\right\}\\ &=\frac{h^{k+1}}{2^{k+1}}\fint_{-1}^{1}\frac{F_i(\tau)}{\tau-c_m}d\tau\\ &=-\frac{h^{k+1}}{2^k}\phi_{ik}(c_m).\end{aligned}$$

The rest parts of this lemma can be proved similarly as the proof of lemma3.3.

Lemma 3.5 Under the assumption of Lemma 3.3 and for $k>i+1$, there holds that

$$\phi_{ik}(c_j)=-\frac{2^k}{h^{k+1}}\mathop{=\!\!\!\!\!\!\int}_{t_{j-1}}^{t_j}\frac{M_{ik}^j(t,s)}{(k-i)!(t-s)^2}dt, \tag{32}$$

and

$$\begin{aligned}&\int_{-1}^{1}\frac{F_i(\tau)(\tau-c_m)^{k-i-2}}{(k-i)!}d\tau\\ &=\begin{cases} 2\sum_{j=1}^{k_1}C_{2k_1}^{2j-1}\int_{-1}^{1}\frac{\tau^{2j-1}(\tau^2-1)(\tau-c_m)^{k-i-2}}{(k-i)!}d\tau, & i=2k_1,\\ 2\sum_{j=0}^{k_1}C_{2k_1+1}^{2j}\int_{-1}^{1}\frac{\tau^{2j}(\tau^2-1)(\tau-c_m)^{k-i-2}}{(k-i)!}d\tau, & i=2k_1+1.\end{cases}\end{aligned} \tag{33}$$

The proof of this lemma can be obtained in a way similar to that of Lemma 3.3 or 3.4.

Lemma 3.6 Suppose $f(t) \in C^l[a,b], l \geq 2$. If $s \neq t_j$, for any $j = 1,2,\cdots,n$, then there holds

$$
\begin{aligned}
f(t) - f_L(t) = \quad & \sum_{i=1}^{l-1}\sum_{k=i}^{l-1} \frac{(-1)^{i+1} f^{(k+1)}(s)}{h(i+1)!} \frac{M_{ik}^j(t,s)}{(k-i)!} \\
& + \sum_{i=1}^{l-2} \frac{(-1)^{i+1}}{h(i+1)!} \frac{f^{(l)}(\xi_{ij}) - f^{(l)}(s)}{(l-i-1)!} M_{i,l-1}^j(t,s) \\
& + \frac{(-1)^l}{hl!} \tilde{M}_l^j(t), \xi_{ij} \in (t_{j-1},t_j),
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
\tilde{M}_l^j(t) = \quad & (t-t_{j-1})(t-t_j)\left[f^{(l)}(\eta_j)(t-t_{j-1})^{l-1} - f^{(l)}(\zeta_j)(t-t_j)^{l-1}\right] \\
& - f^{(l)}(s) M_{l-1,l-1}^j(t,s), \eta_j, \zeta_j \in (t_{j-1},t_j).
\end{aligned} \tag{35}
$$

Proof: By performing Taylor expansion for $f(t_j$ and $f(t_{j-1})$ at the point t, and by noting that $f(t) \in C^l[a,b]$, we have

$$
f(t_{j-1}) = f(t) + \sum_{i=1}^{l-1} \frac{f^{(i)}(t)}{i!}(t_{j-1}-t)^i + \frac{f^{(l)}(\eta_j)}{l!}(t_{j-1}-t)^l
$$

and

$$
f(t_j) = f(t) + \sum_{i=1}^{l-1} \frac{f^{(i)}(t)}{i!}(t_j-t)^i + \frac{f^{(l)}(\zeta_j)}{l!}(t_j-t)^l,
$$

where $\eta_j, \zeta_j \in (t_{j-1},t_j)$. Thus,

$$
\begin{aligned}
f(t) - f_L(t) = \quad & (t-t_{j-1})(t-t_j)\left\{\sum_{i=1}^{l-2} \frac{(-1)^{i+1} f^{(i+1)}(t)}{h(i+1)!}\left[(t-t_{j-1})^i - (t-t_j)^i\right]\right\} \\
& + (t-t_{j-1})(t-t_j)\frac{(-1)^l}{hl!}\left[f^{(l)}(\eta_j)(t-t_{j-1})^{l-1} - f^{(l)}(\zeta_j)(t-t_j)^{l-1}\right].
\end{aligned} \tag{36}
$$

On the other hand, for $i = 1,\cdots,l-2$, we have

$$
\begin{aligned}
f^{(i+1)}(t) \quad & = f^{(i+1)}(s) + f^{(i+2)}(s)(t-s) + \cdots + \frac{f^{(l)}(\xi_{ij})}{(l-i-1)!}(t-s)^{l-i-1} \\
& = \sum_{k=i}^{l-1} \frac{f^{(k+1)}(s)(t-s)^{k-i}}{(k-i)!} + \frac{f^{(l)}(\xi_{ij}) - f^{(l)}(s)}{(l-i-1)!}(t-s)^{l-i-1},
\end{aligned} \tag{37}
$$

where $\xi_{ij} \in [t_{j-1}, s)$ or $(s, t_{j-1}]$. Combining Eq. 36 and Eq. 37 leads to Eq. 34. Here, the definition of $M^j_{ik}(t,s)$ has been used.
Define

$$H_m(t) = f(t) - f_L(t) - \sum_{i=1}^{l-1}\sum_{k=i}^{l-1} \frac{(-1)^{i+1} f^{(k+1)}(s)}{h(i+1)!} \frac{M^m_{ik}(t,s)}{(k-i)!}, \ \ t \in (t_{m-1}, t_m). \tag{38}$$

Lemma 3.7 Under the assumption of Theorem 2.1, there holds that

$$\left| \neq\!\!\!\!\!\!\int_{t_{m-1}}^{t_m} \frac{H_m(t)}{(t-s)^2} dt \right| \leq C\left|\ln \gamma(h,s)\right| h^{l-1}. \tag{39}$$

Proof: Since $f(t) \in C^l[a,b]$, by Taylor expansion, we have

$$|H_m^{(i)}(t)| \leq Ch^{l-i}, \ \ i = 0,1,2.$$

By the definition of finite-part integral, we have

$$\begin{aligned} \neq\!\!\!\!\!\!\int_{t_{m-1}}^{t_m} \frac{H_m(t)}{(t-s)^2} dt = & \frac{hH_m(s)}{(s-t_{m-1})(t_m-s)} + H_m'(s) \ln \frac{t_m - s}{s - t_{m-1}} \\ & + \int_{t_{m-1}}^{t_m} \frac{H_m(t) - H_m(s) - H_m'(s)(t-s)}{(t-s)^2} dt. \end{aligned} \tag{40}$$

Now, we estimate the right hand side of Eq. 40 term by term. Since $H_m(t_m) = 0$, we have

$$\begin{aligned} \left| \frac{hH_m(s)}{(s-t_{m-1})(t_m-s)} \right| &= \left| \frac{h[H_m(s) - H_m(t_m)]}{(s-t_{m-1})(t_m-s)} \right| \\ &= \left| \frac{hH_m'(\xi_m)}{(s-t_{m-1})} \right| \leq Ch^{l-1}, \ \ \xi_m \in (s, t_m), \end{aligned} \tag{41}$$

$$\left| H_m'(s) \ln \frac{t_m - s}{s - t_{m-1}} \right| \leq C\left|\ln \gamma(h,s)\right| h^{l-1} \tag{42}$$

and

$$\begin{aligned} \left| \int_{t_{m-1}}^{t_m} \frac{H_m(t) - H_m(s) - H_m'(s)(t-s)}{(t-s)^2} dt \right| &= \left| \int_{t_{m-1}}^{t_m} \frac{1}{2} H_m''(\eta_m) dt \right| \\ &\leq Ch^{l-1}, \ \ \eta_m \in (t_{m-1}, t_m). \end{aligned} \tag{43}$$

Combining Eq. 41, Eq. 42 and Eq. 43 leads to Eq. 39 and the proof is completed.

Lemma 3.8 Under the assumption of Theorem 2.1, we have

$$\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{M}_l^j(t)}{(t-s)^2}\,dt\right|\leq C\gamma^{-1}(h,s)\frac{h^{l-1}}{l!} \tag{44}$$

and

$$\begin{aligned}&\left|\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(t,s)}{(t-s)^2}\,dt\right|\\ &\leq\begin{cases}C\dfrac{h^{l-1}}{(l-1)!}(|\ln\gamma(h,s)|+|\ln h|), & i=l-2,\\[2ex] C\dfrac{h^{l-1}}{(l-i-1)!}, & i<l-2.\end{cases}\end{aligned} \tag{45}$$

Proof: From Eq. 35, we see that $\tilde{M}_l^j(t)\leq Ch^{l+1}$, and thus

$$\begin{aligned}\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{M}_l^j(t)}{(t-s)^2}dt\right| &\leq C\frac{h^l}{l!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{1}{(t-s)^2}dt\\ &=C\frac{h^l}{l!}\left(\frac{1}{s-t_{m-1}}-\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-t_m}\right)\\ &\leq C\gamma^{-1}(h,s)\frac{h^{l-1}}{l!}.\end{aligned}$$

Now, we estimate Eq. 45. If $i=l-2$, then

$$\begin{aligned}&\left|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{l-1}}{h(l-1)!}\int_{t_{j-1}}^{t_j}(f^{(l)}(\xi_{il})-f^{(l)}(s))\frac{M_{l-2,l-1}^j(t,s)}{(t-s)^2}dt\right|\\ &\leq C\frac{h^{l-1}}{(l-1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\left|\frac{1}{t-s}\right|dt\\ &\leq C\frac{h^{l-1}}{(l-1)!}\left(\int_a^{t_{m-1}}\frac{1}{s-t}dt+\int_{t_m}^{b}\frac{1}{t-s}dt\right)\\ &=C\frac{h^{l-1}}{(l-1)!}\ln\frac{(b-s)(s-a)}{(t_m-s)(s-t_{m-1})}\\ &\leq C\frac{h^{l-1}}{(l-1)!}(|\ln\gamma(h,s)|+|\ln h|).\end{aligned}$$

If $i < l-2$, by Eq. 7, we have

$$\begin{aligned}
&\left|\sum_{j=1,,j\neq m}^{n}\frac{(-1)^{i+1}}{h(i+1)!}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{il})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^{j}(t,s)}{(t-s)^2}dt\right| \\
&=\left|\sum_{j=1,,j\neq m}^{n}\frac{(-1)^{i+1}h^l}{2^{l+1}(i+1)!}\int_{-1}^{1}\frac{f^{(l)}(\xi(\tau)_{il})-f^{(l)}(c_j)}{(l-i-1)!}F_i(\tau)(\tau-c_j)^{l-i-3}d\tau\right| \\
&\leq C\frac{h^{l-1}}{(l-i-1)!}.
\end{aligned}$$

We finished the proof.

3.1 Proof of Theorem 2.1

Proof: By Lemma 3.6, we have

$$\begin{aligned}
&\left(\int_{a}^{t_{m-1}}+\int_{t_m}^{b}\right)\frac{f(t)-f_L(t)}{(t-s)^2}dt \\
&=\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f(t)-f_L(t)}{(t-s)^2}dt \\
&=\sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{M_{ik}^{j}(t,s)}{(t-s)^2}dt \\
&+\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^{j}(t,s)}{(t-s)^2}dt \\
&+\sum_{j=1,j\neq m}^{n}\frac{(-1)^n}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{M}_l^{j}(t)}{(t-s)^2}dt.
\end{aligned} \tag{46}$$

From the definition of $H_m(t)$ in Eq. 38

$$\begin{aligned}
\,=\!\!\!\!\!\!\int_{t_{m-1}}^{t_m}\frac{f(t)-f_L(t)}{(t-s)^2}dt = &\,=\!\!\!\!\!\!\int_{t_{m-1}}^{t_m}\frac{H_m(t)}{(t-s)^2}dt \\
&+\sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\,=\!\!\!\!\!\!\int_{t_{m-1}}^{t_m}\frac{M_{ik}^{m}(t,s)}{(t-s)^2}dt
\end{aligned} \tag{47}$$

Putting Eq. 46 and Eq. 47 together yields

$$\begin{aligned}
\int\!\!\!\!\!\!= _a^b \frac{f(t)-f_L(t)}{(t-s)^2}dt &= \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\sum_{j=1}^{n}\int\!\!\!\!\!\!= _{t_{j-1}}^{t_j}\frac{M_{ik}^j(t,s)}{(t-s)^2}dt \\
&\quad+\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(t,s)}{(t-s)^2}dt \\
&\quad+\int\!\!\!\!\!\!= _{t_{m-1}}^{t_m}\frac{H_m(t)}{(t-s)^2}dt+\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{M}_l^j(t,s)}{(t-s)^2}dt \\
&:= \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^i h^k f^{(k+1)}(s)}{(i+1)!2^k}\sum_{j=1}^{n}\phi_{ik}(c_j)+R_n(s), \qquad (48)
\end{aligned}$$

where

$$R_n(s)=R_n^{(1)}(s)+R_n^{(2)}(s)$$

and

$$R_n^{(1)}(s)=\int\!\!\!\!\!\!= _{t_{m-1}}^{t_m}\frac{H_m(t)}{(t-s)^2}dt,$$

$$\begin{aligned}
R_n^{(2)}(s)= &\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{M_{i,l-1}^j(t,s)}{(t-s)^2}dt \\
&+\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{M}_l^j(t)}{(t-s)^2}dt.
\end{aligned}$$

By Lemma 3.7 and Lemma 3.8, we have

$$|R_n(s)|\le C(|\ln h|+\gamma^{-1}(h,s))h^{l-1}.$$

For $\phi_{ik}(c_j)$ $k>i+1$, from the lemma 3.5 we know that there are no singular kernel and can be considered as the Riemann integral.
For $k=i,i+1$, by the definition of $b_i^{n,m}(\tau)$ in Eq. 19, Lemma 3.1 and Lemma 3.2, there exists a function $a_i(\tau)$ such that

$$a_i(\tau)=\lim_{n\to\infty}b_i^{n,m}(\tau)=\begin{cases}\dfrac{(-1)^{i-1}}{i!}c_{i-1,i}(\tau)+\dfrac{(-1)^i}{(i+1)!}c_{ii}(\tau), & i>1,\\[2ex] -\dfrac{1}{2}c_{11}(\tau), & i=1.\end{cases} \qquad (49)$$

The proof is complete.

4 Extrapolation method

In the above sections, we have proved that the error functional of the trapezoidal rule has the following asymptotic expansion

$$E_n(f) = \sum_{i=1}^{l-1} \frac{h^i}{2^i} f^{(i+1)}(s)a_i(\tau) + R_n(s). \tag{50}$$

It is easy to see that the error functional depends on the value of $a_i(\tau)$. Now we present algorithm for the given s.

Assume there exists positive integer n_0 such that

$$m_0 := \frac{n_0(s-a)}{b-a}$$

is a positive number. We first partition $[a,b]$ into n_0 equal subinterval to get a mesh denoted by Π_1 with mesh size $h_1 = (b-a)/n_0$. Then we refine Π_1 to get mesh Π_2 with mesh size $h_2 = h_1/2$. In this way, we get a series of meshes $\{\Pi_j\}(j = 1,2,\cdots)$ in which Π_j is refined from Π_{j-1} with mesh size denoted by h_j. The extrapolation scheme is presented in Tab. 1.

Table 1: Extrapolation scheme of $T_i^{(j)}$

$T(h_1) = T_1^{(1)}$				
$T(h_2) = T_1^{(2)}$	$T_2^{(1)}$			
$T(h_3) = T_1^{(3)}$	$T_2^{(2)}$	$T_3^{(1)}$		
$T(h_4) = T_1^{(4)}$	$T_2^{(3)}$	$T_3^{(2)}$	$T_4^{(1)}$	
$T(h_5) = T_1^{(5)}$	$T_2^{(4)}$	$T_3^{(3)}$	$T_4^{(2)}$	$T_5^{(1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

For a given $\tau \in (-1,1)$, define

$$s_j = s + \frac{\tau+1}{2}h_j, \quad j = 1,2,\cdots. \tag{51}$$

and

$$T(h_j) = I_{2^{j-1}n_0}(f,s_j). \tag{52}$$

We present the following extrapolation algorithm:

Step one:

$$\text{Compute } T_1^{(j)} = T(h_j), \quad j = 1,\cdots,m.$$

Step two:

$$\text{Compute } T_i^{(j)} = T_{i-1}^{(j+1)} + \frac{T_{i-1}^{(j+1)} - T_{i-1}^{(j)}}{2^{i-1}-1}, \quad i = 2, \cdots, m \quad j = 1, \cdots, m-i.$$

Theorem 4.1 Under the asymptotic expansion of Theorem 2.1, for a given τ and the series of meshes defined by Eq. 51, we have

$$|I(f,s) - T_i^{(j)}| \leq Ch^i \tag{53}$$

and a posteriori asymptotic error estimate is given by

$$|\frac{T_{i-1}^{(j+1)} - T_{i-1}^{(j)}}{2^{i-1}-1}| \leq Ch^{i-1}.$$

Proof: By the asymptotic expansion of Eq. 50 for a given τ, we have

$$\begin{aligned} I(f,s) - T(h_j) &= I(f,s) - I(f,s_j) + I(f,s_j) - T(h_j) \\ &= I(f,s) - I(f,s_j) + \sum_{i=1}^{l-1} \frac{h_j^i}{2^i} a_i(\tau) f^{(i+1)}(s_j) + O(h_j^{l-1}). \end{aligned} \tag{54}$$

By the definition of finite-part and Eq. 51, for the first two part of Eq. 54 by Taylor expansion for $I(f;s_j)$ at the point s,

$$\begin{aligned} I(f;s_j) \quad = \quad & I(f;s) + I'(f;s)\frac{\tau+1}{2}h_j + \frac{I''(f;s)}{2!}(\frac{\tau+1}{2}h_j)^2 \\ & + \cdots + \frac{I^{(l-2)}(f;s)}{(l-2)!}(\frac{\tau+1}{2}h_j)^{l-2} + o((h_j)^{l-2}), \end{aligned} \tag{55}$$

where we have used the Lemma 2 in [Wu and Yu (1999);Yu (2002)].
Similarly we also expand $f^{(i+1)}(s_j)$ at point s, then we have

$$\begin{aligned} f^{(i+1)}(s_j) \quad = \quad & f^{(i+1)}(s) + f^{(i+2)}(s)\frac{\tau+1}{2}h_j + \frac{f^{(i+3)}(s)}{2!}(\frac{\tau+1}{2}h_j)^2 \\ & + \cdots + \frac{f^{(l)}(s)}{(l-i-1)!}(\frac{\tau+1}{2}h_j)^{l-i-1} + o((h_j)^{l-i-1}). \end{aligned} \tag{56}$$

Putting Eq. 55, Eq. 56 and Eq. 54 together, we have

$$I(f,s) - T(h_j) = \sum_{i=1}^{l-2} b_i(s,\tau) h_j^i + O(h_j^{l-1}), \tag{57}$$

where

$$b_i(s,\tau) = f^{(i+1)}(s)\sum_{k=1}^{i}\frac{a_k(\tau)}{2^k}\left(\frac{\tau+1}{2}\right)^{i-k}\frac{1}{(i-k)!} - \frac{(\tau+1)^i}{2^i i!}I^{(i)}(f,s), \tag{58}$$

then $b_i(s,\tau)$ is a constant for given τ. By Eq. 57, we also have

$$I(f,s) - T(h_{j+1}) = \sum_{i=1}^{l-2} b_i(s,\tau)h_{j+1}^i + O(h_{j+1}^{l-1}). \tag{59}$$

By Eq. 57 and Eq. 59, with $h_j = 2h_{j+1}$ we have

$$\begin{aligned} I(f,s) &= 2T(h_{j+1}) - T(h_j) + \sum_{i=2}^{l-2} b_i(s,\tau)\left(\frac{1}{2^{i-1}}-1\right)h_j^i + O(h_j^{l-1}) \\ &= T_2^{(j)} + \sum_{i=2}^{l-2} b_i(s,\tau)\left(\frac{1}{2^{i-1}}-1\right)h_j^i + O(h_j^{l-1}), \end{aligned} \tag{60}$$

which implies

$$I(f,s) - T_2^{(j)} = \sum_{i=2}^{l-2} b_i(s,\tau)\left(\frac{1}{2^{i-1}}-1\right)h_j^i + O(h_j^{l-1}) \tag{61}$$

and

$$T_2^{(j)} = 2T(h_{j+1}) - T(h_j). \tag{62}$$

By Eq. 57 and Eq. 59, we also have

$$T(h_{j+1}) - T(h_j) = \sum_{i=1}^{l-2} b_i(s,\tau)\left(\frac{1}{2^i}-1\right)h_j^i + O(h_j^{l-1}). \tag{63}$$

In order to obtain accuracy $O(h^3)$, we should continue to use extrapolation process again. A posteriori asymptotic error estimate can be similarly got. In this way, we continue extrapolation process and finished the proof.

5 Numerical Example

In this section, we present example to confirm our theoretical analysis.

Example 1 We consider the finite-part integral

$$⨎_0^1 \frac{x^4+1}{(x-s)^2}dt = 4s^2 + 2s + \frac{4}{3} + \frac{s+1}{s(s-1)} + 4s^3\log\frac{1-s}{s} \tag{64}$$

Table 2: Extrapolation value of the trapezoidal rule $s = 0.25$

	$\tau = -2/3$	h^2−extra	h^3−extra
32	-4.427994656e+0		
64	-4.470949523e+0	-4.513904391e+0	
128	-4.492714408e+0	-4.514479293e+0	-4.514670927e+0
256	-4.503668423e+0	-4.514622438e+0	-4.514670154e+0
512	-4.509163295e+0	-4.514658166e+0	-4.514670075e+0

Table 3: Errors of the trapezoidal rule $s = 0.25$

	$\tau = -2/3$	h^2−extra	h^3−extra
32	-8.667540960e-2		
64	-4.372054216e-2	-7.656747194e-4	
128	-2.195565741e-2	-1.907726621e-4	8.613570168e-7
256	-1.100164219e-2	-4.762696573e-5	8.826638886e-8
512	-5.506770788e-3	-1.189938672e-5	9.806290002e-9

Table 4: A posteriori error of the trapezoidal rule $s = 0.25$

	a posteriori error	a posteriori error	a posteriori error
32			
64	-4.295486744e-2		
128	-2.176488475e-2	-1.916340191e-4	
256	-1.095401522e-2	-4.771523212e-5	-1.104415183e-7
512	-5.494871401e-3	-1.190919300e-5	-1.120858555e-8

Table 5: Errors of the trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

	$\tau = -2/3$	$\tau = 2/3$	$\tau = 0$
100	4.135192716e-1	2.200095045e+0	1.345661969e+0
200	2.047486574e-1	1.055159313e+0	6.570399556e-1
400	1.018774233e-1	5.170413867e-1	3.247134719e-1
800	5.081520627e-2	2.559654336e-1	1.614220085e-1
1600	2.537681635e-2	1.273534599e-1	8.047938958e-2

and let $s = 0.25$ and 0.9. We choose the series $s_j = s + (\tau + 1)h_j/2$ with $\tau = 0, \pm\frac{2}{3}$, where s_j is defined as Eq. 51. Numerical results shows that $\tau = -\frac{2}{3}$ is the best one.

In Tab. 2 and Tab. 6, we present the extrapolation value of $s = 0.25$ and $s = 0.9$

Table 6: Extrapolation value of trapezoidal rule $s = 0.9$

	$\tau = -2/3$	h^2-extra	h^3-extra
100	-2.155840392e+1		
200	-2.134963330e+1	-2.114086269e+1	
400	-2.124676207e+1	-2.114389083e+1	-2.114490022e+1
800	-2.119569985e+1	-2.114463763e+1	-2.114488657e+1
1600	-2.117026146e+1	-2.114482307e+1	-2.114488488e+1

Table 7: Errors of the trapezoidal rule $s = 0.9$

	$\tau = -2/3$	h^2-extra	h^3-extra
100	4.135192716e-1		
200	2.047486574e-1	-4.021956765e-3	
400	1.018774233e-1	-9.938107202e-4	1.557129472e-5
800	5.081520627e-2	-2.470107994e-4	1.922507508e-6
1600	2.537681635e-2	-6.157357297e-5	2.388358382e-7

Table 8: A posteriori error of the trapezoidal rule $s = 0.9$

	a posteriori error	a posteriori error	a posteriori error
100			
200	-2.087706142e-1		
400	-1.028712341e-1	3.028146045e-3	
800	-5.106221707e-2	7.467999207e-4	-1.364878721e-5
1600	-2.543838992e-2	1.854372264e-4	-1.683671670e-6

respectively. In Tab. 3 and Tab. 7, error of the trapezoidal rule are h, h^2 and h^3 for the extrapolation results which agrees with the theoretic analysis. We also obtain a posterior estimation in Tab. 4 and Tab. 8.

For a given s, we can find the starting meshes n_0 theoretically. But in actual computation if n_0 is too big, the extrapolation algorithm is not proper to be adopted. For example with $s = \frac{1}{\sqrt{2}}$ we can not find the proper starting meshes. There are many methods to solve the problem, one methods is by moving the starting meshes a little then make the singular point be located at the mesh point. In fact, it is not difficult to extend our methods to the quasi-uniform meshes and the proof is similarly to Theorem 2.1.

Example 2 We still consider the finite-part integral Eq. 64 for s is always placed at the mesh point in such a way that with two meshes shorter or longer at the end of the interval. Then we refine the quasi-uniform mesh and get the results as follow.

Table 9: Errors of the trapezoidal rule $s_j = s + (\tau+1)h_j/2$

	$\tau = -2/3$	$\tau = 2/3$	$\tau = 0$
32	1.431192486e-1	7.611952535e-1	5.727236676e-1
64	6.931541967e-2	3.571260464e-1	2.761101316e-1
128	3.410556423e-2	1.730661634e-1	1.355831409e-1
256	1.691584560e-2	8.520188248e-2	6.718450459e-2
512	8.423825331e-3	4.227332480e-2	3.344182282e-2

Table 10: Extrapolation value of the trapezoidal rule $s = 1/\sqrt{2}$

	$\tau = -2/3$	h^2-extra	h^3-extra
32	-4.884663520e+0		
64	-4.810859691e+0	-4.737055862e+0	
128	-4.775649836e+0	-4.740439980e+0	-4.741568020e+0
256	-4.758460117e+0	-4.741270399e+0	-4.741547205e+0
512	-4.749968097e+0	-4.741476077e+0	-4.741544636e+0

Table 11: Errors of the trapezoidal rule $s = 1/\sqrt{2}$

	$\tau = -2/3$	h^2-extra	h^3-extra
32	1.431192486e-1		
64	6.931541967e-2	-4.488409286e-3	
128	3.410556423e-2	-1.104291206e-3	2.374815431e-5
256	1.691584560e-2	-2.738730250e-4	2.933035312e-6
512	8.423825331e-3	-6.819494059e-5	3.644208721e-7

Table 12: A posteriori error of the trapezoidal rule $s = 1/\sqrt{2}$

	a posteriori error	a posteriori error	a posteriori error
32			
64	7.380382896e-2		
128	3.520985544e-2	-3.384118080e-3	
256	1.718971863e-2	-8.304181809e-4	2.081511899e-5
512	8.492020272e-3	-2.056780844e-4	2.568614440e-6

Numerical results of the extrapolation value, extrapolation errors and a posteriori error are presented in Tab. 10, Tab. 11 and Tab. 12 respectively. One can see that the convergence orders are h, h^2 and h^3, which is in good agreement with our theoretical analysis.

The tables show that our methods not only possess high accuracy and low computational complexity, but also extrapolation error and a posterior error estimate coincide with the theoretic analysis very well.

6 Concluding remarks

In this paper, we have shown both theoretically and numerically that the error function of the trapezoidal rule has the expression of Eq. 13. Based on the the asymptotic expansion of the error function, we present extrapolation algorithm to accelerate convergence rate. In order to avoid the compute function $a_i(\tau)$, we choose a series to approximate the singular point with the local coordinate for a given τ. Moreover, by the extrapolation method we not only obtain a high order of accuracy, but also a posteriori error estimate is conveniently derived. As we known, function $f(t)$ in the boundary element methods is unknown, with the help of a posteriori error estimate an satisfactory accuracy can be easily obtained.

The extrapolation methods has been extensively studied for solving partial differential equations and singular integral equations, see Lü and Huang (2003) and Lü and Huang (2006). The results in this paper show a possible way to improve the accuracy of the computation for Hadamard finite-part integral. Compare with the indirect methods, this methods is simple and effective. Furthermore, it is possible to extend the approach in this paper to the Cauchy principal value integral and hypersingular integral on circle.

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References

Albuquerque, E. L.; Aliabadi, M. H. (2008): A boundary element formulation for boundary only analysis of thin shallow shells. *CMES:Computer Modeling in Engineering & Sciences,* vol. 29, no. 2, pp. 63-73.

Andrews, L. C. (2002): *Special Functions of Mathematics for Engineers.* McGraw-Hill, Inc.

Brunner, H.(2004): *Collocation Methods for Volterra Integral and Related Functional Equations* Cambridge University Press.

Choi, U.J.; Kim, S.W.; Yun, B.I.(2004): Improvement of the asymptotic behaviour of the Euler-Maclaurin formula for Cauchy principal value and Hadamard finite-part integrals *Int. J. Numer. Methods.*, vol.61, pp. 496-513.

Du, Q.K. (2001): Evaluations of certain hypersingular integrals on interval. *Int. J. Numer. Methods.*, vol. 512, pp. 1195-1210.

Elliott, D.; Venturino, E. (1997): Sigmoidal transformations and the Euler-Maclaurin expansion for evaluating certain Hadamard finite-part integrals. *Numer. Math.*, vol. 77, pp. 453-465.

He,X.F.; Lim,K.M.; Lim, S. L. (2008): Fast BEM solvers for 3D soisson-type equations. *CMES:Computer Modeling in Engineering & Sciences,* vol. 35, no. 1, pp. 21-48.

Hasegawa, T. (2004): Uniform approximations to finite Hilbert transform and its derivative. *J. Comput. Appl. Math.*, vol. 163, pp. 127-138.

Hui, C.Y.; Shia, D.(1999): Evaluations of hypersingular integrals using Gaussian quadrature. *Int. J. Numer. Methods.*, vol. 44, pp. 205-214.

Ioakimidis, N.I.(1985): On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives. *Math. Comp.*, vol 44, pp. 191-198.

Kim, P.; Jin, U.C. (2003): Two trigonometric quadrature formulae for evaluating hypersingular integrals. *Inter. J. Numer. Methods.*, vol. 56, pp. 469-486.

Liem, C.B.; Lü, T.; Shih, T.M. (1995): *The Splitting Extrapolation Method.* World Scientific, Singapore.

Linz, P. (1985): On the approximate computation of certain strongly singular integrals. *Computing.*, vol. 352, pp. 345–353.

Liu, D.J; Yu, D. H.(2008): The coupling method of natural boundary element and mixed finite element for stationary Navier-Stokes equation in unbounded domains. *CMES:Computer Modeling in Engineering & Sciences,* vol. 37, no. 3, pp. 305-330.

Liu, C.S.(2007): A highly accurate solver for the mixed-boundary potential problem and singular problem in arbitrary plane domain. *CMES:Computer Modeling in Engineering & Sciences,* vol. 20, No. 2, pp. 111-122.

Lü, T.; Huang, Y. (2006): Extrapolation method for solving weakly singular nonlinear Volterra integral equations of the second kind. *J. Math. Anal. Appl.*, vol. 324, pp. 223–237.

Lü, T.; Huang, Y. (2003): A generalization of discrete Gronwall inequality and its application to weakly singular Volterra integral equality of the second kind. *J. Math. Anal. Appl.*, vol.282, pp. 56-62.

Monegato, G.(1994): Numerical evaluation of hypersingular integrals. *J. Comput. Appl. Math.*, vol.50, pp. 9-31.

Owatsiriwong, A.; Phansri, B.; Park, K.H.(2008): A cell-less BEM formulation for 2D and 3D elastoplastic problems using particular integrals. *CMES:Computer Modeling in Engineering & Sciences,* vol.31, no.1, pp. 37-59.

Sanz, J.A.; Solis, M.; Dominguez, J.(2007): Hypersingular BEM for piezoelectric solids: formulation and applications for fracture mechanics. *CMES:Computer Modeling in Engineering & Sciences,* vol.17, no.3, pp. 215-229.

Sidi, A.(2003): *Practical Extrapolation Methods Theory and Applications*. Cambridge University Press.

Wu, J.M.; Sun, W.W. (2008): The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval. *Numer. Math.*, vol. 109, pp. 143–165.

Wu, J.M.; Sun, W.W. (2005): Newton-Cotes formulae for the numerical evaluation of certain hypersingular integral. *Computing.*, vol.75, pp. 297-309.

Wu, J.M.; Sun, W.W. (2005): The superconvergence of the composite trapezoidal rule for Hadamard finite part integrals., *Numer. Math.*, vol. 102, pp. 343–363.

Wu, J.M.; Lü,Y. (2005): A superconvergence result for the second order Newton-Cotes formula for certain finite part integrals. *IMA Journal of Numerical Analysis.*,vol.25, pp. 253-263.

Wu, J.M.; Yu,D.H.(1999): The approximate computation of hypersingular integrals on interval., *Chinese J. Numer. Math. Appl.*, vol.21 pp. 25-33.

Yu, D. H.; Huang,H.Y.(2008): The artificial boundary method for a nonlinear interface problem on unbounded domain. *CMES:Computer Modeling in Engineering & Sciences,* vol.35, no.3, pp. 227-252.

Yu, D. H.(2002): *Natural Boundary Integrals Method and its Applications*. Kluwer Academic Publishers.

Yu, D. H.(1993): *Mathematical Theory of Natural Boundary Element Method*. Science Press, Beijing.

Yu,D.H.(1992): The approximate computation of hypersingular integrals on interval., *Numer. Math. J. Chinese Univ.*,vol.1, pp. 114-127.

Yu,D.H.(1985): Approximation of boundary conditions at infinity for a harmonic equation., *J. Comp. Math.*,vol. 3, no. 3, pp. 219-227.

Young, D.L.;Chen, K.H.; Chen,J.T.;Kao, J.H. (2007): A modified method of fundamental solutions with source on the boundary for solving laplace equations with circular and arbitrary domains. *CMES:Computer Modeling in Engineering & Sciences,* vol.19, no.3, pp.197-221.

Zhou, Y.T.; Li,X.; Yu, D. H.(2008): Integral method for contact problem of bonded plane material with arbitrary cracks., *CMES:Computer Modeling in Engineering & Sciences,* vol. 36, no.2, pp. 147-172.