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An alternating iterative MFS algorithm for the Cauchy problem for the modified Helmholtz equation

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Abstract We investigate the numerical implementation of the alternating iterative algorithm originally proposed by Kozlov et al. (Comput Math Math Phys 31:45–52) for the Cauchy problem associated with the two-dimensional modified Helmholtz equation using a meshless method. The two mixed, well-posed and direct problems corresponding to every iteration of the numerical procedure are solved using the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter is chosen according to the generalized cross-validation criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, is also presented. The iterative MFS algorithm is tested for Cauchy problems for the two-dimensional modified Helmholtz operator to confirm the numerical convergence, stability and accuracy of the method.

Keywords Modified Helmholtz equation · Inverse problem · Cauchy problem · Iterative method of fundamental solutions (MFS) · Regularization

1 Introduction

Helmholtz-type equations are often used to describe the vibration of a structure [1], the acoustic cavity problem [3], the radiation wave [19], the scattering of a wave [17], the problem of heat conduction in fins [28], the Debye-Hückel theory [7], the linearization of the Poisson–Boltzmann equation [29], etc. In many engineering problems, either the boundary conditions are often incomplete, or the geometry of the domain under investigation is not completely known, or the so-called wave number, $\kappa > 0$, that characterises the Helmholtz-type equation is unknown. These are *inverse problems* and it is well known that they are generally ill-posed, in the sense that the existence, uniqueness and stability of their solutions are not always guaranteed [16]. A classical example of an inverse boundary value problem associated with Helmholtz-type equations is represented by the *Cauchy problem*. In this case, boundary conditions are incomplete, in the sense that a part of the boundary of the solution domain is over-specified by prescribing on it both the primary field and its normal derivative, while the remaining boundary is under-specified and boundary conditions on the latter boundary have to be determined. The uniqueness of the Cauchy problem is guaranteed without the necessity of removing the eigenvalues for the Laplacian operator, as it happens in the case of direct problems for the Helmholtz equation (see [4]). However, the Cauchy problem suffers from the non-existence and instability of the solution.

Over the last decade, many theoretical and numerical studies have been devoted to the Cauchy problem associated with Helmholtz-type equations. DeLillo et al. [8] detected the source of acoustical noise inside the cabin of a midsize aircraft from measurements of the acoustical pressure field inside the cabin by solving a linear Fredholm integral equation of the first kind and they extended this study to three-dimensional problems (see [9]). The alternating iterative algorithm of Kozlov et al. [27], which reduces the Cauchy problem to solving a sequence of well-posed boundary value problems, was implemented numerically using the boundary

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element method (BEM) for the two-dimensional modified Helmholtz equation by Marin et al. [41]. Marin et al. [42] used the conjugate gradient method (CGM), in conjunction with the BEM, in order to solve the same inverse problem for both the Helmholtz and the modified Helmholtz equations. Four regularization methods for the stable solution of the Cauchy problem associated with Helmholtz-type equations, namely the Tikhonov regularization, the singular value decomposition (SVD), the CGM and the alternating iterative algorithm of [27], were compared by Marin et al. [43]. The Landweber–Fridman method and the BEM were used to solve the Cauchy problem for two-dimensional Helmholtz and modified Helmholtz equations with L^2-boundary data by Marin et al. [44]. Jin and Zheng [24] solved some inverse boundary value problems for the Helmholtz equation using the boundary knot method and a SVD regularization and they also extended this method to some inverse problems associated with the inhomogeneous Helmholtz equation [25]. The numerical solution for the Cauchy problem for two- and three-dimensional Helmholtz-type equations by employing the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method and SVD, was investigated by Marin and Lesnic [39] and Marin [33], and Jin and Zheng [26], respectively. Some spectral regularization methods and a modified Tikhonov regularization method to stabilize the Cauchy problem for the Helmholtz equation at fixed frequency were proposed by Xiong and Fu [60], while Jin and Marin [23] employed the plane wave method and the SVD to solve stably the same problem. Wei et al. [59], Qin and Wen [51] and Qin et al. [52] reduced the Cauchy problem associated with Helmholtz-type equations to a moment problem and also provided an error estimate and convergence analysis for the latter. Qin and Wei [48,50] proposed two regularization methods, namely a modified Tikhonov regularization method and a truncation method, for the stable approximate solution to the Cauchy problem for the Helmholtz equation and they also presented convergence and stability results under suitable choices of the regularization parameter. The quasi-reversibility method and a truncation method were used to solve the Cauchy problem for the modified Helmholtz equation in a rectangular domain by Qin and Wei [49], who also analysed the stability and convergence of the proposed regularization procedures. Shi et al. [53] addressed a fourth-order modified method for the solution of the Cauchy problem associated with the modified Helmholtz equation in an infinite strip domain and they also provided convergence estimates under the suitable choices of regularization parameters and the a priori assumption on the bounds of the exact solution. Recently, the Cauchy problem for two-dimensional Helmholtz-type equations with L^2-boundary data was approached by combining the BEM with the minimal error method by Marin [36].

The MFS is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of solutions to linear partial differential equations. Like the BEM, the MFS is applicable when a fundamental solution of the governing PDE is explicitly known. Since its introduction as a numerical method in the late 1970s by Mathon and Johnston [45], it has been successfully applied to a large variety of physical problems, an account of which may be found in the survey papers [6,12–14].

The ease of implementation of the MFS and its low computational cost make it an ideal candidate for inverse problems as well. For these reasons, the MFS, in conjunction with various regularization methods (e.g. the Tikhonov regularization method, Morozov's discrepancy principle, singular value decomposition), have been used increasingly over the last decade for the numerical solution of inverse problems. For example, the Cauchy problem associated with the heat conduction equation [10,21,22,30,35,37,54,57,58,61], linear elasticity [32,38], steady-state heat conduction in functionally graded materials (FGMs) [33], Helmholtz-type equations [26,34,39], Stokes problems [5], the biharmonic equation [40] etc. have been successfully addressed by employing the MFS.

To our knowledge, the MFS has not, as yet, been applied iteratively to the numerical solution of the Cauchy problem associated with the modified Helmholtz equation. Due to this fact and also encouraged by the recent results obtained by Marin [37], who implemented the alternating iterative algorithm of Kozlov et al. [27], in conjunction with the MFS, for two-dimensional harmonic Cauchy problems, we decided to extend, in this paper, the work of Marin [37] to Cauchy problems for the modified Helmholtz operator in two dimensions. At every iteration, two mixed, well-posed and direct problems are solved using the MFS, in conjunction with the Tikhonov regularization method. For each of the aforementioned direct problems, the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, is also presented. The iterative MFS algorithm is then tested for Cauchy problems for the modified Helmholtz operator in two-dimensional simply and doubly connected domains with smooth boundaries.

2 Mathematical formulation

Consider an open bounded domain $\Omega \subset \mathbb{R}^d$, with d the dimension of the space where the problem is posed, usually $d \in \{1, 2, 3\}$, and assume that Ω is bounded by a curve $\partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and

$\Gamma_1 \cap \Gamma_2 = \emptyset$. In this work, in order to refer to a specific physical problem, we shall consider Helmholtz-type equations in the context of heat transfer problems (see [28]). We therefore assume that the temperature field, $u(\mathbf{x})$, satisfies the modified Helmholtz equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, namely

$$\left(\nabla^2 - \kappa^2\right) u(\mathbf{x}) = 0, \quad \mathbf{x} = (x_1, \ldots, x_d) \in \Omega, \tag{1}$$

where $\kappa > 0$, $\nabla^2 \equiv \sum_{i=1}^{d} \partial_i \, \partial_i$ and $\partial_i \equiv \partial / \partial x_i$. The partial differential equation (2) models the heat conduction in a fin where u is the dimensionless local fin temperature, $\kappa^2 = h / \left(\widetilde{k} \ \delta_f\right)$, h is the surface heat transfer coefficient $[\mathrm{W} / \left(\mathrm{m}^2\,\mathrm{K}\right)]$, $\widetilde{k}$ is the thermal conductivity of the fin $[\mathrm{W} / (\mathrm{m\,K})]$ and δ_f is the half-fin thickness [m].

We now let $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), \ldots, n_d(\mathbf{x}))^{\mathsf{T}}$ be the outward normal vector at $\mathbf{x} \in \partial\Omega$ and $q(\mathbf{x}) \equiv \nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$. In the direct problem formulation, the knowledge of the constant κ, the location, shape and size of the entire boundary $\partial\Omega$, the temperature and/or the normal heat flux on the entire boundary $\partial\Omega$ gives the corresponding Dirichlet, Neumann, or mixed boundary conditions which enable one to determine the unknown boundary conditions, as well as the temperature distribution in the solution domain.

A different and more interesting situation arises when it is possible to measure both the temperature and the normal heat flux on a part of the boundary $\partial\Omega$, say Γ_1, and this leads to the mathematical formulation of the Cauchy problem consisting of the partial differential equations (2) and the boundary conditions

$$u(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad q(\mathbf{x}) = \widetilde{q}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_1, \tag{2}$$

where $\widetilde{u} \in H^{1/2}(\Gamma_1)$ and $\widetilde{q} \in \left(H^{1/2}(\Gamma_1)\right)^*$ are prescribed temperature and normal heat flux, respectively. In the above formulation of the boundary conditions (2), it can be seen that the boundary Γ_1 is over-specified by prescribing both the temperature $u\big|_{\Gamma_1} = \widetilde{u}$ and the normal heat flux $q\big|_{\Gamma_1} = \widetilde{q}$, while the boundary Γ_1 is under-specified since both the temperature $u\big|_{\Gamma_2}$ and the normal heat flux $q\big|_{\Gamma_2}$ are unknown and have to be determined. We also assume that data are chosen such that there exists a solution to this Cauchy problem. This solution is unique according to the so-called unique continuation properties for elliptic equations. A necessary condition for the Cauchy problem given by Eqs. (1)–(2) to be identifiable is that $\mathrm{meas}(\Gamma_1) \geq \mathrm{meas}(\Gamma_2)$.

This inverse problem is much more difficult to solve both analytically and numerically than the direct problem, since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well known that this solution is unstable with respect to small perturbations into the data on Γ_1, see Hadamard [16]. Thus the problem is ill-posed and we cannot use a direct approach, such as the least-squares method, in order to solve the system of linear equations which arises from the discretisation of the partial differential equation (1) and the boundary conditions (2). Therefore, regularization methods are required in order to solve accurately the inverse problem (1)–(2) for the modified Helmholtz equation.

3 Description of the algorithm

Kozlov et al. [27] proposed the following iterative algorithm for the simultaneous reconstruction of the unknown temperature $u\big|_{\Gamma_2}$ and normal heat flux $q\big|_{\Gamma_2}$ on the under-specified boundary:

Step 1 (i) If $k = 1$ then specify an initial boundary temperature guess on Γ_2, namely $u^{(2k-1)} \in H^{1/2}(\Gamma_2)$.

(ii) If $k > 1$ then solve the following mixed, well-posed, direct problem:

$$\left(\nabla^2 - \kappa^2\right) u^{(2k-1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{3a}$$

$$u^{(2k-1)}(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{3b}$$

$$q^{(2k-1)}(\mathbf{x}) = q^{(2k-2)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{3c}$$

to determine $u^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $u^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$

Step 2 Having constructed the approximation $u^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$\left(\nabla^2 - \kappa^2\right) u^{(2k)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{4a}$$

$$q^{(2k)}(\mathbf{x}) = \widetilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{4b}$$

$$u^{(2k)}(\mathbf{x}) = u^{(2k-1)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{4c}$$

is solved to determine $u^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $q^{(2k)}(\mathbf{x}) \equiv \nabla u^{(2k)}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 3 Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Let $H^1(\Omega)$ be the Sobolev space and $H^{1/2}(\partial\Omega)$ be the space of traces on $\partial\Omega$ corresponding to $H^1(\Omega)$, see e.g. Lions and Magenes [31]. We denote by $H^{1/2}(\Gamma_i)$ the space of functions from $H^{1/2}(\partial\Omega)$ that are bounded on Γ_i and by $\left(H^{1/2}(\Gamma_i)\right)^*$ the dual space of $H^{1/2}(\Gamma_i)$, for $i = 1, 2$. Kozlov et al. [27] showed that if $\partial\Omega$ is smooth, $\widetilde{u} \in H^{1/2}(\Gamma_1)$ and $\widetilde{q} \in \left(H^{1/2}(\Gamma_1)\right)^*$, then the alternating iterative algorithm based on steps $1 - 3$ produces two sequences of approximate solutions $\left\{u^{(2k-1)}\right\}_{k\geq 1}$ and $\left\{u^{(2k)}\right\}_{k\geq 1}$ which both converge in $H^1(\Omega)$ to the solution u of the Cauchy problem (1)–(2) for any initial guess $u^{(1)} \in H^{1/2}(\Gamma_2)$, provided that a solution to this Cauchy problem exists. Furthermore, the alternating iterative algorithm has a regularizing character. Also, the same conclusion holds if at the step 1 one specifies an initial guess for the unknown normal heat flux on Γ_2, i.e. $\mathrm{q}^{(1)} \in \left(\mathrm{H}^{1/2}(\Gamma_2)\right)^*$, instead of an initial guess for the temperature, $\mathrm{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$, and we modify steps 1 and 2 accordingly.

4 Method of fundamental solutions

4.1 MFS approximation

The fundamental solution of the modified Helmholtz equation in two-dimensions is given by, see Fairweather and Karageorghis [12], and Marin and Lesnic [39]

$$G(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{2\pi} K_0 \left(\kappa \left\|\mathbf{x} - \boldsymbol{\xi}\right\|\right), \quad \mathbf{x} \in \overline{\Omega}, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{5}$$

where $\boldsymbol{\xi}$ is a singularity (or source point) and K_0 is the modified Bessel function of the second kind of order zero. The main idea of the MFS consists of approximating the temperature in the solution domain by a linear combination of fundamental solutions with respect to M singularities $\boldsymbol{\xi}^{(j)}$, $j = 1, \ldots, M$, in the form

$$\mathrm{u}(\mathbf{x}) \approx \mathrm{u}_M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j \, G(\mathbf{x}, \boldsymbol{\xi}^{(j)}), \quad \mathbf{x} \in \overline{\Omega}, \tag{6}$$

where $\mathbf{c} = [\mathrm{c}_1, \ldots, \mathrm{c}_M]^{\mathsf{T}}$ and $\boldsymbol{\xi} \in \mathbb{R}^{2M}$ is a vector containing the coordinates of the singularities $\boldsymbol{\xi}^{(j)}$, $j = 1, \ldots, M$.

On taking into account the definitions of normal heat flux and the fundamental solution for the two-dimensional modified Helmholtz equation (5) then the normal heat flux, through a curve defined by the outward unit normal vector $\mathbf{n}(\mathbf{x})$, can be approximated on the boundary $\partial\Omega$ by

$$\mathrm{q}(\mathbf{x}) \approx \mathrm{q}_M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j \, H(\mathbf{x}, \boldsymbol{\xi}^{(j)}), \quad \mathbf{x} \in \partial\Omega, \tag{7}$$

where

$$H(\mathbf{x}, \boldsymbol{\xi}) = -\frac{\kappa}{2\pi} K_1 \left(\kappa \left\|\mathbf{x} - \boldsymbol{\xi}\right\|\right) \left[\frac{\mathbf{x} - \boldsymbol{\xi}}{\left\|\mathbf{x} - \boldsymbol{\xi}\right\|} \cdot \mathbf{n}(\mathbf{x})\right], \quad \mathbf{x} \in \overline{\Omega}, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Omega}. \tag{8}$$

Here K_1 is the modified Bessel function of the second kind of order one.

Next, we select the N_1 MFS collocation points $\left\{\mathbf{x}^{(i)}\right\}_{i=1}^{N_1}$ on the boundary Γ_1 and the N_2 MFS collocation points $\left\{\mathbf{x}^{(i)}\right\}_{i=N_1+1}^{N_1+N_2}$ on the boundary Γ_2, such that the total number of MFS collocation points used to discretise the boundary $\partial\Omega$ of the solution domain Ω is given by $N = N_1 + N_2$.

According to the MFS approximations (6) and (7), the discretised versions of the the boundary value problems (3a)–(3c) and (4a)–(4c) recast as

$$\mathbf{A}^{(1)} \, \mathbf{c}^{(2k-1)} = \mathbf{b}^{(2k-1)}, \quad k > 1, \tag{9}$$

and

$$\mathbf{A}^{(2)} \, \mathbf{c}^{(2k)} = \mathbf{b}^{(2k)}, \quad k \geq 1, \tag{10}$$

respectively. Here the components of the MFS matrices and right-hand side vectors corresponding to Eqs. (9) and (10) are given by

$$\mathrm{A}_{ij}^{(1)} = \begin{cases} G(\mathbf{x}^{(i)}, \boldsymbol{\xi}^{(j)}), & i = 1, \ldots, N_1, \\ & j = 1, \ldots, M, \\ H(\mathbf{x}^{(i)}, \boldsymbol{\xi}^{(j)}), & i = N_1 + 1, \ldots, N_1 + N_2, \\ & j = 1, \ldots, M, \end{cases} \tag{11a}$$

$$\mathrm{b}_i^{(2k-1)} = \begin{cases} \widetilde{\mathrm{u}}(\mathbf{x}^{(i)}), & i = 1, \ldots, N_1, \\ \mathrm{q}^{(2k-2)}(\mathbf{x}^{(i)}), & i = N_1 + 1, \ldots, N_1 + N_2, \end{cases} \tag{11b}$$

and

$$\mathrm{A}_{ij}^{(2)} = \begin{cases} H(\mathbf{x}^{(i)}, \boldsymbol{\xi}^{(j)}), & i = 1, \ldots, N_1, \\ & j = 1, \ldots, M, \\ G(\mathbf{x}^{(i)}, \boldsymbol{\xi}^{(j)}), & i = N_1 + 1, \ldots, N_1 + N_2, \\ & j = 1, \ldots, M, \end{cases} \tag{12a}$$

$$\mathrm{b}_i^{(2k)} = \begin{cases} \widetilde{\mathrm{q}}(\mathbf{x}^{(i)}), & i = 1, \ldots, N_1, \\ \mathrm{u}^{(2k-1)}(\mathbf{x}^{(i)}), & i = N_1 + 1, \ldots, N_1 + N_2, \end{cases} \tag{12b}$$

respectively.

Each of Eqs. (9) and (10) represents a system of N linear algebraic equations with M unknowns, namely the MFS coefficients $\mathbf{c}^{(2k-1)} = \left[\mathrm{c}_1^{(2k-1)}, \ldots, \mathrm{c}_M^{(2k-1)}\right]^{\mathsf{T}}$ and $\mathbf{c}^{(2k)} = \left[\mathrm{c}_1^{(2k)}, \ldots, \mathrm{c}_M^{(2k)}\right]^{\mathsf{T}}$, respectively. It should be noted that in order to uniquely determine the solutions $\mathbf{c}^{(2k-1)} \in \mathbb{R}^M$ and $\mathbf{c}^{(2k)} \in \mathbb{R}^M$ to the systems of linear algebraic equations (9) and (10), respectively, the number N of MFS boundary collocation points on the boundary $\partial\Omega$ and the number M of singularities must satisfy the inequality $M \leq N$. However, the systems of linear algebraic equations (9) and (10) cannot be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution for noisy Cauchy data on Γ_1.

4.2 MFS boundary collocation points and singularities

In order to implement the MFS, the location of the singularities has to be determined and this is usually achieved by considering either the static or the dynamic approach. In the static approach, the singularities are pre-assigned and kept fixed throughout the solution process, whilst in the dynamic approach, the singularities and the unknown coefficients are determined simultaneously during the solution process (see Fairweather and Karageorghis [12]). Thus the dynamic approach transforms the inverse problem into a more difficult nonlinear ill-posed problem which is also computationally much more expensive. The advantages and disadvantages of the MFS with respect to the location of the fictitious sources are described at length in Heise [20] and Burgess and Maharejin [2].

Recently, Gorzelańczyk and Kołodziej [15] thoroughly investigated the performance of the MFS with respect to the shape of the pseudo-boundary on which the source points are situated, proving that, for the same number of boundary collocation points and sources, more accurate results are obtained if the shape of the pseudo-boundary is similar to that of the boundary of the solution domain. Therefore, we have decided to employ the static approach in our computations, at the same time accounting for the findings of Gorzelańczyk and Kołodziej [15].

5 Regularization

It is well-known that the MFS discretisation matrices $\mathbf{A}^{(i)}$, $i = 1, 2$, are severely ill-conditioned. The accurate and stable solutions of Eqs. (9) and (10) are very important for obtaining physically meaningful numerical results. Regularization methods are among the most popular and successful methods for solving stably and accurately ill-conditioned matrix equations (see Hansen [18] and Tikhonov and Arsenin [55]). In this section we present a classical regularization procedure for obtaining stable solutions to the systems of linear algebraic equations (9) and (10), as well as details regarding the optimal choice of the regularization parameter.

5.1 Tikhonov regularization method

Consider the following system of linear algebraic equations

$$\mathbf{A}\,\mathbf{c} = \mathbf{b}, \tag{13}$$

where $N \geq M$, $\mathbf{A} \in \mathbb{R}^{N\times M}$, $\mathbf{c} \in \mathbb{R}^{M}$ and $\mathbf{b} \in \mathbb{R}^{N}$. Note that Eq. (13) may describe each of the MFS systems of linear equations (9) and (10), provided that

$$\mathbf{A} = \mathbf{A}^{(1)}, \quad \mathbf{c} = \mathbf{c}^{(2k-1)}, \quad \mathbf{b} = \mathbf{b}^{(2k-1)}, \quad k > 1, \tag{14}$$

and

$$\mathbf{A} = \mathbf{A}^{(2)}, \quad \mathbf{c} = \mathbf{c}^{(2k)}, \quad \mathbf{b} = \mathbf{b}^{(2k)}, \quad k \geq 1, \tag{15}$$

respectively. The Tikhonov zeroth-order regularized solution to the generically written system of linear algebraic equations (13) is sought as (see [55])

$$\mathbf{c}_\lambda : \quad \mathcal{F}_\lambda\left(\mathbf{c}_\lambda\right) = \min_{\mathbf{c}\in\mathbb{R}^M} \mathcal{F}_\lambda\left(\mathbf{c}\right), \tag{16}$$

where $\mathcal{F}_\lambda$ represents the Tikhonov zeroth-order regularization functional given by (see [55])

$$\begin{aligned} &\mathcal{F}_\lambda(\cdot) : \mathbb{R}^M \longrightarrow [0, \infty), \\ &\mathcal{F}_\lambda\left(\mathbf{c}\right) = \left\|\mathbf{A}\,\mathbf{c} - \mathbf{b}\right\|^2 + \lambda^2 \left\|\mathbf{c}\right\|^2, \end{aligned} \tag{17}$$

and $\lambda > 0$ is the regularization parameter to be prescribed. Formally, the Tikhonov regularized solution $\mathbf{c}_\lambda$ of the problem (13) is given as the solution of the normal equation

$$\left(\mathbf{A}^\mathsf{T}\mathbf{A} + \lambda^2\mathbf{I}_M\right)\,\mathbf{c} = \mathbf{A}^\mathsf{T}\,\mathbf{b}, \tag{18}$$

where $\mathbf{I}_M \in \mathbb{R}^{M\times M}$ is the identity matrix. More precisely, $\mathbf{c}_\lambda$ can be expressed as

$$\mathbf{c}_\lambda = \mathbf{A}^\dagger\,\mathbf{b}, \quad \mathbf{A}^\dagger \equiv \left(\mathbf{A}^\mathsf{T}\mathbf{A} + \lambda^2\mathbf{I}_M\right)^{-1}\,\mathbf{A}^\mathsf{T}. \tag{19}$$

If the Cauchy data on the over-specified boundary Γ_1 are noisy and hence the right-hand side of Eq. (13) is corrupted by noise, i.e.

$$\left\|\mathbf{b}^\epsilon - \mathbf{b}\right\| \leq \epsilon, \tag{20}$$

then the following stability estimate holds, see [11],

$$\|\mathbf{c}_\lambda^\epsilon - \mathbf{c}_\lambda\| \leq \frac{\epsilon}{\lambda}, \tag{21}$$

where $\mathbf{c}_\lambda$ is given by Eq. (19).

To summarize, the Tikhonov regularization method solves a constrained minimization problem using a smoothness norm in order to provide a stable solution which fits the data and also has a minimum structure.

5.2 Selection of the optimal regularization parameter

The performance of regularization methods depends crucially on the suitable choice of the regularization parameter. One extensively studied criterion is the discrepancy principle (see e.g. Morozov [46]). Although this criterion is mathematically rigorous, it requires a reliable estimation of the amount of noise added into the data which may not be available in practical problems. Heuristic approaches are preferable in the case when no a priori information about the noise is available. For the Tikhonov zeroth-order regularization method, several heuristical approaches have been proposed, including the L-curve criterion (see Hansen [18]), and the generalized cross-validation (GCV) (see Wahba [56]). In this paper, we employ the GCV criterion to determine the optimal regularization parameter, λ_{opt}, for the Tikhonov zeroth-order regularization method, namely

$$\lambda_{\mathrm{opt}}: \quad \mathcal{G}\left(\lambda_{\mathrm{opt}}\right) = \min_{\lambda>0} \mathcal{G}\left(\lambda\right). \tag{22}$$

Here

$$\mathcal{G}(\cdot) : (0, \infty) \longrightarrow [0, \infty),$$
$$\mathcal{G}(\lambda) = \frac{\left\| \mathbf{A}\,\mathbf{c}_\lambda - \mathbf{b}^\epsilon \right\|^2}{\left[\mathrm{trace}\left(\mathbf{I}_N - \mathbf{A}\,\mathbf{A}^\dagger\right)\right]^2}, \tag{23}$$

where $\mathbf{c}_\lambda$ is given by Eq. (19) with $\mathbf{b} = \mathbf{b}^\epsilon$.

6 Numerical results and discussion

In this section, we present the performance of the proposed numerical method, namely the alternating iterative MFS described in Sects. 3 and 4. To do so, we solve numerically the Cauchy geometric problem given by Eqs. (1)–(2) for the two-dimensional modified Helmholtz equation in the geometries described below.

6.1 Examples

For the examples investigated in this paper, we consider the following analytical solution for the temperature

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x}) = \exp\left(\mathrm{a}_1\, \mathrm{x}_1 + \mathrm{a}_2\, \mathrm{x}_2\right), \ \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \overline{\Omega}, \tag{24a}$$

and the corresponding analytical normal heat flux

$$\mathrm{q}^{(\mathrm{an})}(\mathbf{x}) = \left[\mathrm{a}_1\, \mathrm{n}_1(\mathbf{x}) + \mathrm{a}_2\, \mathrm{n}_2(\mathbf{x})\right] \mathrm{u}^{(\mathrm{an})}(\mathbf{x}), \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \partial\Omega. \tag{24b}$$

The geometries considered herein and constants involved in the analytical solutions (24a) and (24b) are given by:

Example 1 (Simply connected domain with a smooth boundary) We consider the unit disk

$$\Omega = \left\{ \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathbb{R}^2 \,\middle|\, \rho(\mathbf{x}) < \mathrm{r} \right\},$$

where $\rho(\mathbf{x}) = \sqrt{\mathrm{x}_1^2 + \mathrm{x}_2^2}$ is the radial polar coordinate of $\mathbf{x}$ and $\mathrm{r} = 1.0$. Here $\kappa = 1.0$, $\mathrm{a}_1 = 0.5$, $\mathrm{a}_2 = \sqrt{\kappa^2 - \mathrm{a}_1^2} = \sqrt{3}/2$, $\Gamma_1 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, \pi/2 \le \theta(\mathbf{x}) \le 2\pi\right\}$ and $\Gamma_2 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, 0 < \theta(\mathbf{x}) < \pi/2\right\}$, where $\theta(\mathbf{x})$ is the angular polar coordinate of $\mathbf{x}$.

Example 2 (Doubly connected domain with a smooth boundary) We consider the annular domain

$$\Omega = \left\{ \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathbb{R}^2 \,\middle|\, \mathrm{r}_{\mathrm{int}} < \rho(\mathbf{x}) < \mathrm{r}_{\mathrm{out}} \right\},$$

where $\mathrm{r}_{\mathrm{int}} = 0.5$ and $\mathrm{r}_{\mathrm{out}} = 1.0$. Here $\kappa = 2.0$, $\mathrm{a}_1 = 1.0$, $\mathrm{a}_2 = -\sqrt{\kappa^2 - \mathrm{a}_1^2} = -\sqrt{3}$, $\Gamma_1 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, \rho(\mathbf{x}) = \mathrm{r}_{\mathrm{out}}\right\}$ and $\Gamma_2 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, \rho(\mathbf{x}) = \mathrm{r}_{\mathrm{int}}\right\}$.

The inverse problems investigated in this paper have been solved using the uniform distribution of both the MFS boundary collocation points $\mathbf{x}^{(i)}$, $i = 1, \ldots, N$, and the singularities $\boldsymbol{\xi}^{(j)}$, $j = 1, \ldots, M$. Furthermore, the numbers of MFS boundary collocation points N_1 and N_2 corresponding to the over- and under-specified boundaries Γ_1 and Γ_2, respectively, the number of singularities M and the distance d_{S} between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_{\mathrm{S}}$ on which the singularities are situated, were set to:

(i) $N_1/3 = N_2 = N/4 \in \{10, 20, 30\}$, $M = N$ and $d_{\mathrm{S}} = 3.0$ for Example 1;
(ii) $N_1/2 = N_2 = N/3 \in \{20, 30, 40\}$, $M = N_1 + N_2/2 = 5N/6$, while $d_{\mathrm{S}} = 0.5$ and $d_{\mathrm{S}} = 2.0$ for the inner and outer boundaries, respectively, for Example 2.

6.2 Initial guess

An arbitrary real valued function $\mathrm{u}^{(1)} \in H^{1/2}(\Gamma_2)$ may be specified as an initial guess for the unknown temperature on the under-specified boundary Γ_2. In order to improve the rate of convergence of the iterative algorithm, one may choose a real valued function which ensures the continuity of the boundary temperature at the common endpoints of the over- and under-specified boundaries Γ_1 and Γ_2, respectively, and which is also linear with respect to the angular polar coordinate θ (see e.g. Marin et al. [41]). More precisely, for Example 1 the following initial guess for the unknown temperature on Γ_2 may be chosen:

$$\mathrm{u}^{(1)}(\mathbf{x}) = \frac{\theta(\mathbf{x}^{(2)}) - \theta(\mathbf{x})}{\theta(\mathbf{x}^{(2)}) - \theta(\mathbf{x}^{(1)})}\, \mathrm{u}^{(\mathrm{an})}(\mathbf{x}^{(1)}) + \frac{\theta(\mathbf{x}) - \theta(\mathbf{x}^{(1)})}{\theta(\mathbf{x}^{(2)}) - \theta(\mathbf{x}^{(1)})}\, \mathrm{u}^{(\mathrm{an})}(\mathbf{x}^{(2)}), \quad \mathbf{x} \in \Gamma_2, \tag{25}$$

where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are the common endpoints of the over- and under-specified boundaries, i.e. $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \left\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right\}$. However, in the general situation when the over- and under-specified boundaries have no common points, as is the case of Example 2, one cannot use the procedure described above. Therefore, in this case, the initial guess for the unknown temperature on the under-specified boundary Γ_2 is chosen as

$$\mathrm{u}^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_2. \tag{26}$$

In this study, we have decided to use the initial guess (26). In this way, the most general situations regarding the geometry of the solution domain are accounted for and the robustness of the alternating iterative algorithm with respect to the initial guess for the unknown temperature on Γ_2 is also tested.

6.3 Convergence of the algorithm

If N_i MFS collocation points, $\{\mathbf{x}^{(\ell)}\}_{\ell=1}^{N_i}$, are considered on the boundary $\Gamma_i \subset \partial\Omega$ then the *root mean square error* (RMS error) associated with the real valued function $f(\cdot) : \Gamma_i \longrightarrow \mathbb{R}$ on Γ_i is defined by

$$\mathrm{RMS}_{\Gamma_i}(f) = \sqrt{\frac{1}{N_i}\sum_{\ell=1}^{N_i} f(\mathbf{x}^{(\ell)})^2}, \tag{27}$$

In order to investigate the convergence of the algorithm, at every iteration, $k \geq 1$, we evaluate the following accuracy errors corresponding to the temperature and normal heat flux on the under-specified boundary, Γ_2, which are defined as *relative RMS errors*, i.e.

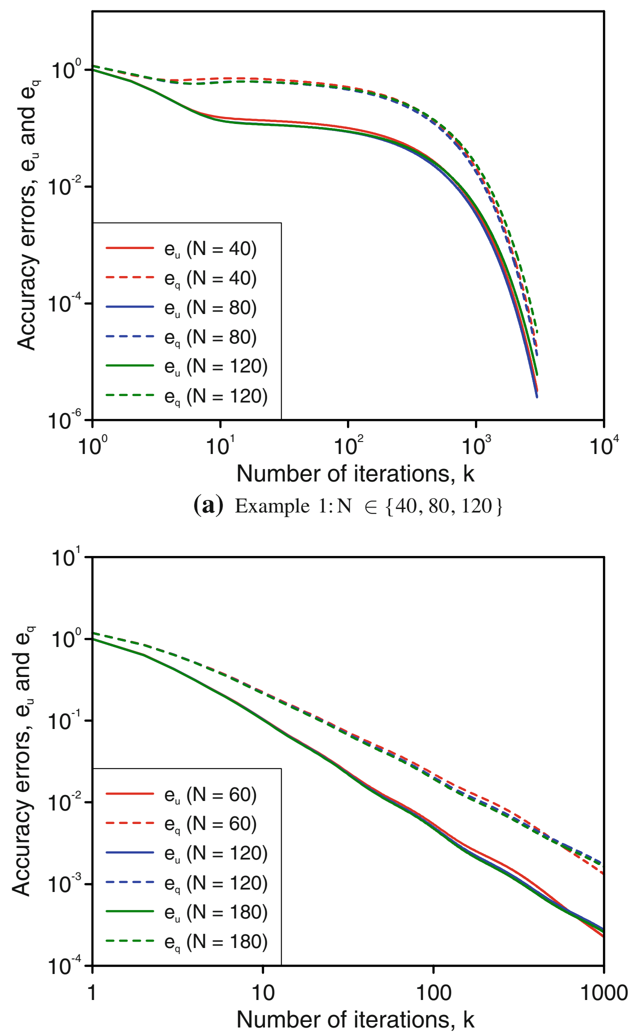
$$\mathrm{e_u}(k) = \frac{\mathrm{RMS}_{\Gamma_2}(\mathrm{u}^{(2k-1)} - \mathrm{u}^{(\mathrm{an})})}{\mathrm{RMS}_{\Gamma_2}(\mathrm{u}^{(\mathrm{an})})} = \frac{\|\mathrm{u}^{(2k-1)}|_{\Gamma_2} - \mathrm{u}^{(\mathrm{an})}|_{\Gamma_2}\|_2}{\|\mathrm{u}^{(\mathrm{an})}|_{\Gamma_2}\|_2}, \quad k \geq 1, \tag{28a}$$

and

$$\mathrm{e_q}(k) = \frac{\mathrm{RMS}_{\Gamma_2}(\mathrm{q}^{(2k)} - \mathrm{q}^{(\mathrm{an})})}{\mathrm{RMS}_{\Gamma_2}(\mathrm{q}^{(\mathrm{an})})} = \frac{\|\mathrm{q}^{(2k)}|_{\Gamma_2} - \mathrm{q}^{(\mathrm{an})}|_{\Gamma_2}\|_2}{\|\mathrm{q}^{(\mathrm{an})}|_{\Gamma_2}\|_2}, \quad k \geq 1, \tag{28b}$$

where $\mathrm{u}^{(2k-1)}$ and $\mathrm{q}^{(2k)}$ are the temperature and normal heat flux on the boundary Γ_2 retrieved after k iterations by solving the well-posed, mixed, direct, boundary value problems (3a)–(3c) and (4a)–(4c), respectively. The error in predicting the temperature inside the solution domain, Ω, may also be evaluated, but it has an evolution similar to that of the errors $\mathrm{e_u}$ and $\mathrm{e_q}$ given by Eqs. (28a) and (28b), respectively, and hence this is not pursued herein.

Figure 1a and b display the accuracy errors $\mathrm{e_u}$ and $\mathrm{e_q}$ as functions of the number of iterations, k, obtained using exact Cauchy data on the over-specified boundary, Γ_1, and various numbers of MFS collocation points, for the inverse problems given by Examples 1 and 2, respectively. It can be seen from these figures that both errors $\mathrm{e_u}$ and $\mathrm{e_q}$ decrease even after a large numbers of iterations, e.g. $k = 3000$ and $k = 1000$ for Examples 1 and 2, respectively, and as expected $\mathrm{e_u} < \mathrm{e_q}$ for all MFS discretisations employed, i.e. normal heat fluxes are more inaccurate than temperatures. Furthermore, as N increases, the errors $\mathrm{e_u}$ and $\mathrm{e_q}$ decrease showing that $N \geq 80$ for Example 1 and $N \geq 120$ in the case of Example 2 ensure a sufficient discretisation for the accuracy to be achieved.

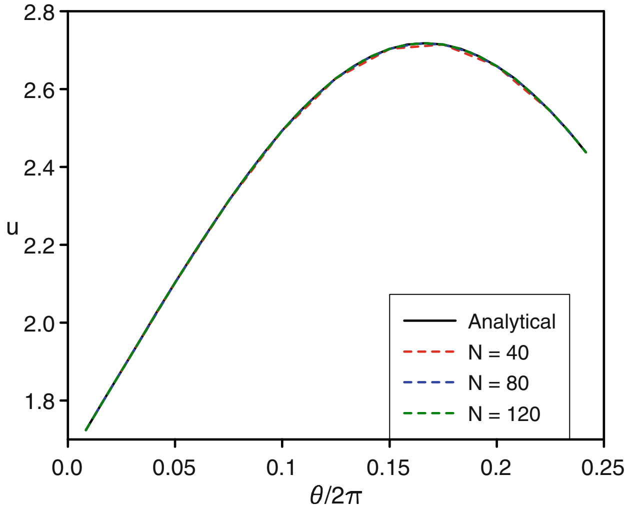


(a) Example 1: N $\in \{40, 80, 120\}$

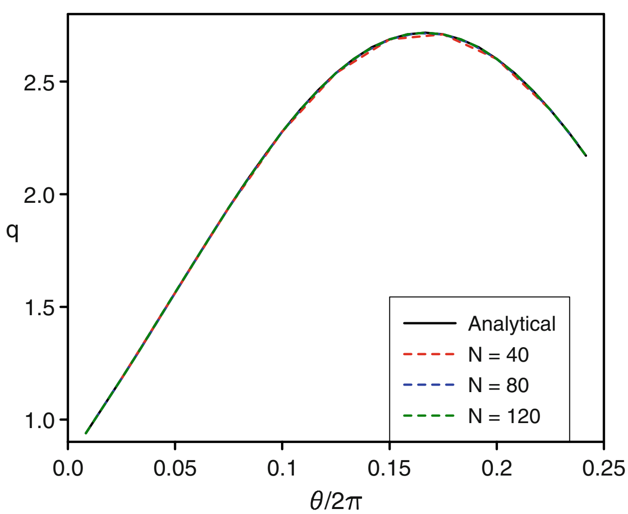
(b) Example 2: N $\in \{60, 120, 180\}$

Fig. 1 The accuracy errors, $\mathrm{e_u}$ and $\mathrm{e_q}$, as functions of the number of iterations, k, obtained using exact Cauchy data on Γ_1 and various numbers of MFS boundary collocation points, for **a** Example 1, and **b** Example 2

The analytical and numerical solutions for the temperature $\mathrm{u}|_{\Gamma_2}$ and the normal heat flux $\mathrm{q}|_{\Gamma_2}$ obtained with exact Cauchy data after $k = 3000$ iterations, for the Cauchy problem given by Example 1 are presented in Fig. 2a and b, respectively. Fig. 3a and b illustrate the analytical and numerical values for the temperature $\mathrm{u}|_{\Gamma_2}$ and the normal heat flux $\mathrm{q}|_{\Gamma_2}$, respectively, retrieved with exact Cauchy data after $k = 1000$ iterations, in the case of Example 2. From these figures, it can be seen that the accuracy in predicting both the temperature distribution and normal heat flux on the boundary Γ_2

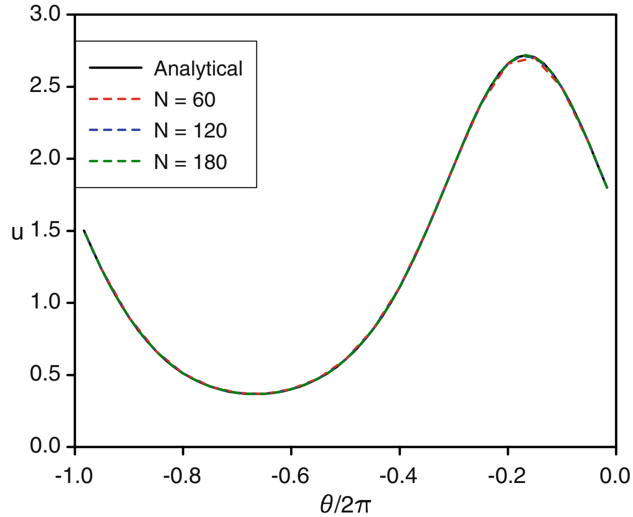


(a) Example 1: Temperatures on Γ_2

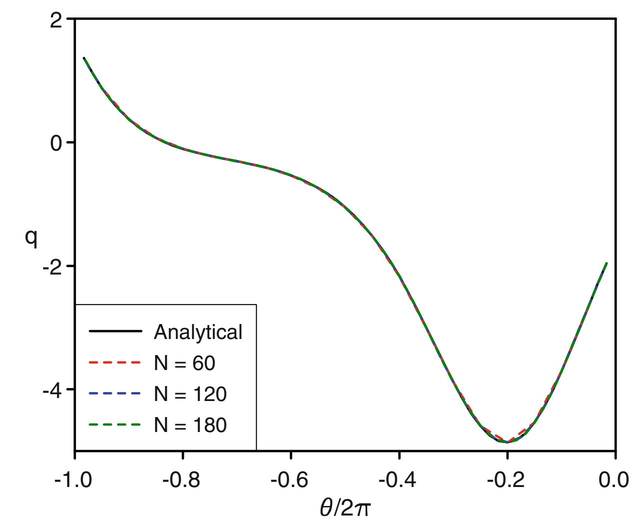


(b) Example 1: Normal heat fluxes on Γ_2

Fig. 2 The analytical and numerical **a** temperatures u, and **b** normal heat fluxes q, on the under-specified boundary Γ_2, obtained using exact Cauchy data on Γ_1, $k = 300$ iterations and various numbers of MFS boundary collocation points, namely $N \in \{40, 80, 120\}$, for Example 1



(a) Example 2: Temperatures on Γ_2



(b) Example 2: Normal heat fluxes on Γ_2

Fig. 3 The analytical and numerical **a** temperatures u, and **b** normal heat fluxes q, on the under-specified boundary Γ_2, obtained using exact Cauchy data on Γ_1, $k = 300$ iterations and various numbers of MFS boundary collocation points, namely $N \in \{60, 120, 180\}$, for Example 2

is very good. As expected, the errors in predicting the normal heat flux $q|_{\Gamma_2}$ are larger than the errors in predicting the temperature $u|_{\Gamma_2}$ since the normal heat flux contains higher-order derivatives of the latter.

From Figs. 1, 2, 3, it can be concluded that the MFS-based alternating iterative algorithm described in Sects. 3 and 4 produces an accurate and convergent numerical solution for both the missing boundary temperature and normal heat flux with respect to increasing the number of iterations, k, and the number of MFS boundary collocation points, N, provided that exact input Cauchy data are used. However, exact data are seldom available in practice since measurement errors always include noise in the prescribed boundary conditions and this is investigated next.

6.4 Stopping criterion

Once the convergence, with respect to increasing the number of iterations, k, and the number of MFS boundary collocation points, N, of the numerical solution to the exact solution has been established, we fix $N = 80$ and $N = 120$ for Examples 1 and 2, respectively, and investigate the stability of the numerical solution. In what follows, the temperature, $u|_{\Gamma_1} = u^{(an)}|_{\Gamma_1}$, and/or the normal heat flux, $q|_{\Gamma_1} = q^{(an)}|_{\Gamma_1}$, on the over-specified boundary have been perturbed as

$$\widetilde{u}^{\epsilon}|_{\Gamma_1} = u|_{\Gamma_1} + \delta u, \quad \delta u = \texttt{G05DDF}(0, \sigma_u),$$
$$\sigma_u = \max_{\Gamma_1} |u| \times (p_u/100), \tag{29}$$

and

$$\widetilde{q}^{\epsilon}|_{\Gamma_1} = q|_{\Gamma_1} + \delta q, \quad \delta q = \texttt{G05DDF}(0, \sigma_q),$$
$$\sigma_q = \max_{\Gamma_1} |q| \times \left(p_q/100\right), \tag{30}$$

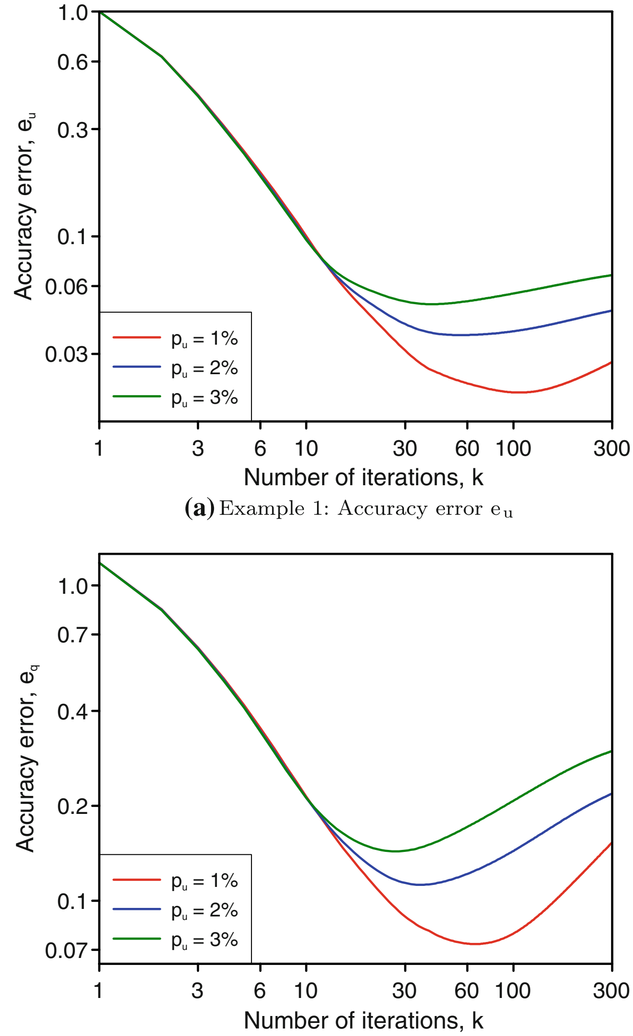
respectively. Here δu and δq are Gaussian random variables with mean zero and standard deviations σ_u and σ_q, respectively, generated by the NAG subroutine G05DDF [47], while $p_u\%$ and $p_q\%$ are the percentages of additive noise included into the input boundary temperature, $u\big|_{\Gamma_1}$, and normal heat flux, $q\big|_{\Gamma_1}$, respectively, in order to simulate the inherent measurement errors.

Figure 4a and b present the accuracy errors e_u and e_q, respectively, for various levels of Gaussian random noise $p_u \in \{1\%, 2\%, 3\%\}$ added into the temperature data $u\big|_{\Gamma_1}$. From these figures it can be seen that as p_u decreases then e_u and e_q decrease. However, the errors in predicting the temperature and the normal heat flux on the under-specified boundary Γ_2 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion must be used in order to terminate the iterative process at the point where the errors in the numerical solutions start increasing.

After each iteration, k, we evaluate the following convergence error which is associated with the temperature on the over-specified boundary, Γ_1, namely

$$\begin{aligned} E_u(k) &= \frac{\mathrm{RMS}_{\Gamma_1}(u^{(2k)} - \widetilde{u}^{\epsilon})}{\mathrm{RMS}_{\Gamma_1}(\widetilde{u}^{\epsilon})} \\ &= \frac{\| u^{(2k)}|_{\Gamma_1} - \widetilde{u}^{\epsilon}|_{\Gamma_1} \|_2}{\| \widetilde{u}^{\epsilon}|_{\Gamma_1} \|_2}, \quad k \geq 1, \end{aligned} \tag{31}$$

where $u^{(2k)}$ is the temperature on the over-specified boundary Γ_1 retrieved numerically after k iterations by solving the well-posed, mixed, direct, boundary value problem (4a)–(4c). This error E_u should tend to zero as the sequences $\left\{u^{(2k-1)}\right\}_{k\geq 1}$ and $\left\{u^{(2k)}\right\}_{k\geq 1}$ tend to the analytical solution, $u^{(an)}$, in the space $H^1(\Omega)$ and hence they are expected to provide an appropriate stopping criterion. If we investigate the error E_u obtained at every iteration for Example 2 for various levels of Gaussian random noise added into the input temperature data $u\big|_{\Gamma_1}$, we obtain the curves graphically represented in Fig. 5. By comparing Figs. 4 and 5, it can be noticed that the convergence error E_u reaches a plateau region at around the



(a) Example 1: Accuracy error e_u

(b) Example 1: Accuracy error e_q

Fig. 4 The accuracy errors **a** e_u, and **b** e_q, as function of the number of iterations, k, obtained using $N = 120$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 2

same number iterations as the number of iterations required for the accuracy errors e_u and e_q attain their corresponding minimum. Therefore, for noisy Cauchy data a natural stopping criterion ceases the MFS alternating iterative algorithm at the optimal number of iterations, k_{opt}, at which the plateau region is reached by the convergence error, E_u.

As mentioned in the previous section, for exact data the iterative process is convergent with respect to increasing the number of iterations, k, since the accuracy errors e_u and e_q keep decreasing even after a large number of iterations, see Figs. 1a and b. It should be noted in this case that a stopping criterion is not necessary since the numerical solution is convergent with respect to increasing the number of iterations. Nonetheless, even in this case the errors E_u, e_u and e_q have a similar behaviour and the error E_u may be used to stop the

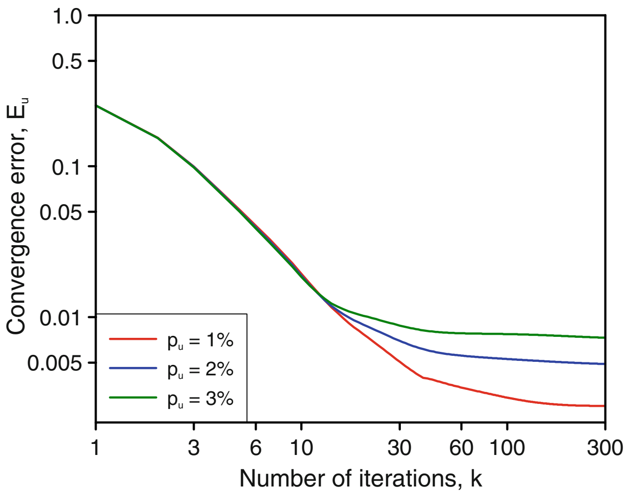


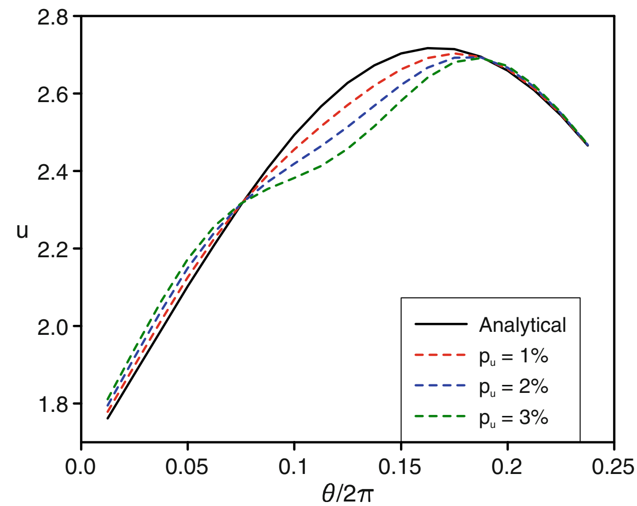
Fig. 5 The convergence error, E_u, as a function of the number of iterations, k, obtained using $N = 120$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 2

iterative process at the point where the rate of convergence is very small and no substantial improvement in the numerical solution is obtained even if the iterative process is continued. Therefore, it can be concluded that the regularizing stopping criterion proposed is very efficient in locating the point where the errors start increasing and the iterative process should be ceased.

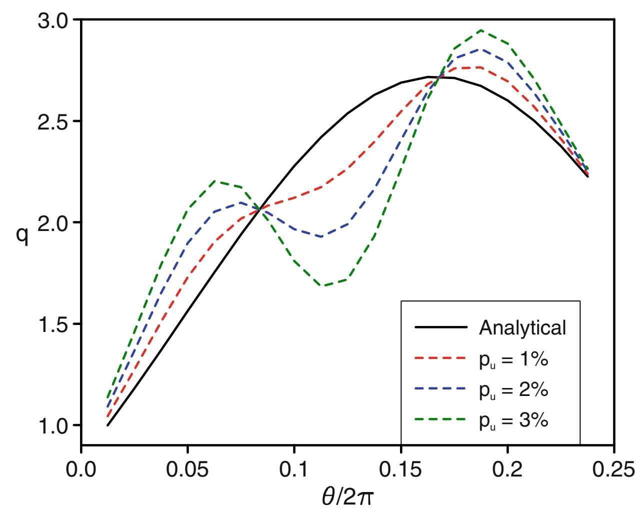
6.5 Stability of the algorithm

Based on the stopping criterion described in Sect. 6.4, the analytical and numerical values for the temperature, u, and normal heat flux, q, on the under-specified boundary Γ_2, obtained using various levels of noise added into the temperature data on the over-specified boundary Γ_1 for Example 1, are illustrated in Fig. 6a and b, respectively. From Fig. 6a it can be seen that the accuracy in predicting the missing boundary temperature, $u|_{\Gamma_2}$, is reasonable and the numerical solution converges to the exact solution as the level of noise, p_u, added into the input Dirichlet data decreases. However, the numerical solutions obtained for the unknown normal heat flux on the under-specified boundary Γ_2 are very poor approximations for their exact values, as can be seen from Fig. 6b, at the same time exhibiting an oscillatory behaviour. The reason for this is that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 \neq \emptyset$ and it is well known that the gradient of the temperature possesses singularities at the points where the data changes from temperature boundary conditions to normal heat flux boundary conditions, even if the temperature and the flux data are of class $\mathcal{C}^\infty$.

The proposed MFS-alternating iterative algorithm, in conjunction with the stopping criterion introduced in the previous section, works very well for the Cauchy problem



(a) Example 1: Temperatures on Γ_2

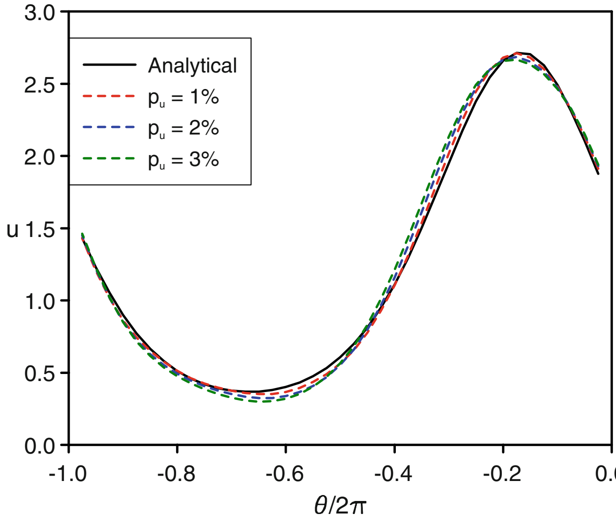


(b) Example 1: Normal heat fluxes on Γ_2

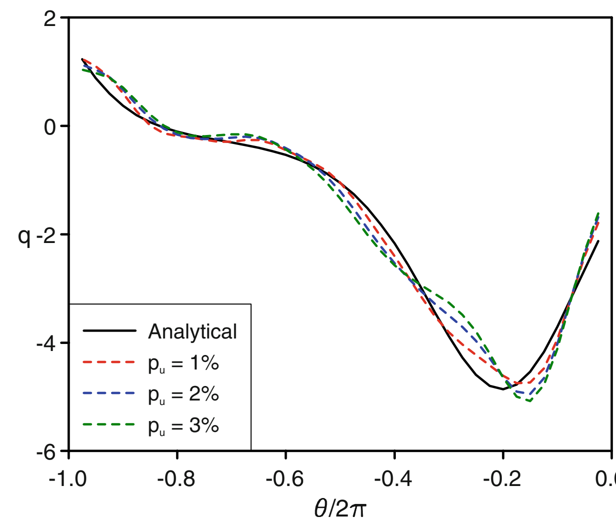
Fig. 6 The analytical and numerical **a** temperatures u, and **b** normal heat fluxes q, on the under-specified boundary Γ_2, obtained using $N = 116$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 1

associated with the modified Helmholtz equation in a doubly connected domain with a smooth boundary, such as the annulus investigated in Example 2. Figure. 7a and b show the numerical results for the temperature and normal heat flux on the boundary Γ_2, obtained using the stopping criterion introduced in Sect. 6.4, $M = N = 80$ and various amounts of noise added into the Dirichlet data, namely $p_u \in \{1\%, 2\%, 3\%\}$, in comparison with their corresponding analytical values, in the case of Example 2.

In the case of Example 2, very good results have also been retrieved for both the unknown temperature, $u|_{\Gamma_2}$, and normal heat flux, $q|_{\Gamma_2}$, when using the stopping criterion

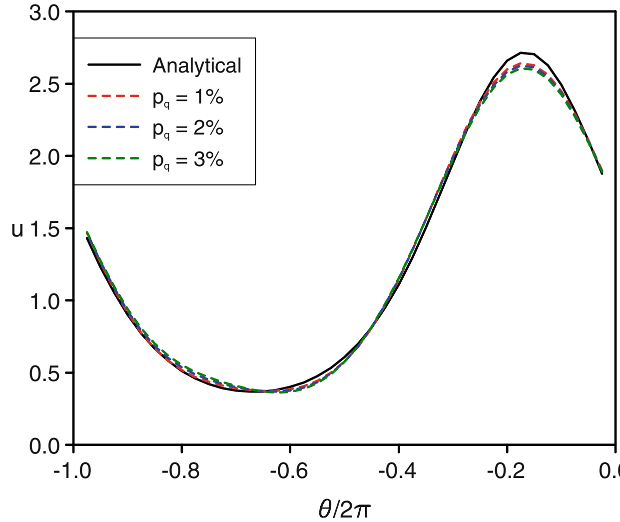


(a) Example 2: Temperatures on Γ_2

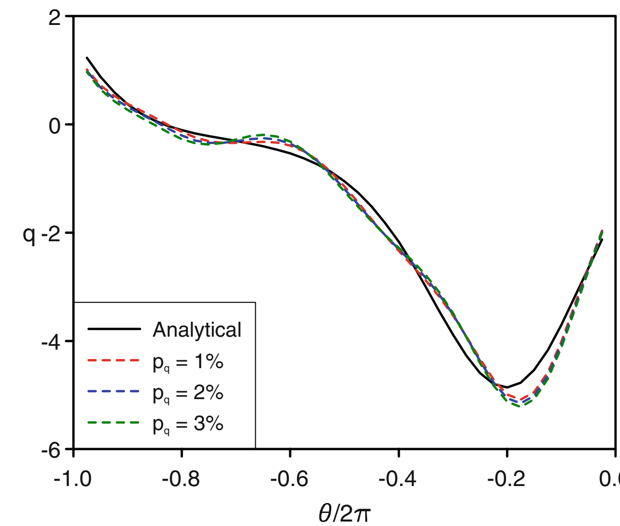


(b) Example 2: Normal heat fluxes on Γ_2

Fig. 7 The analytical and numerical **a** temperatures u, and **b** normal heat fluxes q, on the under-specified boundary Γ_2, obtained using $N = 80$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 2



(a) Example 2: Temperatures on Γ_2



(b) Example 2: Normal heat fluxes on Γ_2

Fig. 8 The analytical and numerical **a** temperatures u, and **b** normal heat fluxes q, on the under-specified boundary Γ_2, obtained using $N = 100$ MFS boundary collocation points and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%, 3\%, 5\%\}$, for Example 2

described in Sect. 6.4, $M = N = 100$ and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%, 2\%, 3\%\}$, and these are presented in Fig. 8a and b, respectively. By comparing Figs. 7 and 8 we can conclude that, as expected, the numerical results obtained using the proposed MFS alternating iterative algorithm, in conjunction with the aforementioned stopping criterion, are more sensitive to perturbations in the normal heat flux on the over-specified boundary than to noisy boundary temperature on Γ_1.

From the numerical results presented in this section, it can be concluded that the stopping criterion developed in Sect. 6.4 has a regularizing effect and the numerical solution obtained by the iterative MFS described in this paper is convergent and stable with respect to increasing the number of MFS boundary collocation points and decreasing the level of noise added into the Cauchy input data, respectively.

7 Conclusions

In this paper, the alternating iterative algorithm of Kozlov et al. [27] was implemented, for the Cauchy problem associated with the two-dimensional modified Helmholtz equation, using a meshless method. The two mixed, well-posed and direct problems corresponding to every iteration of the numerical procedure were solved using the MFS, in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter was selected according to the GCV criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, was also presented. The MFS-based iterative algorithm was tested for Cauchy problems associated with the modified Helmholtz operator in two-dimensional simply and doubly connected domains with smooth boundaries.

From the numerical results presented in this study, it can be concluded that the proposed method is consistent, accurate, convergent with respect to increasing the number of MFS boundary collocation points and stable with respect to decreasing the amount of noise added into the Cauchy data. It should be mentioned that, for a simply connected domain, although the missing boundary temperatures are predicted within a reasonable accuracy, the numerical results obtained using the present algorithm for the unknown normal heat flux are very poor approximations for their exact values. One possible disadvantage of the MFS-based iterative algorithm is related to the optimal choice of the regularization parameter associated with the Tikhonov regularization method which requires, at each step of the alternating iterative algorithm of Kozlov et al. [27], additional iterations with respect to the regularization parameter. However, this inconvenience can be overcome by introducing relaxation procedures in the MFS iterative algorithm and this is currently being under investigation.

The implementation of the MFS in an iterative manner can be extended to the alternating iterative method of Kozlov et al. [27] for two-dimensional Cauchy problems associated with elliptic partial differential operators whose fundamental solutions are available, such as the Navier-Lamé system of linear elasticity and anisotropic heat conduction, as well as other iterative algorithms and similar three-dimensional inverse problems, but these are deferred as future work.

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