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# Evaluation of the degenerate scale for BIE in plane elasticity and antiplane elasticity by using conformal mapping

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ABSTRACT

For a better understanding for the formulation of the degenerate scale problem by using the complex variable, preliminary knowledge is introduced. Formulation for the degenerate scale problem is based on the direct usage of the complex variable and the conformal mapping. After using the conformal mapping, the vanishing displacement condition is assumed on the boundary of unit circle. The complex potentials on the mapping plane are sought in a form of superposition of the principal part and the complementary part. The principal part of the complex potentials is given beforehand, and the complementary part plays a role for compensating the displacement along the boundary from the principal part. After using the appropriate complex potentials, the boundary displacement becomes one term with the form of $g(R)-c$ ($g(R)$ a function of $R$), where $R$ denotes a length parameter. By letting the vanishing displacement on the boundary, or $g(R)-c=0$, the degenerate scale "$R$" is obtained. For four cases, the elliptic contour, the triangle contour, the square contour and the ellipse-like contour, the degenerate scales are evaluated in a closed form. For the case of antiplane elasticity, similar degenerate scale problems are solved.



## 1. Introduction

The boundary integral equation (abbreviated as BIE) was widely used in elasticity, and the fundamental for BIE could be found from Refs. [1–3]. Heritage and early history of the boundary element method was summarized more recently [4]. However, some difficult points for the BIE remain.

The degenerate scale problem in BIE is a particular boundary value problem in elasticity as well as in Laplace equation. It is known that the BIE in plane elasticity is generally formulated on the usage of the Somigliana's identity [3]. When the Dirichlet problem is formulated for the exterior domain, a BIE is obtained. In the Dirichlet problem for the exterior domain, the boundary displacements are the input data and the boundary tractions are the investigated arguments. If the assumed boundary displacements vanish, the BIE becomes a homogenous equation, or the right hand term in the equation is equal to zero. Such formulated homogenous BIE has been shown to yield a non-trivial solution for boundary tractions (or $\sigma_{ij}\neq 0$ at the boundary points), when the adopted configuration is equal to a degenerate scale. Since the displacement and stress condition at infinity, for example $u_i=O(\ln R)$, $\sigma_{ij}=O(1/R)$, or $u_i=O(1/R)$, $\sigma_{ij}=O(1/R^2)$, has not been defined exactly, it is expected to have a non-trivial solution from the viewpoint of mathematics and mechanics. This point will be discussed in detail later. Alternatively speaking, the relevant non-homogenous BIE has a non-unique solution when the degenerate scale is reached. From the viewpoint of engineering, the non-unique solution is an illogical one. Therefore, one must avoid meeting illogical solution caused by occurrence of the degenerate scale. Simply because the kernel from the displacement of fundamental solution does not depend on the normal of the boundary, the relevant homogenous equation for the Dirichlet problem for the interior domain must be the same style as that for the exterior domain. For the Laplace equation, the similar degenerate scale problem also exists.

The boundary integral equation is used for a ring region with the vanishing displacements along the boundary. In some particular scale of the configuration, the corresponding homogenous equation has non-trivial solution for the boundary tractions [5,6]. In fact, if the displacements are vanishing at the boundary of ring region, the stresses must also be equal to zero. Therefore, the obtained result seriously violates the basic property of elasticity.

Numerical technique for evaluating the degenerate scale was also suggested by using two sets of some particular solution in the geometry of normal scale [7].

Mathematical analysis of the degenerate scale problems for an elliptic-domain problem in elasticity was presented [8]. The analysis depends on the usage of the Airy stress function,

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which in turn is expressed through two complex potentials. By using the complex variable and the elliptic coordinate, a closed form solution for the degenerate scale was obtained. Both components of the displacement vanish for the boundary distribution when the degenerate scale occurs [8].

It is proved that there are either two single critical values or one double critical value for any domain boundary. For two circle holes in an infinite plate, degenerate scale was also studied [9].

Numerical procedure was developed to evaluate the degenerate scale directly from the zero value of a determinant. In the time of evaluating the eigenvalue, or the degenerate scale, the eigenfunction was also obtained as well [10,11]. In those papers, the influence for the used scale to the obtained solution is examined numerically. It is assumed that $\lambda$ is the degenerate scale (or critical value) for size $a$. If the input data for the displacements are from an exact solution in closed form, and one chooses the $a=\lambda-\varepsilon$, $a=\lambda$ or $a=\lambda+\varepsilon$ ($\varepsilon=0.00002$ a small value), the obtained tractions along the boundary are seriously deviated from exact solution in closed form. In addition, if one chooses $a=0.9\lambda$ or $a=1.1\lambda$, small deviation in the computation is found.

It is found in this paper that, if the conformal mapping function for a notch configuration is known beforehand, the relevant degenerate scale (or the eigenvalue, or the critical value) and the solution for displacements and stresses (or the eigenfunction) can be obtained in closed form. Clearly, the present study does not rely on the numerical computation.

It was pointed out that the influence matrix of the weakly singular kernel (logarithmic kernel) might be singular for the Dirichlet problem of Laplace equation when the geometry is reaching some special scale [12]. In this case, one may obtain a non-unique solution. The non-unique solution is not physically realizable. From the viewpoint of linear algebra, the problem also originates from rank deficiency in the influence matrices [12]. By using a particular solution in the normal scale, the degenerate scale can be evaluated numerically [12]. The degenerate scale problem in BIE for the two-dimensional Laplace equation was studied by using degenerate kernels and Fourier Series [13]. In the circular domain case, the degenerate scale problem in BIE for the two-dimensional Laplace equation was studied by using degenerate kernels and circulants, where the circulants mean the influence matrices for the discrete system had a particular character [14].

This paper evaluates the degenerate scale for BIE in plane elasticity and antiplane elasticity by using the complex variable and the conformal mapping. Basic equations in the complex variable method of plane elasticity are compactly addressed. From the formulation of an exterior problem in plane elasticity, the background for existence of the degenerate scale is discussed in detail.

After using the conformal mapping, the vanishing displacement condition is assumed on the boundary of unit circle. The complex potentials on the mapping plane are sought in a form of superposition of the principal part and the complementary part. It is found from detail derivation that there are some complex potentials that satisfy the vanishing displacement condition along the boundary when the degenerate scale is reached. For four cases, the elliptic contour, the triangle contour, the square contour and the ellipse-like contour, the degenerate scales are evaluated. For the case of antiplane elasticity, similar degenerate scale problems are solved.

## 2. Preliminary knowledge

Basic equations in the complex variable method of plane elasticity are compactly addressed. From the formulation of an exterior problem in plane elasticity, the background for existence of the degenerate scale is discussed in detail. Formulation of the degenerate scale problem based on BIE is also introduced. Emphasis is on the usage of a new suggested kernel. Comparison between two techniques, from the usage of complex variable and the BIE, is stated in brief.

### 2.1. *The representation form of the displacement–stress field in plane elasticity*

The following analysis depends on the complex variable function method in plane elasticity [15]. In the method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces $(X, Y)$ and the displacements $(u, v)$ are expressed in terms of two complex potentials $\phi_1(z)$ and $\psi_1(z)$ such that

$$\sigma_x+\sigma_y=4\,Re\,\phi_1'(z),$$
$$\sigma_y-\sigma_x+2i\sigma_{xy}=2[\bar{z}\phi_1''(z)+\psi_1'(z)], \tag{1}$$

$$f=-Y+iX=\phi_1(z)+z\overline{\phi_1'(z)}+\overline{\psi_1(z)}, \tag{2}$$

$$2G(u+iv)=\kappa\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)}, \tag{3}$$

where $z=x+iy$ denotes complex variable, $G$ is the shear modulus of elasticity, $\kappa=(3-\nu)/(1+\nu)$ is for the plane stress problems, $\kappa=3-4\nu$ is for the plane strain problems, and $\nu$ is the Poisson's ratio. In the present study, the plane strain condition is assumed thoroughly. In the following, we occasionally rewrite the displacements "$u$", "$v$" as $u_1$, $u_2$, $\sigma_x$, $\sigma_y$, $\sigma_{xy}$ as $\sigma_{11}$, $\sigma_{22}$, $\sigma_{12}$, and "$x$", "$y$" as $x_1$, $x_2$, respectively.

For any notch, for example, the elliptic notch in an infinite plate with resultant forces $P_x$ and $P_y$ applied on the contour and vanishing remote tractions (Fig. 1(a)), the relevant complex potentials can be expressed in the following form [15]:

$$\phi_1(z)=A_2\ \ln z+a_0+\sum_{k=1}^{\infty}\frac{a_k}{z^k},$$
$$\psi_1(z)=B_2\ \ln z+b_0+\sum_{k=1}^{\infty}\frac{b_k}{z^k}, \tag{4}$$

where

$$B_2=-\kappa\bar{A}_2,\quad A_2=-\frac{P_x+iP_y}{2\pi(\kappa+1)}. \tag{5}$$

The coefficients $a_k$ and $b_k$ ($k=1, 2, \ldots$) cane be evaluated from the solution of the boundary value problem, and two constants $a_0$ and $b_0$ represent the rigid translation of the body. Since two constants $a_0$ and $b_0$ are involved in Eq. (4), the complex potentials shown by Eq. (4) are expressed in an impure deformable form. Simply deleting two constants $a_0$ and $b_0$ in Eq. (4), we have:

$$\phi_1(z)=A_2\ \ln z+\sum_{k=1}^{\infty}\frac{a_k}{z^k},\quad \psi_1(z)=B_2\ \ln z+\sum_{k=1}^{\infty}\frac{b_k}{z^k}. \tag{6}$$

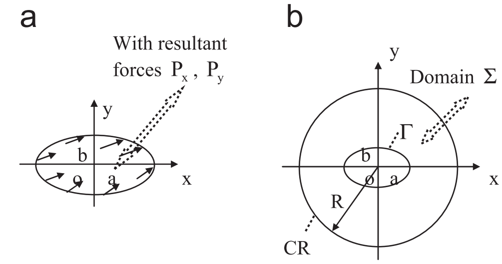


**Fig. 1.** (a) An elliptic hole in infinite plate with non-equilibrated loadings on the contour. (b) A ring-shaped domain $\Sigma$ bounded by an elliptic contour $\Gamma$ and a large circle "CR".

The complex potentials shown by Eq. (6) are expressed in a pure deformable form.

Assume that the two complex potentials are expressed in the form:

$$\phi_1(z) = \sum_{j=1}^{M} \alpha_j F_j(z), \quad \psi_1(z) = \sum_{j=1}^{M} \beta_j G_j(z)$$

where $\alpha_j$, $\beta_j$ real coefficients. (7)

If all the pairs $\phi_1(z) = F_j(z)$, $\psi_1(z) = 0$ and $\phi_1(z) = 0$, $\psi_1(z) = G_j(z)$ ($j = 1, 2, \ldots, M$) cause non-vanishing stress anywhere excluded some insulated singular points, the complex potentials are defined as the pure deformable form. Therefore, the pair $\phi_1(z) = \mathrm{i}\gamma z + c_1$, $\psi_1(z) = c_2$ (where $\gamma$ denotes real constant, and $c_1$, $c_2$, denote two complex constants) is represented in impure deformable form.

### 2.2. A particular boundary value problem for exterior domain

Let us consider a particular boundary value problem in the region $\Sigma$ bounded by an ellipse $\Gamma$ and a large circle "CR" (Fig. 1(b)). In the problem, the following boundary condition is assumed:

$$2G(u + \mathrm{i}v)|_{\Gamma} = 0. \quad (8)$$

The usage of the Clapeyron's theorem for the bounded region between two boundaries $\Gamma$ and "CR" will yield (Fig. 1(b)):

$$\frac{1}{2}\int_{\Gamma} \sigma_{ij} n_j u_i \,\mathrm{d}s + \frac{1}{2}\int_{\mathrm{CR}} \sigma_{ij} n_j u_i \,\mathrm{d}s = \iint_{\Sigma} w(e_{ij}) \,\mathrm{d}x\,\mathrm{d}y, \quad (9)$$

where $w(e_{ij})$ is the strain density function. Substituting Eq. (8) into (9) yields:

$$\iint_{\Sigma} w(e_{ij}) \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_{CR} \sigma_{ij} n_j u_i \,\mathrm{d}s. \quad (10)$$

In the first case, we assume that the radius of circular boundary "$R$" takes a definite value, for example $R = 1000$. In addition, we assume $u_i = 0$ or $\sigma_{ij} n_j = 0$ along the boundary "CR". In this case, from the mentioned condition on the contour $\Gamma$ and Eq. (10) we have:

$$\iint_{\Sigma} w(e_{ij}) \,\mathrm{d}x\,\mathrm{d}y = 0. \quad (11)$$

Since the strain energy $w(e_{ij})$ always takes a positive value, therefore, the relevant strains must be vanishing, or $e_{ij} = 0$, and the displacements must be a rigid motion. After considering the condition $u_i|_{\Gamma} = 0$ shown by Eq. (8), the displacements must be vanishing ($u_i \equiv 0$) on the entire domain $\Sigma$.

In the second case, we assume that the radius of circular boundary "$R$" is a variable. There are two subclasses for consideration. In the first subclass, we assume that the loadings applied on the elliptic contour are in equilibrium, or $A_2 = 0$ and $B_2 = 0$ in the complex potentials shown by Eq. (6). Therefore, from Eqs. (1) and (3), we have $u_i = \mathrm{O}(1/R)$ (from the expression $u_i = \mathrm{O}(1/z)$) and $\sigma_{ij} = \mathrm{O}(1/R)^2$. Therefore, we have:

$$\lim_{R\to\infty} \frac{1}{2}\int_{\mathrm{CR}} \sigma_{ij} n_j u_i \,\mathrm{d}s = 0. \quad (12)$$

In this case, Eq. (11) and the relevant conclusion are still valid.

In the second subclass, we assume that the loadings applied on the elliptic contour are not in equilibrium, or $A_2 \neq 0$ and $B_2 \neq 0$ in the complex potentials shown by Eq. (6). Therefore, from Eqs. (1), (3) and (6), we have $u_i = \mathrm{O}(\ln R)$ and $\sigma_{ij} = \mathrm{O}(1/R)$, and one generally obtains:

$$\frac{1}{2}\int_{\mathrm{CR}} \sigma_{ij} n_j u_i \,\mathrm{d}s \neq 0, \quad (13)$$

$$\iint_{\Sigma} w(e_{ij}) \,\mathrm{d}x\,\mathrm{d}y \neq 0. \quad (14)$$

Eq. (12) is derived from the fact that $\int_{\mathrm{CR}} \sigma_{ij} n_j u_i \,\mathrm{d}s = \int_0^{2\pi} \mathrm{O}(\ln R)\mathrm{O}(1/R) R \,\mathrm{d}\theta$ is generally not equal to zero when $R \to \infty$. In addition, Eq. (14) reveals that the strain energy $w(e_{ij})$ is not vanishing anywhere, or $w(e_{ij}) \neq 0$ for $(x, y) \in \Sigma$. In addition, $w(e_{ij}) \neq 0$ is equivalent to $u_i \neq 0$ anywhere for $(x, y) \in \Sigma$.

From Eq. (14), we see that it is possible to have a non-vanishing displacements $u_i \neq 0$ for $(x, y) \in \Sigma$ under the boundary condition shown by Eq. (8). Some detail analysis shows that non-vanishing displacements (or $u_i \neq 0$) exist for $(x, y) \in \Sigma$ under the condition shown by Eq. (8) when the scale is reaching some critical value [5–8,12].

Therefore, the degenerate scale problem can be defined as follows. One wants to find a particular scale such that the exterior problem with vanishing displacements on the contour, or $2G(u + \mathrm{i}v)|_{\Gamma} = 0$, has a non-trivial solution for displacements and stresses (or $u_i \neq 0$, $\sigma_{ij} \neq 0$, for $(x, y) \in \Sigma$).

### 2.3. Formulation for the exterior BVP (boundary value problem) and the degenerate scale problem

In order to formulate the BIE (boundary integral equation), the first step is to derive a fundamental solution, which is defined by some concentrated forces applied at the point $z = t$ (Fig. 2(a)). For the fundamental field caused by concentrated force at the point $z = t$, the relevant complex potentials are as follows [15]:

$$\phi(z) = F \ln(z - t), \quad \psi(z) = -\kappa \bar{F} \ln(z - t) - \frac{F\bar{t}}{z - t}, \quad (15)$$

where

$$F = -\frac{P_x + \mathrm{i}P_y}{2\pi(\kappa + 1)}, \quad (16)$$

where $P_x + \mathrm{i}P_y$ is the concentrated force applied at the point $z = t$. Note that the complex potentials shown by Eq. (15) is expressed in a pure deformable form. From the complex potentials shown by Eq. (15), a simple derivation will lead to the kernel $U_{ij}^{*1}(\xi, x)$ used later.

Clearly, in the first step, the point "$t$" is an inner point in the region, then it approaches the boundary point $\xi$. That is to say the point "$t$" has a relation with the boundary point $\xi$. In addition, the point "$z$" has a relation with "$x$" in the kernel $U_{ij}^{*1}(\xi, x)$.

Without losing generality, we can introduce the BIE for the region between the elliptic contour $\Gamma$ and a large circle "CR" (Fig. 2(b)). The observation point $\xi$ is assumed on the elliptic contour $\xi \in \Gamma$. For the plane strain case, the suggested BIE can be

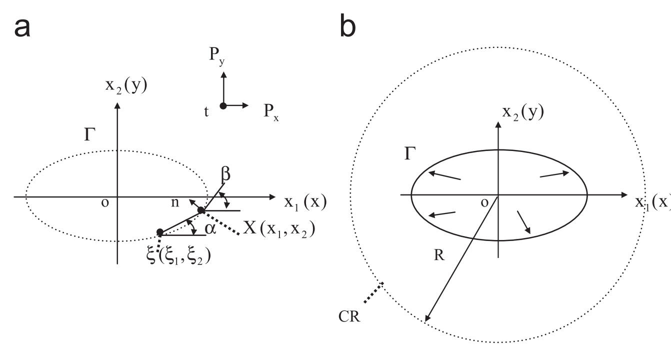


**Fig. 2.** (a) A concentrated force applied at the point $z = t$, or the loading condition for the fundamental field. (b) Some loadings having resultant forces applied on the elliptic contour, or the loading condition for the physical field.

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written as follows:

$$\frac{1}{2}u_i(\xi)+\int_\Gamma P^*_{ij}(\xi,x)u_j(x)\,ds(x) = \int_\Gamma U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x)+D^{*1}_{i(CR)}(\xi),\quad i=1,2,\xi\in\Gamma, \tag{17}$$

where $D^{*1}_{i(CR)}(\xi)$ is a mutual work difference integral (abbreviated as MWDI) on a large circle and is defined by

$$D^{*1}_{i(CR)}(\xi)=-\int_{CR}P^*_{ij}(\xi,x)u_j(x)\,ds(x)+\int_{CR}U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x) \tag{18}$$

and the kernel $P^*_{ij}(\xi,x)$ is defined by [3]

$$P^*_{ij}(\xi,x)=-\frac{1}{4\pi(1-\nu)}\frac{1}{r}\{(r_{,1}n_1+r_{,2}n_2)((1-2\nu)\delta_{ij}+2r_{,i}r_{,j}) +(1-2\nu)(n_ir_{,j}-n_jr_{,i})\}, \tag{19}$$

where Kronecker deltas $\delta_{ij}$ is defined as, $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$, and

$$r_{,1}=\frac{x_1-\xi_1}{r}=\cos\alpha,\quad r_{,2}=\frac{x_2-\xi_2}{r}=\sin\alpha,\quad n_1=-\sin\beta,\quad n_2=\cos\beta, \tag{20}$$

where the angles $\alpha$ and $\beta$ are indicated in Fig. 2(a).

The kernel $P^*_{ij}(\xi,x)$ can be obtained in a usual way [3]. The kernel $U^{*1}_{ij}(\xi,x)$ may be obtained from the fundamental stress field (Fig. 2(a)). From the complex potentials shown by Eq. (15), a simple derivation will result in the following kernel [10]:

$$U^{*1}_{ij}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}-0.5\delta_{ij}\}. \tag{21}$$

It was proved that if the kernel $U^{*1}_{ij}(\xi,x)$ or the complex potentials shown by Eq. (15) is used, we have [10]:

$$D^{*1}_{i(CR)}(\xi)=-\int_{CR}P^*_{ij}(\xi,x)u_j(x)\,ds(x)+\int_{CR}U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x) =0. \tag{22}$$

Therefore, the BIE shown by Eq. (17) can be modified to

$$\frac{1}{2}u_i(\xi)+\int_\Gamma P^*_{ij}(\xi,x)u_j(x)\,ds(x) = \int_\Gamma U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x),\quad i=1,2,\xi\in\Gamma. \tag{23}$$

It is assumed that if the displacements are assumed on the boundary of the elliptic contour, and we want to find the boundary traction acted. This demand will formulate the Dirichlet problem of plane elasticity for exterior region. From Eq. (23), the integral equation for Dirichlet problem can be expressed as follows:

$$\int_\Gamma U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x)=\tilde g_i(\xi)\quad i=1,2,\xi\in\Gamma. \tag{24}$$

In addition, the right hand term in Eq. (24) is defined by

$$\tilde g_i(\xi)=\frac{1}{2}\tilde u_i(\xi)+\int_\Gamma P^*_{ij}(\xi,x)\tilde u_j(x)\,ds(x)\quad i=1,2,\xi\in\Gamma. \tag{25}$$

In Eq. (25), $\tilde u_j(x)$ ($j=1,2$) are the assumed boundary displacement which are given beforehand, and they can be considered as the input data in the problem.

Letting $\tilde g_i(\xi)=0$ in Eq. (24), the following homogenous equation is obtained as follows:

$$\int_\Gamma U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x)=0,\quad i=1,2,\xi\in\Gamma. \tag{26}$$

Note that the condition $\tilde g_i(\xi)=0$ is derived from the condition $u_j(x)=0$ ($x\in\Gamma$). Alternatively speaking, we have assumed that the displacements along the contour are vanishing.

On the basis of Eq. (26), the degenerate scale problem can be formulated as follows. One wants to find a particular scale such that the integral Eq. (26) has a non-trivial solution $p_j(x)\neq 0$ (for $j=1,2$, $x\in\Gamma$). Here, $p_j=0$ ($j=1,2$) is a trivial solution.

After making discretization, Eq. (26) can be rewritten in a matrix form as follows:

$$\mathbf{U}(\mathbf{a})\mathbf{p}=0, \tag{26a}$$

where the vector $\mathbf{p}$ denotes the tractions applied on the boundary, or $p_j$ ($j=1,2$), In Eq. (26a), $\mathbf{U}(\mathbf{a})$ represents the influence matrix in which each element is a function of the size "$a$".

In the numerical solution, one can change the "$a$" value gradually such that the following condition is satisfied [10,11] as follows:

$$\det\mathbf{U}(\mathbf{a})=0. \tag{26b}$$

The solution for "$a$" from Eq. (26b) is denoted by $a=a_d$ which in turn is called the degenerate scale. Therefore, if the real size coincides with the degenerate scale we have a non-trivial solution for $\mathbf{p}$, or $\mathbf{p}\neq 0$, or $p_j\neq 0$ for $j=1,2$.

Clearly, if the condition (26b) is satisfied for $a=a_d$ (here $a_d$ denotes the degenerate scale), the influence matrix $\mathbf{U}(\mathbf{a})$ is rank deficiency [16].

From above-mentioned analysis we see that the following three statements: (1) if $a=a_d$, Eq. (26) has a non-trivial solution for $p_j$ ($j=1,2$); (2) if $a=a_d$, the condition $\det\mathbf{U}(\mathbf{a})=0$ shown by Eq. (26b) is satisfied; and (3) if $a=a_d$, the influence matrix $\mathbf{U}(\mathbf{a})$ is rank deficiency, have the same meaning.

For the elliptic contour case, the problem was solved by using elliptic coordinates [8]. In addition, several numerical techniques were suggested to solve this problem [7,10–12].

On the other hand, one can also expresses the complex potentials for the fundamental solution as follows:

$$\phi(z)=F\ln(z-t),\quad \psi(z)=-\kappa\bar F\ln(z-t)-\frac{F\bar t}{z-t}+\bar F. \tag{27}$$

Note that, the complex potentials shown by Eq. (27) are expressed in an impure deformable form because of the involved constant $\bar F$. From Eq. (27), a simple derivation will result in the following kernel [3]:

$$U^{*2}_{ij}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}\}, \tag{28}$$

where the kernel $U^{*2}_{ij}(\xi,x)$ is generally used in available text book [3].

Similarly, based on the kernel $U^{*2}_{ij}(\xi,x)$, the following integral equation is introduced as follows:

$$\frac{1}{2}u_i(\xi)+\int_\Gamma P^*_{ij}(\xi,x)u_j(x)\,ds(x) = \int_\Gamma U^{*1}_{ij}(\xi,x)p_j(x)\,ds(x)+D^{*2}_{i(CR)}(\xi),\quad i=1,2,\xi\in\Gamma, \tag{29}$$

where $D^{*2}_{i(CR)}(\xi)$ is a mutual work difference integral (abbreviated as MWDI) on a large circle and is defined by

$$D^{*2}_{i(CR)}(\xi)=-\int_{CR}P^*_{ij}(\xi,x)u_j(x)\,ds(x)+\int_{CR}U^{*2}_{ij}(\xi,x)p_j(x)\,ds(x). \tag{30}$$

It was proved that if the applied loadings are not in equilibrium, then $D^{*2}_{i(CR)}(\xi)$ is not equal to zero in general, or $D^{*2}_{i(CR)}(\xi)\neq 0$ [10,17,18]. Therefore, the term $D^{*2}_{i(CR)}(\xi)$ cannot be dropped in Eq. (29), or Eq. (29) cannot be modified further [10].

The usage of the kernels ($U^{*1}_{ij}(\xi,x)$ or $U^{*2}_{ij}(\xi,x)$) will influence the result of degenerate scale. For the case of circular hole, or $R=a=b$, the degenerate scale is $R_{cr}=1$ from three sources: (a) by using an elliptic coordinates in Ref. [8]; (b) by using the conformal mapping technique; and (c) a numerical solution by using the fundamental solution shown by Eq. (15) with the relevant $U_{ij}^{*1}(\xi,x)$ [10]. However, if the kernel $U_{ij}^{*2}(\xi,x)$ is used, other researcher obtained a different result $R_{cr}=\exp(1/2\kappa)$ [5,6]. This is to say the usage of the kernel $U_{ij}^{*2}(\xi,x)$ is inadequate for evaluating the degenerate scale. It is also found that the kernel $U_{ij}^{*2}(\xi,x)$ cannot be used for the case of non-equilibrated loading on the contour.

## 3. Evaluation of the degenerate scale for boundary value problem in plane elasticity and by using the conformal mapping

Formulation for degenerate scale problem is based on the direct usage of the complex variable and the conformal mapping. After using the conformal mapping, the vanishing displacement condition is assumed on the boundary of unit circle. The complex potentials on the mapping plane are sought in a form of superposition of the principal part and the complementary part. The principal part of the complex potentials is given beforehand, and the complementary part plays a role for compensating the displacement along the boundary from the principal part. After using the appropriate complex potentials, the boundary displacement becomes one term with the form of $g(R)-c$ ($g(R)$ a function of $R$), where $R$ denotes a length parameter and $c$ is a constant. By letting the vanishing displacement on the boundary, or $g(R)-c=0$, the degenerate scale "$R$" is obtained. For four cases, the elliptic contour, the triangle contour, the square contour and the ellipse-like contour, the degenerate scales are evaluated.

### 3.1. Evaluation of the degenerate scale for the elliptic contour case

Assume that there is an ellipse with half-axis lengths $(a, b)$ and the elliptical contour and its exterior region (in $z$-plane) is mapped into a unit circle and its exterior region (in $\varsigma$-plane) (Fig. 3(a,b)):

$$z=\omega(\varsigma)=R\left(\varsigma+\frac{m}{\varsigma}\right),\quad \text{where } R=\frac{a+b}{2},\quad m=\frac{a-b}{a+b}. \tag{31}$$

After using the conformal mapping, the following function are introduced as follows:

$$\phi(\varsigma)=\phi_1(z)|_{z=\omega(\zeta)},\quad \psi(\varsigma)=\psi_1(z)|_{z=\omega(\zeta)},\quad \phi_1'(z)=\frac{\phi'(\varsigma)}{\omega'(\varsigma)},$$

$$\phi_1''(z)=\frac{\phi''(\varsigma)\omega'(\varsigma)-\phi'(\varsigma)\omega''(\varsigma)}{(\omega'(\varsigma))^3},\quad \psi_1'(z)=\frac{\psi'(\varsigma)}{\omega'(\varsigma)}. \tag{32}$$

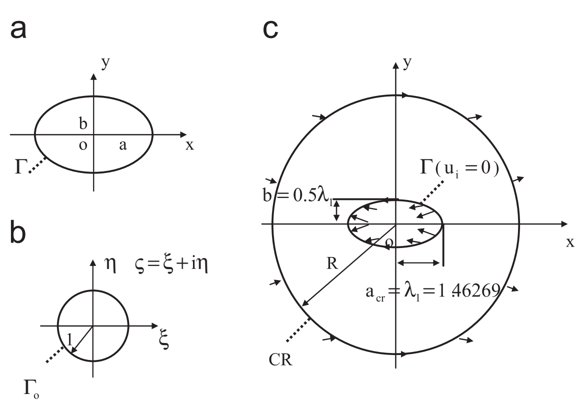


**Fig. 3.** (a) An elliptic hole in an infinite plate, on the $z$-plane, (b) the mapping of elliptic hole and its exterior region, on the $\varsigma$-plane, and (c) degenerate scale in the case of $b/a=0.5$ and relevant displacements and loading along the boundaries.

Therefore, the stresses, resultant forces and displacements can be expressed as follows:

$$\sigma_x+\sigma_y=4\,Re\left(\frac{\phi'(\varsigma)}{\omega'(\varsigma)}\right),$$

$$\sigma_y-\sigma_x+2i\sigma_{xy}=2\left(\frac{\overline{\omega(\varsigma)}[\phi''(\varsigma)\omega'(\varsigma)-\phi'(\varsigma)\omega''(\varsigma)]}{(\omega'(\varsigma))^3}+\frac{\psi'(\varsigma)}{\omega'(\varsigma)}\right), \tag{33}$$

$$f=-Y+iX=\phi(\varsigma)+\frac{\omega(\varsigma)\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}+\overline{\psi(\varsigma)}, \tag{34}$$

$$2G(u+iv)=\kappa\phi(\varsigma)-\frac{\omega(\varsigma)\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}}-\overline{\psi(\varsigma)}. \tag{35}$$

In the following derivation, it is preferable to express the displacement in the form:

$$2G(u-iv)=\kappa\overline{\phi(\varsigma)}-\frac{\overline{\omega(\varsigma)}\phi'(\varsigma)}{\omega'(\varsigma)}-\psi(\varsigma). \tag{36}$$

From a general analysis in plane elasticity, if there is a resultant force applied on the contour, the complex potentials can be expressed as follows [15]:

$$\phi_1(z)=\phi_{1p}(z)+\phi_{1c}(z),\quad \psi_1(z)=\psi_{1p}(z)+\psi_{1c}(z)$$
$$\text{with } \phi_{1p}(z)=A\ln z,\quad \psi_{1p}(z)=-\kappa\bar{A}\ln z, \tag{37}$$

where $A=A_1+iA_2$, and $\phi_{1c}(z)$, $\psi_{1c}(z)$ are two analytic functions.

Since at remote place, $z\approx R\varsigma$, the relevant complex potentials can be expressed as follows:

$$\phi(\varsigma)=\phi_p(\varsigma)+\phi_c(\varsigma),\quad \psi(\varsigma)=\psi_p(\varsigma)+\psi_c(\varsigma)$$
$$\text{with } \phi_p(z)=A(\ln\varsigma+\ln R),\quad \psi_p(\varsigma)=-\kappa\bar{A}(\ln\varsigma+\ln R), \tag{38}$$

where $\phi_c(\varsigma)$ and $\psi_c(\varsigma)$ are two analytic functions. The two functions can be expressed in the following form:

$$\phi_c(\varsigma)=\sum_{k=1}^{\infty}\frac{c_k}{\varsigma^k},\quad \psi_c(\varsigma)=\sum_{k=1}^{\infty}\frac{d_k}{\varsigma^k}. \tag{39}$$

After substituting Eq. (38) into Eq. (36), and letting the variable $\varsigma$ be on the unit circle (using the property $\bar{\varsigma}=1/\varsigma$ on unit circle), the displacements at the boundary point can be expressed as follows:

$$2G(u-iv)|_\Gamma=\kappa\overline{\phi_c(\varsigma)}-\frac{\varsigma(m\varsigma^2+1)\phi_c'(\varsigma)}{\varsigma^2-m}-\psi_c(\varsigma)+2\kappa\bar{A}\ln R$$
$$-\left(m+\frac{1+m^2}{\varsigma^2-m}\right)A\quad \text{for } \varsigma=\exp(i\theta)\in\Gamma_o \text{ or } z\in\Gamma. \tag{40}$$

One may write one term in Eq. (40) in an explicit form:

$$\frac{\varsigma(m\varsigma^2+1)\phi_c'(\varsigma)}{\varsigma^2-m}=\frac{\varsigma(m\varsigma^2+1)}{\varsigma^2-m}\left(\sum_{k=1}^{\infty}\frac{-kc_k}{\varsigma^{k+1}}\right). \tag{41}$$

That is to say the function shown by Eq. (41) has estimation $(\varsigma(m\varsigma^2+1)\phi_c'(\varsigma))/(\varsigma^2-m)=O(1/\varsigma)$ at infinity, or it is an analytic function in the region outside the unit circle. By using this property, we can let

$$\kappa\overline{\phi_c(\varsigma)}=0\quad \text{for } \varsigma=\exp(i\theta)\in\Gamma_o, \tag{42}$$

$$-\psi_c(\varsigma)-\frac{\varsigma(m\varsigma^2+1)\phi_c'(\varsigma)}{\varsigma^2-m}-\frac{1+m^2}{\varsigma^2-m}A=0$$
$$\text{for } \varsigma=\exp(i\theta)\in\Gamma_o. \tag{43}$$

Using the result in Appendix, from Eqs. (42) and (43) we will find

$$\phi_c(\varsigma)=0,\quad \psi_c(\varsigma)=-\frac{1+m^2}{\varsigma^2-m}A. \tag{44}$$

Finally, from Eqs. (38) and (44) we have:

$$\phi(\varsigma) = A(\ln \varsigma + \ln R), \quad \psi(\varsigma) = -\kappa\bar{A}(\ln \varsigma + \ln R) - \frac{1+m^2}{\varsigma^2 - m}A. \tag{45}$$

Substituting Eqs. (42) and (43) into Eq. (40) yields:

$$2G(u - iv)|_\Gamma = 2\kappa\bar{A}\ln R - mA \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \tag{46}$$

The following boundary condition is imposed to the problem:

$$2G(u - iv)|_\Gamma = 0 \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \tag{47}$$

Comparing Eq. (46) with Eq. (47), we find the following equation:

$$2\kappa\bar{A}\ln R - mA = 0. \tag{48}$$

There are two cases under consideration. In the first case, we assume that $A = 1$ in Eq. (48), and obtain:

$$2\kappa \ln R - m = 0. \tag{49}$$

By letting $\delta = b/a$, and $m = (a-b)/(a+b) = (1-\delta)/(1+\delta)$, the critical size for $R$ is obtained as follows:

$$R_{cr1} = \exp\left(\frac{m}{2\kappa}\right) \quad \text{or} \quad R_{cr1} = \exp\left(\frac{1-\delta}{2\kappa(1+\delta)}\right) \quad \text{where } \delta = b/a. \tag{50}$$

In addition, from the relation $R = a(1+\delta)/2$, we will critical value for "$a$":

$$a_{cr1} = \lambda_1 = \frac{2}{1+\delta}\exp\left(\frac{1-\delta}{2\kappa(1+\delta)}\right). \tag{51}$$

Clearly, the two half-axis take the value $a = \lambda_1$ and $b = \delta\lambda_1$, then the condition shown by Eq. (47) will be fulfilled. In this case, by letting $A = 1$ in Eq. (45) two complex potentials will be as follows:

$$\phi(\varsigma) = (\ln \varsigma + \ln R), \quad \psi(\varsigma) = -\kappa(\ln \varsigma + \ln R) - \frac{1+m^2}{\varsigma^2 - m}. \tag{52}$$

The displacements and stresses derived from the complex potentials shown by Eq. (52) correspond to the eigenfunction for the eigenvalue $R_{cr1}$.

For a real case of $b/a = 0.5$, the degenerate scale and the loading condition are shown in Fig. 3(c). Particular features of this case are as follows. The critical value is $a_{cr} = \lambda_1 = 1.46269$, $b = 0.5\lambda_1$. The loadings on the inner boundary $\Gamma$ and on the outer circle "CR" are in equilibrium, and the displacements are vanishing along the inner boundary $\Gamma$.

Similarly, if we assume $A = i$ in Eq. (48), we have:

$$2\kappa \ln R + m = 0, \tag{53}$$

$$R_{cr2} = \exp\left(-\frac{1-\delta}{2\kappa(1+\delta)}\right) \quad \text{where } \delta = b/a, \tag{54}$$

$$a_{cr2} = \lambda_2 = \frac{2}{1+\delta}\exp\left(-\frac{1-\delta}{2\kappa(1+\delta)}\right). \tag{55}$$

Clearly, the two half-axis take the value $a = \lambda_2$ and $b = \delta\lambda_2$, then the condition shown by Eq. (47) can also be fulfilled. In the second case, by letting $A = i$ in Eq. (45) two complex potentials will be as follows:

$$\phi(\varsigma) = i(\ln \varsigma + \ln R), \quad \psi(\varsigma) = i\left(\kappa(\ln \varsigma + \ln R) - \frac{1+m^2}{\varsigma^2 - m}\right). \tag{56}$$

Clearly, the displacements and stresses derived from the complex potentials shown by Eq. (56) correspond to the eigenfunction for the eigenvalue $R_{cr2}$.

### 3.2. Basic steps used for evaluating the degenerate scale

For the elliptic notch case, there are four values involved or $a$, $b$, $R$ $(=a(1+m)/2)$ and $m(=(a-b)/(a+b))$, and among them two values are independent. Thus, if $m$ is give beforehand, one can choose $R$ as a parameter for degenerate scale. The necessary steps for evaluating the degenerate scale are summarized as follows.

In the first step, we introduce some complex potentials in the form of $\phi_1(z) = \phi_{1p}(z) + \phi_{1c}(z)$ and $\psi_1(z) = \psi_{1p}(z) + \psi_c(z)$, and let the displacements on the hole contour take the following form:

$$2G(u - iv)|_\Gamma = g(R) - c \quad \text{where } \Gamma \text{ the hole contour}, \tag{57}$$

where $g(R)$ is a function of $R$, and $g(R)$ represents a value and does not depend on the location of point $(x, y)$ on the contour $\Gamma$. In Eq. (57), $c$ is a constant. In fact, we have got this term in the form $(2\kappa \ln R - m)$ in Eq. (49) (corresponding to $g(R) - c$).

In the second step, we simply let the following equation to be satisfied

$$g(R) - c = 0 \quad \text{or} \quad g(R) = c. \tag{58}$$

From Eq. (58), we can get the degenerate scale $R$. In fact, we have got the degenerate scale $R_{cr} = \exp(m/2\kappa)$ shown by Eq. (50). Therefore, if $R$ takes the value of degenerate scale (or the solution from Eq. (58)), we have:

$$2G(u - iv)|_\Gamma = 0. \tag{59}$$

This will complete the derivation.

In the derivation, the concept of pure deformable form is particularly important. The above-mentioned complex potentials, the pair $\phi_{1p}(z)$, $\psi_{1p}(z)$ and the pair $\phi_{1c}(z)$, $\psi_{1c}(z)$ must belong to the pure deformable form.

It is known that, for a given stress state, the relevant displacements can be different each other by a translation and a rotation. Therefore, if the used complex potentials are not expressed in the pure deformable form, we will get the equation:

$$g(R) = c_1 \quad \text{or} \quad g(R) = c_2 \quad \text{where } c_1, c_2 \text{ different constants}. \tag{60}$$

This will cause uncertainty in the derivation.

### 3.3. Evaluation of the degenerate scale for triangle contour case

Similarly, the degenerate scale for triangle contour case can also be evaluated (Fig. 4(a)). The approximate mapping function is as follows [15]:

$$z = \omega(\varsigma) = R\left(\varsigma + \frac{1}{3\varsigma^2}\right). \tag{61}$$

Clearly, the mapping configuration using series shown by Eq. (61) is not an exact triangle.

In this case, the two complex potentials are still assumed in the form of Eqs. (37)–(39). After substituting Eqs. (38), (39) and (61) into Eq. (36), and letting the variable $\varsigma$ being on the unit circle (using the property $\bar{\varsigma} = 1/\varsigma$ on unit circle), we will find the displacements at the boundary point as follows:

$$\begin{aligned} 2G(u - iv)|_\Gamma &= \kappa\overline{\phi_c(\varsigma)} - \frac{1}{3}\left(\varsigma^2 + \frac{11\varsigma^2}{3\varsigma^3 - 2}\right)\phi_c'(\varsigma) - \psi_c(\varsigma) \\ &\quad + 2\kappa\bar{A}\ln R - \frac{A}{3}\left(\varsigma + \frac{11\varsigma}{3\varsigma^3 - 2}\right) \\ &\quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \end{aligned} \tag{62}$$

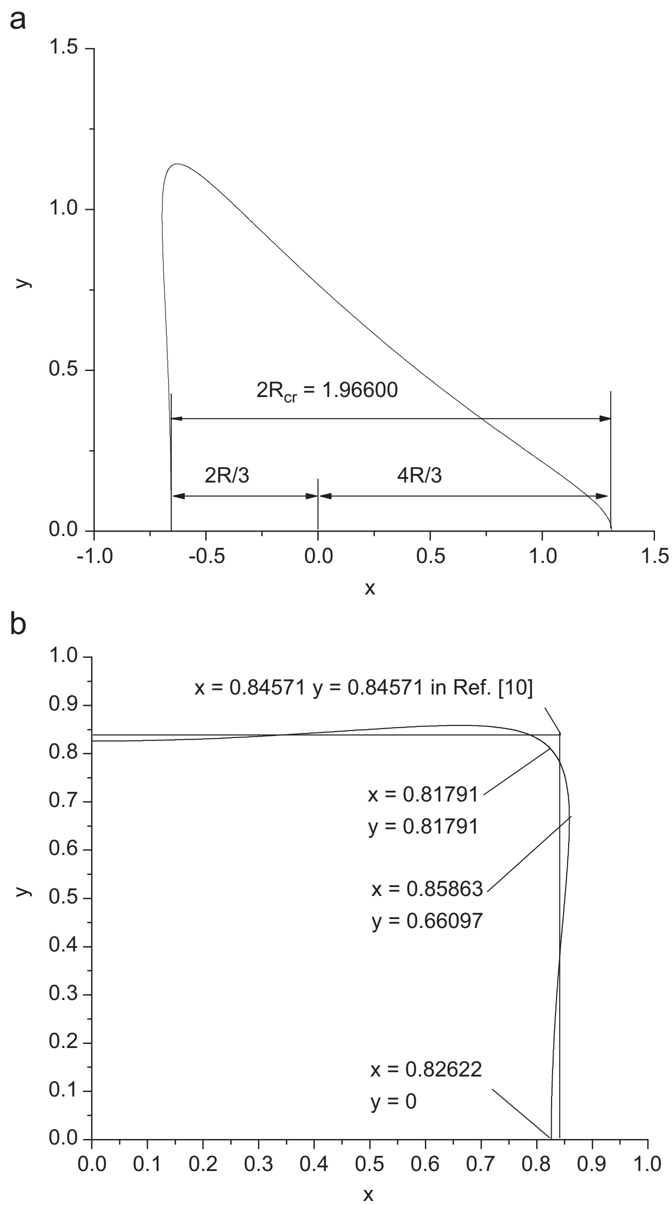


**Fig. 4.** (a) A half of triangle hole with evaluated degenerate scale. (b) A quarter of square hole with evaluated degenerate scale and comparison between numerical result and conformal mapping technique.

One may write one term in Eq. (62) in an explicit form as follows:

$$B(\varsigma) = \frac{1}{3}\left(\varsigma^2 + \frac{11\varsigma^2}{3\varsigma^3 - 2}\right)\phi_c'(\varsigma) = \frac{1}{3}\left(\varsigma^2 + \frac{11\varsigma^2}{3\varsigma^3 - 2}\right)\left(\sum_{k=1}^{\infty}\frac{-kc_k}{\varsigma^{k+1}}\right). \quad (63)$$

Note that, the function $B(\varsigma)$ is now considered as a function defined in the region outside the unit circle. From Eq. (63), we find that the function $B(\varsigma)$ has the following principal part $B_p(\varsigma)$at infinity:

$$B_p(\varsigma) = -\frac{c_1}{3}. \quad (64)$$

Therefore, Eq. (62) can be rewritten as follows:

$$2G(u - iv)|_\Gamma = \kappa\overline{\phi_c(\varsigma)} - \frac{1}{3}\left(\left(\varsigma^2 + \frac{11\varsigma^2}{3\varsigma^3 - 2}\right)\phi_c'(\varsigma) + c_1\right) + \frac{c_1}{3} - \psi_c(\varsigma) + 2\kappa\bar{A}\ln R - \frac{A}{3}\left(\varsigma + \frac{11\varsigma}{3\varsigma^3 - 2}\right)$$
$$\text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \quad (65)$$

From the property of complex variable and result in Appendix, for some terms in Eq. (65) we may assume:

$$\kappa\overline{\phi_c(\varsigma)} - \frac{A\varsigma}{3} = 0 \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o, \quad (66)$$

$$-\psi_c(\varsigma) - \frac{1}{3}\left(\left(\varsigma^2 + \frac{11\varsigma^2}{3\varsigma^3 - 2}\right)\phi_c'(\varsigma) + c_1\right) - \frac{A}{3}\left(\frac{11\varsigma}{3\varsigma^3 - 2}\right) = 0$$
$$\text{for } \varsigma = \exp(i\theta) \in \Gamma_o). \quad (67)$$

From Eqs. (66) and (67) and the result in Appendix, we have:

$$\phi_c(\varsigma) = \frac{\bar{A}}{3\kappa}\frac{1}{\varsigma} \quad \text{and} \quad c_1 = \frac{\bar{A}}{3\kappa}, \quad (68)$$

$$\psi_c(\varsigma) = -\frac{1}{3}\left(\left(\varsigma^2 + \frac{11\varsigma^2}{3\varsigma^3 - 2}\right)\phi_c'(\varsigma) + \frac{\bar{A}}{3\kappa}\right) - \frac{A}{3}\left(\frac{11\varsigma}{3\varsigma^3 - 2}\right). \quad (69)$$

Further, substituting Eqs. (66) and (67) into Eq. (65), we will find:

$$2G(u - iv)|_\Gamma = 2\kappa\bar{A}\ln R + \frac{\bar{A}}{9\kappa} \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \quad (70)$$

As before, by using the condition (47), $2G(u - iv)|_\Gamma = 0$, it follows:

$$\ln R = -\frac{1}{18\kappa^2}. \quad (71)$$

Finally, the degenerate scale is obtained as follows:

$$R_{cr} = \exp\left(-\frac{1}{18\kappa^2}\right) \quad \text{or} \quad R_{cr} = 0.98300$$
$$\text{for } \kappa = 3 - 4\nu = 1.8. \quad (72)$$

Simply because the configuration is symmetric with rotations $\theta = 2\pi/3$ or $4\pi/3$, the critical value is 1.

### 3.4. Evaluation of the degenerate scale for square contour case

Similarly, the degenerate scale for square contour case can also be evaluated (Fig. 4(b)). The approximate mapping function is as follows [15]:

$$z = \omega(\varsigma) = R\left(\varsigma - \frac{1}{6\varsigma^3}\right). \quad (73)$$

Clearly, the mapping configuration using series shown by Eq. (73) is not an exact square.

In this case, the two complex potentials are still assumed in the form of Eqs. (37)–(39). After substituting Eqs. (38), (39) and (73) into Eq. (36), and letting the variable $\varsigma$ being on the unit circle (using the property $\bar{\varsigma} = 1/\varsigma$ on unit circle), we will find the displacements at the boundary point as follows:

$$2G(u - iv)|_\Gamma = \kappa\overline{\phi_c(\varsigma)} + \frac{\varsigma^3}{6}\left(1 - \frac{13}{2\varsigma^4 + 1}\right)\phi_c'(\varsigma) - \psi_c(\varsigma) + 2\kappa\bar{A}\ln R + \frac{\varsigma^2}{6}\left(1 - \frac{13}{2\varsigma^4 + 1}\right)A$$
$$\text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \quad (74)$$

One may write one term in Eq. (74) in an explicit form as follows:

$$B(\varsigma) = \frac{\varsigma^3}{6}\left(1 - \frac{13}{2\varsigma^4 + 1}\right)\phi_c'(\varsigma) = \frac{\varsigma^3}{6}\left(1 - \frac{13}{2\varsigma^4 + 1}\right)\left(\sum_{k=1}^{\infty}\frac{-kc_k}{\varsigma^{k+1}}\right). \quad (75)$$

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Note that, the function $B(\varsigma)$ is now considered as a function defined in the region outside the unit circle. From Eq. (75), we find that the function $B(\varsigma)$ has the following principal part $B_p(\varsigma)$ at infinity:

$$B_p(\varsigma) = -\frac{c_1\varsigma}{6} - \frac{c_2}{3}. \tag{76}$$

Therefore, Eq. (74) can be rewritten as follows:

$$2G(u - iv)|_\Gamma = \kappa\overline{\phi_c(\varsigma)} + \left\{\frac{\varsigma^3}{6}\left(1 - \frac{13}{2\varsigma^4+1}\right)\phi_c'(\varsigma) + \frac{c_1\varsigma}{6} + \frac{c_2}{3}\right\} - \frac{c_1\varsigma}{6} - \frac{c_2}{3} - \psi_c(\varsigma) + 2\kappa\bar{A}\ \ln R + \frac{\varsigma^2}{6}A - \frac{13\varsigma^2}{6(2\varsigma^4+1)}A \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \tag{77}$$

From the property of complex variable and result in Appendix, for some terms in Eq. (77) we may assume that

$$\kappa\overline{\phi_c(\varsigma)} - \frac{c_1\varsigma}{6} + \frac{\varsigma^2}{6}A = 0 \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o, \tag{78}$$

$$-\psi_c(\varsigma) + \left\{\frac{\varsigma^3}{6}\left(1 - \frac{13}{2\varsigma^4+1}\right)\phi_c'(\varsigma) + \frac{c_1\varsigma}{6} + \frac{c_2}{3}\right\} - \frac{13\varsigma^2}{6(2\varsigma^4+1)}A = 0 \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o. \tag{79}$$

From Eqs. (78) and (79) and the result in Appendix, we have:

$$\phi_c(\varsigma) = -\frac{\bar{A}}{6\kappa}\frac{1}{\varsigma^2} \quad \text{with } c_1 = 0,\ c_2 = -\frac{\bar{A}}{6\kappa}, \tag{80}$$

$$\psi_c(\varsigma) = \frac{\varsigma^3}{6}\left(1 - \frac{13}{2\varsigma^4+1}\right)\phi_c'(\varsigma) - \frac{\bar{A}}{18\kappa} - \frac{13\varsigma^2}{6(2\varsigma^4+1)}A. \tag{81}$$

Further, substituting Eqs. (78) and (79) into Eq. (77), we will find that

$$2G(u - iv)|_\Gamma = 2\kappa\bar{A}\ \ln R + \frac{\bar{A}}{18\kappa} \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \tag{82}$$

As before, by using the condition Eq. (47), $2G(u - iv)|_\Gamma = 0$, we find

$$\ln R = -\frac{1}{36\kappa^2}. \tag{83}$$

Finally, the degenerate scale is obtained as follows:

$$R_{cr} = \exp\left(-\frac{1}{36\kappa^2}\right) \quad \text{or} \quad R_{cr} = 0.99146 \quad \text{for } \kappa = 3 - 4v = 1.8. \tag{84}$$

The critical size of the square will be $5R_{cr}/3 = 1.65244$. Simply because the configuration is symmetric with rotations $\theta = \pi/2$ or $\theta = \pi$ or $\theta = 3\pi/2$, the critical value is 1.

In Fig. 4(b), the dashed contour is from the mapping function when the degenerate scale is reached, and $a_1 = 0.82622$ ($a_2 = 0.85863$) denotes the size at the narrow (the wider) portion, respectively. In addition, $a = 0.84571$ is a degenerate scale for square notch from a numerical solution [10].

### 3.5. Evaluation of the degenerate scale for ellipse-like contour case

Similarly, the degenerate scale for ellipse-like contour case can also be evaluated (Fig. 5). The mapping function is as follows:

$$z = \omega(\varsigma) = R\left(\varsigma + \frac{m}{4\varsigma - 1}\right), \quad 0 < m \leqslant 1. \tag{85}$$

In this case, the two complex potentials are still assumed in the form of Eqs. (37)–(39). After substituting Eqs. (38), (39) and (85) into Eq. (36), and letting the variable $\varsigma$ being on the unit circle (using the property $\bar{\xi} = 1/\varsigma$ on unit circle), we will find the displacements at the boundary point as follows:

$$2G(u - iv)|_\Gamma = \kappa\overline{\phi_c(\varsigma)} - g(\varsigma)\phi_c'(\varsigma) - \psi_c(\varsigma) + 2\kappa\bar{A}\ \ln R - \frac{g(\varsigma)}{\varsigma}A \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma, \tag{86}$$

where

$$g(\varsigma) = \frac{h(\varsigma)}{\varsigma - 4}, \quad h(\varsigma) = \frac{-m\varsigma^2 + \varsigma - 4}{\varsigma}\frac{(4\varsigma - 1)^2}{(4\varsigma - 1)^2 - 4m}. \tag{87}$$

Since the function $g(\varsigma)$ has a pole at the point $\varsigma = 4$ ($\varsigma = 4$, in the region outside the unit circle), one needs to make the following modification to two functions:

$$g(\varsigma)\phi_c'(\varsigma) = \left\{g(\varsigma)\phi_c'(\varsigma) - \frac{p_1}{\varsigma - 4}\right\} + \left\{\frac{p_1}{\varsigma - 4} + \frac{p_1}{4}\right\} - \frac{p_1}{4}, \tag{88}$$

$$\frac{g(\varsigma)}{\varsigma} = \left\{\frac{g(\varsigma)}{\varsigma} - \frac{p_2}{\varsigma - 4}\right\} + \left\{\frac{p_2}{\varsigma - 4} + \frac{p_2}{4}\right\} - \frac{p_2}{4}, \tag{89}$$

where

$$p_1 = s_1\phi_c'(4), \quad p_2 = \frac{s_1}{4}, \quad s_1 = h(4) = -\frac{900m}{225 - 4m}. \tag{90}$$

Substituting Eqs. (88) and (89) into Eq. (86) yields:

$$2G(u - iv)|_\Gamma = \kappa\overline{\phi_c(\varsigma)} - \left\{g(\varsigma)\phi_c'(\varsigma) - \frac{p_1}{\varsigma - 4}\right\} - \left\{\frac{p_1}{\varsigma - 4} + \frac{p_1}{4}\right\} + \frac{p_1}{4} - \psi_c(\varsigma) + 2\kappa\bar{A}\ \ln R - \left\{\frac{g(\varsigma)}{\varsigma} - \frac{p_2}{\varsigma - 4}\right\}A - \left\{\frac{p_2}{\varsigma - 4} + \frac{p_2}{4}\right\}A + \frac{p_2}{4}A \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o \text{ or } z \in \Gamma. \tag{91}$$

From the property of complex variable and result in Appendix, for some terms in Eq. (91), we may assume that

$$\kappa\overline{\phi_c(\varsigma)} - \left\{\frac{p_1}{\varsigma - 4} + \frac{p_1}{4}\right\} - \left\{\frac{p_2}{\varsigma - 4} + \frac{p_2}{4}\right\}A = 0 \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o, \tag{92}$$

$$-\psi_c(\varsigma) - \left\{g(\varsigma)\phi_c'(\varsigma) - \frac{p_1}{\varsigma - 4}\right\} - \left\{\frac{g(\varsigma)}{\varsigma} - \frac{p_2}{\varsigma - 4}\right\}A = 0 \quad \text{for } \varsigma = \exp(i\theta) \in \Gamma_o. \tag{93}$$

Note that, two terms in Eq. (92) satisfy the condition$\{\ldots\}|_{\varsigma=0} = 0$. Directly using the result in Appendix, we will find:

$$\kappa\phi_c(\varsigma) = -\frac{1}{4(4\varsigma - 1)}(\bar{p}_1 + p_2\bar{A}), \tag{94}$$

$$\kappa\phi_c'(\varsigma) = \frac{1}{(4\varsigma - 1)^2}(\bar{p}_1 + p_2\bar{A}). \tag{95}$$

Substituting $\varsigma = 4$ into Eq. (95), we can formulate an equation for $\phi_c'(4)$, and it can be evaluated immediately:

$$\phi_c'(4) = \frac{s_2}{4(\kappa^2 - s_2^2)}(s_2A + \kappa\bar{A}) \quad \text{with } s_2 = -\frac{4m}{225 - 4m}, \tag{96}$$

$$p_1 = s_1\phi_c'(4) = \frac{s_1s_2}{4(\kappa^2 - s_2^2)}(s_2A + \kappa\bar{A}). \tag{97}$$

From Eq. (93) and the result in Appendix, we will find:

$$\psi_c(\varsigma) = -\left\{g(\varsigma)\phi_c'(\varsigma) - \frac{p_1}{\varsigma - 4}\right\} - \left\{\frac{g(\varsigma)}{\varsigma} - \frac{p_2}{\varsigma - 4}\right\}A. \tag{98}$$

Note that, the meanings of Eqs. (93) and (98) are different. In the former case, Eq. (93) represents a condition on the circular boundary. However, in Eq. (98), the function $\psi_c(\varsigma)$ is defined in the region outside the unit circle.

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Further, substituting Eqs. (92) and (93) into Eq. (91), we will find:

$$2G(u-\mathrm{i}v)|_{\Gamma}=2\kappa\bar{A}\ln R+\frac{p_1}{4}+\frac{p_2}{4}A$$
$$\text{for } \varsigma=\exp(\mathrm{i}\theta)\in\Gamma_o \text{ or } z\in\Gamma. \quad (99)$$

As before, by using the condition Eq. (47), $2G(u-\mathrm{i}v)|_{\Gamma}=0$, will we find:

$$2\kappa\bar{A}\ln R+\frac{p_1}{4}+\frac{p_2}{4}A=0. \quad (100)$$

If letting $A=1$ in Eq. (100), the first critical value can be obtained as follows:

$$R_{cr1}=\exp\left(\frac{s_1}{32\kappa}\left(-\frac{s_2}{\kappa-s_2}-1\right)\right). \quad (101)$$

If letting $A=i$ in Eq. (100), the second critical value can be obtained as follows:

$$R_{cr2}=\exp\left(\frac{s_1}{32\kappa}\left(-\frac{s_2}{\kappa+s_2}+1\right)\right). \quad (102)$$

For $m=0.1, 0.2, \ldots, 1.0$, two critical values are listed in Table 1. In addition, for $m=0.2, 0.4, \ldots, 1.0$ case, the degenerate sizes are plotted in Figs. 5 and 6, for the $R_{cr1}$ and the $R_{cr2}$ case, respectively.

## 4. Evaluation of the degenerate scale for boundary value problem in antiplane elasticity by using the conformal mapping

Similar to the case of plane elasticity, degenerate scale in the BIE of antiplane elasticity, or for the Laplace equation, can also be studied by using the complex variable and the conformal mapping.

### 4.1. Evaluation of the degenerate scale for elliptic contour case

After using complex potential $\phi_1(z)$ in antiplane elasticity, all the physical quantities can be expressed through $\phi_1(z)$ [19]:

$$Gw(x,y)+\mathrm{i}f(x,y)=\phi_1(z), \quad (103)$$

$$f(x,y)=\int_{z_o}^{z}\sigma_{xz}\,\mathrm{d}y-\sigma_{yz}\,\mathrm{d}x, \quad (104)$$

$$G\frac{\partial w}{\partial x}+\mathrm{i}\frac{\partial f}{\partial x}=\sigma_{xz}-\mathrm{i}\sigma_{yz}=\Phi_1(z)=\phi_1'(z), \quad (105)$$

where $G$ is the shear modulus of elasticity, $w$ is the out of plane displacement, $f$ is the longitudinal resultant force, and $\sigma_{xz}$ and $\sigma_{yz}$ are the stress components, and $z=x+\mathrm{i}y$.

**Table 1**
Two critical values for an ellipse-like notch with the conformal mapping function defined by Eq. (85)

| $m=$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $R_{cr1}$ | 1.0070 | 1.0140 | 1.0211 | 1.0283 | 1.0355 | 1.0428 | 1.0501 | 1.0575 | 1.0650 | 1.0725 |
| $R_{cr2}$ | 0.9931 | 0.9861 | 0.9792 | 0.9723 | 0.9654 | 0.9585 | 0.9516 | 0.9448 | 0.9379 | 0.9311 |

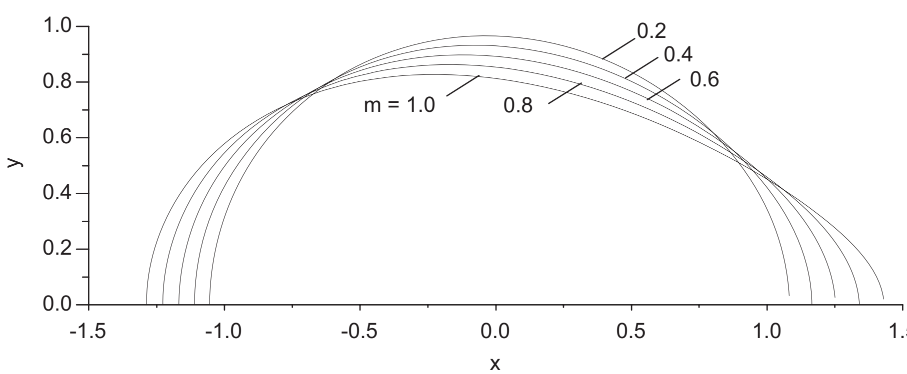


**Fig. 5.** The first degenerate size corresponding to $R_{cr1}$ for an ellipse-like contour using the conformal mapping Eq. (85).

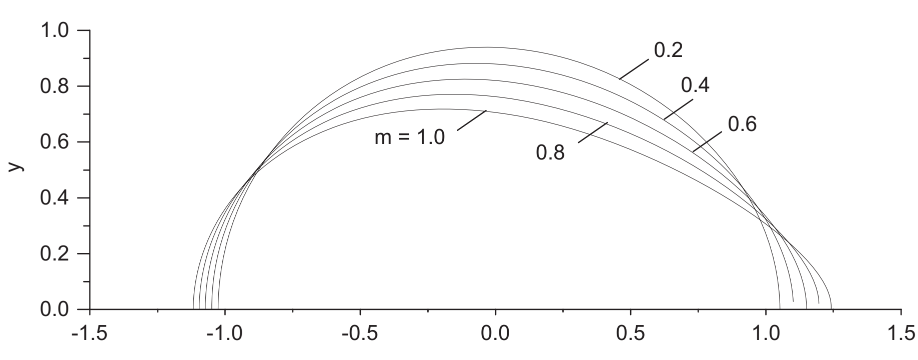


**Fig. 6.** The second degenerate size corresponding to $R_{cr2}$ for an ellipse-like contour using the conformal mapping Eq. (85).

As before, for the elliptic contour case, the following mapping function is introduced:

$$z=\omega(\varsigma)=R\left(\varsigma+\frac{m}{\varsigma}\right)\quad \text{where } R=\frac{a+b}{2},\ m=\frac{a-b}{a+b}. \tag{31}$$

After using the conformal mapping, the following function are introduced:

$$\phi(\varsigma)=\phi_1(z)|_{z=\omega(\zeta)}. \tag{106}$$

Therefore, the displacements, resultant forces and stresses can be expressed as follows:

$$Gw+\mathrm{i}f=\phi(\varsigma)\quad\text{and}\quad 2Gw=\phi(\varsigma)+\overline{\phi(\varsigma)}, \tag{107}$$

$$\sigma_{xz}-\mathrm{i}\sigma_{yz}=\frac{\phi'(\varsigma)}{\omega'(\varsigma)}. \tag{108}$$

It was known that if there is a resultant force in the longitudinal direction applied on the contour, the complex potential can be expressed as follows:

$$\phi_1(z)=A\ln z+\phi_{1c}(z), \tag{109}$$

where $A$ is a real, and $\phi_{1c}(z)$ is an analytic function.

Since at remote place, $z\approx R\varsigma$, the relevant complex potentials can be expressed as follows:

$$\phi(\varsigma)=A(\ln\varsigma+\ln R)+\phi_c(\varsigma), \tag{110}$$

where $\phi_c(\varsigma)$ is an analytic function which can be expressed as follows:

$$\phi_c(\varsigma)=\sum_{k=1}^{\infty}\frac{c_k}{\varsigma^k}. \tag{111}$$

After substituting Eqs. (110) and (111) into Eq. (107), and letting the variable $\varsigma$ being on the unit circle (using the property $\bar{\varsigma}=1/\varsigma$ or $\varsigma\bar{\varsigma}=1$ on unit circle), we will find the displacement at the on the boundary point as follows:

$$2Gw|_\Gamma=2A\ln R+\phi_c(\varsigma)+\overline{\phi_c(\varsigma)}$$
$$\text{for } \varsigma=\exp(\mathrm{i}\theta)\in\Gamma_o \text{ or } z\in\Gamma. \tag{112}$$

It is found that it is appropriate to take $\phi_c(\varsigma)=0$ in Eq. (112). Using this result yielding:

$$2Gw|_\Gamma=2A\ln R\quad\text{for } \varsigma=\exp(\mathrm{i}\theta)\in\Gamma_o \text{ or } z\in\Gamma. \tag{113}$$

As before, the vanishing displacement can be expressed as follows:

$$2Gw|_\Gamma=0\quad\text{for } \varsigma=\exp(\mathrm{i}\theta)\in\Gamma_o \text{ or } z\in\Gamma. \tag{114}$$

From the condition $2Gw|_\Gamma=0$, we will find:

$$\ln R=0. \tag{115}$$

From Eq. (116), the degenerate scale is obtainable:

$$R_{cr}=1. \tag{116}$$

Eq. (116) may be expressed (from the relation $R=(a+b)/2$) in an alternative form:

$$a_{cr}+b_{cr}=2,\quad a_{cr}=\frac{2}{1+\delta}\quad\text{where } \delta=\frac{b_{cr}}{a_{cr}}. \tag{117}$$

This result was obtained in Ref. [12].

### 4.2. *Evaluation of the degenerate scale for triangle and square contour cases*

For the case of square contour, we use the following mapping functions [15]:

$$z=\omega(\varsigma)=R\left(\varsigma+\frac{1}{3\varsigma^2}\right), \tag{61}$$

$$z=\omega(\varsigma)=R\left(\varsigma+\frac{1}{3\varsigma^2}+\frac{1}{45\varsigma^5}\right), \tag{118}$$

$$z=\omega(\varsigma)=R\left(\varsigma+\sum_{n=2}^{N}\frac{c_n}{\varsigma^{3n-4}}\right)$$
$$\text{with } c_n=(-1)^n\frac{(2/3)((2/3)-1)\cdots((2/3)-n+2)}{(3n-4)(n-1)!}. \tag{119}$$

In those functions, the mapping shown by Eq. (119) is more accurate than that shown by the function (118). The technique used in the elliptic contour case can be used in the present case. It is interesting to point out that, for the above-mentioned three mapping cases we always have the same solution $R_{cr}=1$. Therefore, the critical value for height of triangle is obtained as: $h_{cr}=2R_{cr}=2.00000$ (using Eq. (61)), $h_{cr}=92R_{cr}/45=2.04444$ (using Eq. (118)), $h_{cr}=2.05338$ (using Eq. (119) with 200 terms truncated). The previously obtained result was $h_{cr}=2.0700$ [12]. The evaluated degenerate scales for the case of triangle contour are plotted in Fig. 7.

In addition, if the mapping function takes the following form:

$$z=\omega(\varsigma)=R\left(\varsigma+\sum_{k=1}^{\infty}\frac{c_k}{\varsigma^k}\right) \tag{120}$$

the critical value for "$R$" always takes the value $R_{cr}=1$. That is to say, the result $R_{cr}=1$ does not depend on the detail expression for the portion $\sum_{k=1}^{\infty}(c_k/\varsigma^k)$ in Eq. (120)

For the square contour case, we use the following mapping functions [15]:

$$z=\omega(\varsigma)=R\left(\varsigma-\frac{1}{6\varsigma^3}\right), \tag{73}$$

$$z=\omega(\varsigma)=R\left(\varsigma-\frac{1}{6\varsigma^3}+\frac{1}{56\varsigma^7}\right), \tag{121}$$

$$z=\omega(\varsigma)=R\left(\varsigma-\frac{1}{6\varsigma^3}+\frac{1}{56\varsigma^7}-\frac{1}{176\varsigma^{11}}\right), \tag{122}$$

$$z=\omega(\varsigma)=R\left(\varsigma+\sum_{n=2}^{N}\frac{c_n}{\varsigma^{4n-5}}\right)$$
$$\text{with } c_n=-\frac{(1/2)((1/2)-1)\cdots((1/2)-n+2)}{(4n-5)(n-1)!}. \tag{123}$$

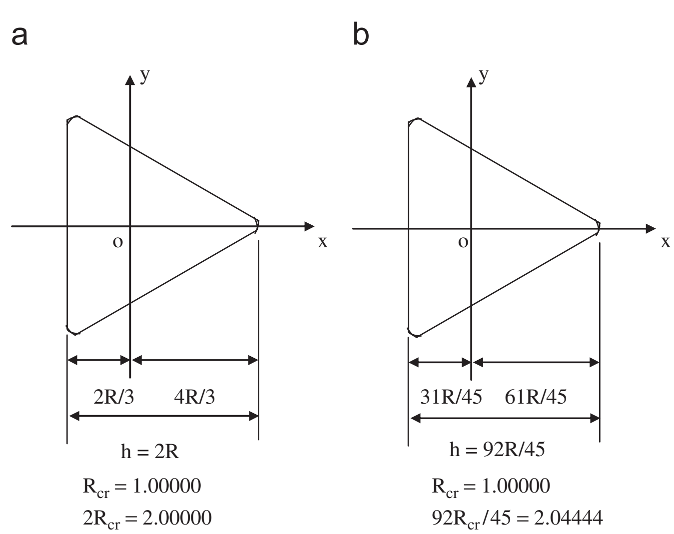


**Fig. 7.** The degenerate size for the case of a triangle contour in antiplane elasticity by using the conformal mapping: (a) Eq. (61) and (b) Eq. (118).

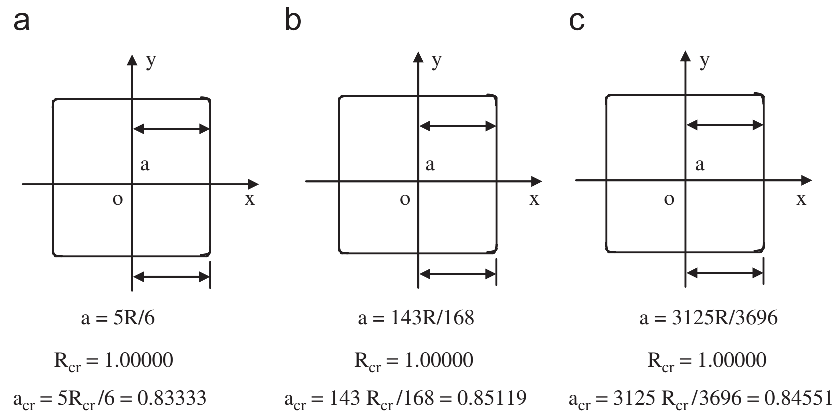


**Fig. 8.** The degenerate size for the case of a square contour in antiplane elasticity by using the conformal mapping; (a) Eqs. (73) and. (121) and (b) Eq. (122).

In those functions, the mapping shown by Eq. (123) is more accurate than that shown by the function (122). It is interesting to point out that, for the above-mentioned four cases of the mapping function we always have the following solution $R_{cr} = 1$. Therefore, the critical value for the half width of square is obtained as follows, $a_{cr} = 5R_{cr}/6 = 0.83333$ (using Eq. (73)), $a_{cr} = 143R_{cr}/168 = 0.85119$ (using Eq. (121)). $a_{cr} = 3125R_{cr}/3696 = 0.84551$ (using Eq. (122)). $a_{cr} = 0.84721$ (using Eq. (123) with 200 terms truncated). The previously obtained result was $a_{cr} = 0.85$ [12]. The evaluated degenerate scales for the case of square contour are plotted in Fig. 8.

## 5. Conclusions

By using the complex variable and the conformal mapping, all the degenerate scale problems in plane elasticity and antiplnae elasticity can be solved in a unified way. The suggested method not only provides the solution for the critical value, but also the eigenfunction.

## Appendix. Evaluation of two analytic functions $f_1(\varsigma)$, $f_2(\varsigma)$ from boundary condition on the unit circle

The following boundary condition on the unit circle is assumed

$$f_1(\varsigma) + \overline{f_2(\varsigma)} - g(\varsigma) - h(\varsigma) = 0 \quad \text{for } \varsigma = \exp(i\theta) \text{ on the unit circle,} \tag{A.1}$$

where $f_1(\varsigma), f_2(\varsigma)$ are the boundary value of two unknown analytic functions. Both functions are defined in the region outside the unit circle, and they are assumed to satisfy $f_1(\infty) = 0$ and $f_2(\infty) = 0$. The function $\overline{f_2(\varsigma)}$ is the conjugate to the function $f_2(\varsigma)$.

In Eq. (A.1), the $g(\varsigma)$ is the boundary value of a given holomorphic function in the region outside the unit circle, and it satisfies the condition $g(\infty) = 0$. That is to say the function $g(\varsigma)$ may have poles in the region inside the unit circle. The following are examples for $g(\varsigma)$:

$$g(\varsigma) = \sum_{k=1}^{\infty} \frac{c_k}{\varsigma^k} \quad \text{or} \quad g(\varsigma) = \sum_{k=1}^{\infty} \frac{d_k}{(\varsigma - 0.5)^k}. \tag{A.2}$$

In Eq. (A.1), the $h(\varsigma)$ is the boundary value of a given holomorphic function in the region inside the unit circle, and it satisfies the condition $h(0) = 0$. That is to say the function $h(\varsigma)$ may have poles in the region outside the unit circle. The following are examples for $h(\varsigma)$:

$$h(\varsigma) = \sum_{k=1}^{\infty} c_k \varsigma_k \quad \text{or} \quad h(\varsigma) = \sum_{k=1}^{\infty} \frac{d_k}{(\varsigma - 2)^k} - \sum_{k=1}^{\infty} \frac{(-1)^k d_k}{2^k}. \tag{A.3}$$

From the function $f_2(\varsigma)$, we may introduce an analytic function [15]:

$$F_2(\varsigma) = \bar{f}_2\left(\frac{1}{\varsigma}\right) = \overline{f_2\left(\frac{1}{\bar{\varsigma}}\right)}. \tag{A.4}$$

Therefore, Eq. (A.1) can be rewritten as follows:

$$f_1(\varsigma) + F_2(\varsigma) - g(\varsigma) - h(\varsigma) = 0 \quad \text{for } \varsigma = \exp(i\theta) \text{ on the unit circle.} \tag{A.5}$$

It was known that for any variable "$t$" in the region outside the unit circle we have [15]:

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f_1(\varsigma)\,d\varsigma}{\varsigma - t} = -f_1(t), \tag{A.6}$$

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{g(\varsigma)\,d\varsigma}{\varsigma - t} = -g(t), \tag{A.7}$$

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{F_2(\varsigma) d\varsigma}{\varsigma - t} = 0 \tag{A.8}$$

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{h(\varsigma)\,d\varsigma}{\varsigma - t} = 0, \tag{A.9}$$

where $\Gamma$ denotes the unit circle for integration. Because the functions $f_1(\varsigma)$ and $g(\varsigma)$ are holomorphic in the region outside the unit circle; therefore, we have equalities shown by Eqs. (A.6) and (A.7).

In addition, because the functions $F_2(\varsigma)$ and $h(\varsigma)$ are holomorphic in the region inside the unit circle, therefore, we have equalities shown by Eqs. (A.8) and (A.9). Therefore, after using the operator $(1/2\pi i)\oint_{\Gamma}[\ldots]d\varsigma/(\varsigma - t)$ to Eq. (A.5), we have:

$$f_1(t) - g(t) = 0 \quad \text{or} \quad f_1(\varsigma) - g(\varsigma) = 0 \quad \text{or} \quad f_1(\varsigma) = g(\varsigma) \quad \text{for } |\varsigma| \geqslant 1. \tag{A.10}$$

By using this result, Eq. (A.1) can be reduced to:

$$\overline{f_2(\varsigma)} - h(\varsigma) = 0 \quad \text{or} \quad f_2(\varsigma) - \overline{h(\varsigma)} = 0 \quad \text{for } \varsigma = \exp(i\theta) \text{ on the unit circle.} \tag{A.11}$$

Similar derivation fro Eq. (A.11) will lead to the following solution:

$$f_2(\varsigma) = H(\varsigma) \quad \text{for } |\varsigma| \geqslant 1, \tag{A.12}$$

where the function $H(\varsigma)$ is defined by

$$H(\varsigma) = \bar{h}\left(\frac{1}{\varsigma}\right) = \overline{h\left(\frac{1}{\bar{\varsigma}}\right)} \quad \text{for } |\varsigma| \geqslant 1. \tag{A.13}$$

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It is seen that under above-mentioned condition, Eq. (A.1) can be decomposed into two equations as follows:

$$f_1(\varsigma) - g(\varsigma) = 0 \quad \text{for } \varsigma = \exp(i\theta) \text{ on the unit circle,} \tag{A.14}$$

$$\overline{f_2(\varsigma)} - h(\varsigma) = 0, \quad \text{for } \varsigma = \exp(i\theta) \text{ on the unit circle.} \tag{A.15}$$

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