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Inverse source identification by Green's function

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ABSTRACT

Based on the use of Green's function, we propose in this paper a new approach for solving specific classes of inverse source identification problems. Effective numerical algorithms are developed to recover both the intensities and locations of unknown point sources from scattered boundary measurements. For numerical verification, several boundary value problems defined on both bounded and unbounded regions of regular shape are given. Due to the use of closed analytic form of Green's function, the efficiency and accuracy of the proposed method can be guaranteed.

1. Introduction

Let Ω be a domain in $\mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$. Consider the following Poisson equation:

$$\Delta u(x,y) = f(x,y) \quad \text{in } \Omega \tag{1.1}$$

with boundary condition

$$B_1[u(x,y)] = g(x,y) \quad \text{on } \Gamma \tag{1.2}$$

where $B_1[u]$ can be Dirichlet, Neumann, or Robin boundary operator. We note here that the domain Ω is not required to be bounded.

When f and g are both given and satisfy some smooth conditions, for instance, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, the above boundary value problem is a well-posed problem for the Poisson Eq. (1.1).

Unfortunately the source term f is not always known. In most practical problems, this source is unattainable except for some of its harmonic components. In real-life applications some scattered measurements of data on the boundary are needed for the recovery of the unknown source term f. This is called inverse source identification problem which is a typical ill-posed problem in the sense of Hadamard [1]. In other words, any small error in the scattered measurement data may induce enormous error to the solution.

Inverse source identification problems arise from many branches of engineering disciplines. For instances, crack determination [2,6], heat source determination [10], inverse heat conduction [16], electromagnetic source identification [39], and Stefan design [17] problems. Theoretical investigation on the inverse source identification problems can be found in the works of [7,31]. In general, the unknown source f in (1.1) can only be recovered from boundary measurements if some a prior knowledge is assumed. For instances, if one of the products in the separation of variables is known [8,36]; or the base area of a cylindrical source is known [8]; or a non-separable type is in the form of a moving front [36], then the boundary condition (1.2) plus some scattered boundary measurements can uniquely determine the unknown source term f. Furthermore, when both u and f are relatively smooth, some standard regularization techniques can be employed, see [14] for more details.

In the last decades, the method of fundamental solutions (MFS) has been developed for solving various partial differential equations [13,15,19,37,48]. The MFS was then extended successfully to solve inverse heat conduction problems [29,30]. The truly meshless computational advantage of the MFS has attracted the attentions of many authors. Recent works on using the MFS for solving inverse problems can be found in literatures such as inverse heat conduction [34,42,43,56]; cavity reconstruction [5]; eigenfrequencies and eigenmodes [3]; elasticity and elastostatics [40,45]; free boundary determination [27]; and Cauchy problems for Laplace [44,53,59,60], Helmholtz [33,41,46], and biharmonic [47] operators. It is well known that the placement of source points in the MFS plays an important role in the accuracy of the method. An optimal placement of the source points is still an open problem. Recently, there are some works on the choice for the location of source points, for instances, the placement of the source points on a circle containing the solution domain [35] or on a fictitious line below the initial time for inverse/backward heat conduction problems [29,30,50]. More references can be found in [51,52]. In particular, placement of source points depending on the shape of contours in using the MFS for solving elastic rod problem has been investigated in [20] with a

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conclusion that smaller error can be obtained if the source points are geometrically located along with the boundary shape of the domain. More recently, an optimal choice strategy based on discrepancy principle has been given in [28]. Applications of the MFS for solving inverse source identification problems can be found in [4,32,57,58].

In this paper we propose a new method based on the use of Green's function to solve inverse source identification problems. The main focus will be placed on the recovery of both intensities and locations of the unknown point source term f for Poisson equations. The proposed method does not require to choose the location of source points as needed in the MFS or generate a mesh as needed in Finite Difference Method (FDM) and Finite Element Method (FEM). In other words, the proposed method has the mesh-free advantage like MFS and choice-free advantage on the location of source points.

2. Statement of the problem

Assume that a potential field $u=u(x,y)$ inside the domain Ω can be generated by M distinct point sources $\{\theta_j=(\xi_j,\eta_j), j=1,\ldots,M\}$ satisfying

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=-\sum_{j=1}^{M} T_j\delta(x-\xi_j,y-\eta_j),\quad (x,y)\in\Omega \tag{2.1}$$

with boundary condition

$$B_1[u(x,y)]=0,\quad (x,y)\in\Gamma \tag{2.2}$$

where T_j denotes the intensity associated with each point source $(\xi_j,\eta_j)\in\Omega$ and $\delta(x,y)$ is the Dirac delta-function.

We first assume that the locations of the point sources $\{(\xi_j,\eta_j)\}$ (which can be pollution sources) are given with unknown intensities $\{T_j\}$ to be recovered from N distinct boundary measurements

$$B_2[u(x_i,y_i)]=\omega_i,\quad (x_i,y_i)\in\Gamma,\quad i=1,\ldots,N \tag{2.3}$$

where (x_i,y_i) are called collocation points and B_2 is a linear boundary operator. The measurement $\hat{\omega}_i$ of the boundary data ω_i usually contains noises with $\hat{\omega}_i=\omega_i+\varepsilon$ in which ε denotes the Gaussian variable with mean zero and variance δ. Here, the magnitude δ also represents the level of noises. The recovery evidently leads to an ill-posed problem.

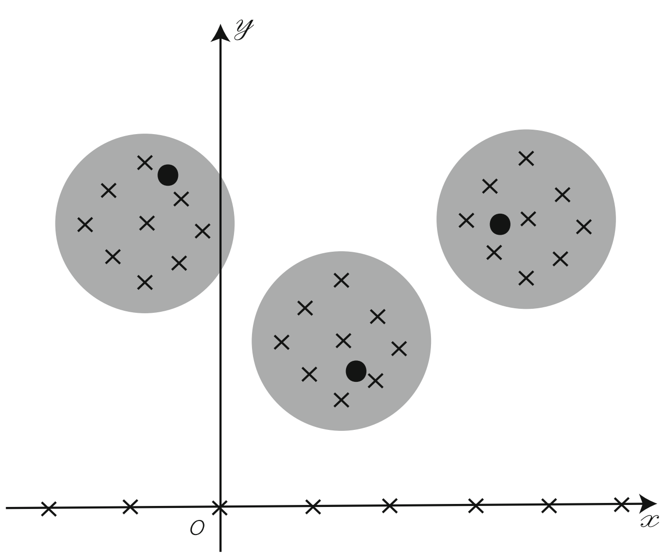

Fig. 1. Locations of point sources.

Secondly, we assume that the locations of the point sources $\theta_j=(\xi_j,\eta_j)$ are not known but an estimated location is given for each unknown point source. In other words, we assume that each point source will be included in a distinct ball inside the domain as

$$\theta_j\in B(\tilde{\theta}_j,r_j)\bigcap\Omega,\quad j=1,2,\ldots,M \tag{2.4}$$

and

$$B(\tilde{\theta}_j,r_j)\bigcap B(\tilde{\theta}_k,r_k)=\emptyset,\quad 1<j<k<M \tag{2.5}$$

where $B(\theta,r)$ denotes the ball centered at θ with radius r. See Fig. 1 for illustration.

The inverse source identification problem is then considered to be the ill-posed boundary value problem governed by Eqs. (2.1)–(2.3). Based on the idea of MFS and the use of Green's function, we devise in the following section a new method to recover the unknown point sources as specified on the right-hand side of Eq. (2.1) in the cases when the locations are either given or can be estimated. It is noted here that the difference between the two cases is significant because the intensities of the point sources have to be fixed in the first case and the problem becomes nonlinear in the second case when the locations of the source points are not given.

3. Methodology based on Green's function

Consider the Green's function $G(x,y;\xi,\eta)$ which satisfies

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=-\delta(x-\xi,y-\eta),\quad (x,y)\in\Omega \tag{3.1}$$

with boundary condition

$$B_1[u(x,y)]=0,\quad (x,y)\in\Gamma \tag{3.2}$$

In this paper we consider both bounded and unbounded domains Ω under different kinds of boundary conditions:

- Dirichlet boundary condition

  $$B_1[u(x,y)]=u(x,y)=0,\quad (x,y)\in\Gamma \tag{3.3}$$

  with Ω being the unbounded half-plane $\{-\infty<x<\infty,\ y>0\}$, whose classical Green's function

  $$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\sqrt{\frac{(x-\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}} \tag{3.4}$$

  can be found in nearly every text on Green's function [49];
- Dirichlet boundary condition

  $$B_1[u(r,\phi)]|_{\Gamma}=u(R,\phi)=0 \tag{3.5}$$

  with Ω being the disk of zero center and radius $R>0$, whose Green's function is given in the form of

  $$G(r,\phi;\rho,\psi)=\frac{1}{4\pi}\ln\frac{R^4-2r\rho R^2\cos(\phi-\psi)+r^2\rho^2}{R^2(r^2-2r\rho\cos(\phi-\psi)+\rho^2)} \tag{3.6}$$

- Robin boundary condition

  $$B_1[u(r,\phi)]|_{\Gamma}=\frac{\partial u(R,\phi)}{\partial r}+\beta u(R,\phi)=0,\quad \beta\geq 0 \tag{3.7}$$

  with Ω being the disk of zero center and radius R, whose Green's function when $\beta>0$ is given by

  $$G(r,\phi;\rho,\psi)=\frac{1}{2\pi}\left[\frac{1}{\beta R}+\ln\frac{R^3}{|z-\zeta||R^2-z\bar{\zeta}|}\right. \tag{3.8}$$

  $$\left.-2\beta R\sum_{m=1}^{\infty}\frac{1}{m(m+\beta R)}\left(\frac{r\rho}{R^2}\right)^m\cos m(\phi-\psi)\right] \tag{3.9}$$

  where $z=r(\cos\phi+i\sin\phi)$ and $\zeta=\rho(\cos\psi+i\sin\psi)$ [49].

Green's functions under other boundary conditions can also be found [49]. For most bounded domains Ω, their Green's functions can be approximated by an superposition of fundamental solutions.

Let $G(x,y;\xi,\eta)$ represent the Green's function for the boundary value problem defined by (3.1) and (3.2). Based on the idea of MFS, we assume that the solution of the inverse source identification problem (2.1), (2.2) and (2.3) can be expressed as the following linear combination:

$$u(x,y)=\sum_{j=1}^{M}T_jG(x-\xi_j,y-\eta_j) \tag{3.10}$$

Hence, as soon as the boundary measurements specified in (2.3) are obtained, the following ill-posed system of linear algebraic equations

$$B_2\left[\sum_{j=1}^{M}T_jG(x_i-\xi_j,y_i-\eta_j)\right]=\omega_i,\quad i=1,\ldots,N \tag{3.11}$$

gives the unknown values of the intensities T_j.

In matrix form, the values of the unknown intensities $\{T_j\}$ can be obtained from solving the following matrix equation:

$$AT=\omega \tag{3.12}$$

where A is a $N\times M$ matrix:

$$A_{ij}=B_2[G(x_i-\xi_j,y_i-\eta_j)]$$

with T and ω representing the corresponding $M\times 1$ and $N\times 1$ vectors, respectively. We now simply take $N=M$ so that the matrix A is a $N\times N$ square matrix whose solution can be found by using standard matrix inversion technique.

The matrix A in Eq. (3.12), however, is severely ill-conditioned. Most standard numerical methods cannot achieve good accuracy in solving the matrix Eq. (3.12) due to the bad conditioning of the matrix A. For illustration, notice that the condition number of the matrix A in the first example given in the next section on numerical verification is 2.2418×10^4 when N is 10. In fact, the condition number of the matrix A increases dramatically with respect to the increase of the total number of point sources as shown in Fig. 2. To obtain stable solutions to these kinds of ill-conditioning systems, some kinds of regularization techniques have to be used [22–25]. In our computations we adapt the standard Tikhonov regularization technique [55] to solve the matrix equation (3.12). The Tikhonov regularized solution T_α for equation (3.12) is defined to be the solution to the following least square problem:

$$\min_T\{\|AT-\omega\|^2+\alpha^2\|T\|^2\} \tag{3.13}$$

where $\|\bullet\|$ denotes the usual Euclidean norm and α is called the regularization parameter.

The determination of a suitable value of the regularization parameter α is crucial and is still under intensive research (refer [54,55]). In our computations we use the L-curve method, which is a kind of noise-free rules, to determine a suitable value of α. The L-curve method was developed by Lawson and Hanson [38] and had been successfully applied by Chen et al. [12] for solving deconvolution problem. Hansen and O'Leary [22] investigated the properties of regularized systems under different values of regularization parameter α. The L-curve method is sketched in the following:

Define a curve

$$L=\{(\log(\|T_\alpha\|^2),\ \log(\|AT_\alpha-\omega\|^2)),\alpha>0\}. \tag{3.14}$$

The curve is known as L-curve and a suitable regularization parameter α is the one that corresponds to a regularized solution near the "corner" of the L-curve [23–25].

In our computations we use the Matlab code developed by Hansen [26] for solving the discrete ill-conditioned system (3.12). The regularized solution of (3.12) is denoted by T_α^*.

Bakushinskii in his paper [9] proved that the convergence of the regularized solution T_α cannot be guaranteed in using noise-free rules. To the knowledge of the authors, there is still no convergence proof available in using other methods, such as Morozov's discrepancy principle and monotone error rule, to determine the regularization parameter [54]. Nevertheless, these methods all perform well in practice (for instances, see [18,21]). The numerical results given in the following section indicate that the proposed scheme is feasible and efficient.

A nonlinear system will be resulted if the locations $\{\theta_j\}$ of the point sources are unknown. It is well known that this kind of nonlinear system is difficult to solve. We propose in the following to linearize the nonlinear system to obtain a stable solution:

Let

$$\Theta:=\bigcup_{j=1}^{M}B(\tilde\theta_j,r_j)\bigcap\Omega \tag{3.15}$$

The union set Θ must contain all exact positions of the point sources. By choosing some addition collocation points $(v_j)_{j=1}^n$ uniformly distributed in Θ, the recovery of the unknown intensities $\{T_j\}$ can be obtained by solving:

$$\frac{\partial^2u(x,y)}{\partial x^2}+\frac{\partial^2u(x,y)}{\partial y^2}=-\sum_{j=1}^{n}\tau_j\delta(z-v_j),\quad z=(x,y)\in\Omega \tag{3.16}$$

with boundary condition

$$B_1[u(x,y)]=0,\quad (x,y)\in\Gamma \tag{3.17}$$

Assuming that the solution to Eqs. (3.16) and (3.17) can be represented by

$$u(x,y)=\sum_{j=1}^{n}\tau_jG(z-v_j) \tag{3.18}$$

the unknown intensity τ_j of the point source located at v_j can then be obtained by using similarly the above proposed method. We then transform the solution τ_j in each ball back to a single source point as follow: The j th unknown source intensity T_j associated with each ball $B(\tilde\theta_j,r_j)\bigcap\Omega$ is approximated by $\hat T_j$ as

$$\hat T_j:=\sum_{j:v_j\in B(\tilde\theta_j,r_j)\bigcap\Omega}\tau_j \tag{3.19}$$

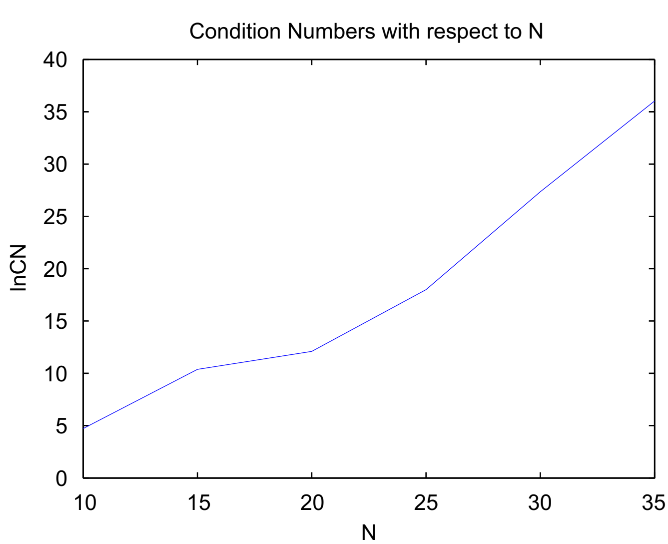

Fig. 2. Condition numbers with respect to N.

corresponding to the computed location of θ_j

$$\hat{\theta}_j := \frac{1}{\hat{T}_j} \sum_{j: v_j \in B(\tilde{\theta}_j, r_j) \bigcap \Omega} \tau_j v_j \tag{3.20}$$

In this way we transfer the nonlinear problem to a finite sequence of linear problems. In the next section, the numerical results illustrate that the proposed method for solving this nonlinear problem is fast and effective.

4. Numerical examples

For numerical verification, we apply in this section the proposed method derived in Section 3 to solve the inverse source identification problems for both bounded and unbound domains under several types of boundary conditions. In the computations we compare the accuracy of the approximation by using the root-mean-square-error:

$$\mathrm{RMSE} = \sqrt{\frac{1}{M}\sum_{j=1}^{M}(T^*_{\alpha;j} - T_j)^2} \tag{4.1}$$

Example 1 Consider the Dirichlet boundary value problem when the domain Ω is the unbounded half-plane $\{-\infty < x < \infty, y > 0\}$ under the Dirichlet boundary condition $B_1[u(x,y)]|_\Gamma = u(x,0) = 0$ and Neumann boundary measurement condition $B_2[u(x,y)]_\Gamma = \partial u(x,0)/\partial y$. The Green's function is given by (3.4).

For the first case when the locations of the point sources are known. We fix M point sources $\{(\xi_i, \eta_i)\}$ in the domain $[-1,1] \times [0,1.5]$. The intensities $\{T_j\}$ of these point sources are generated uniformly in [0,5]. The $N = M$ collocation points $\{(x_i, y_i)\}$ are chosen uniformly on the boundary $[-1,1] \times \{0\}$. From the representation formulae of u given in (3.10), we can then compute the values of $\hat{\omega_i}$ with δ representing the level of noises. The approximate intensities $\{\hat{T}_i\}$ can then be obtained by using the Tikhonov regularization method mentioned in the last section. The comparison between the approximate $\hat{T}_i$ and exact T_i shown in Table 1 indicates that our proposed method can recover very well the intensities of all the point sources. It has also been observed that the method achieves a better approximation at the point sources which are closer to the boundary of the solution domain. This can be explained by the ill-posed nature of inverse source identification problem.

Different noise levels δ have been used in our computations. The numerical approximation of the solution $\{T_j\}$ under different noise level δ is displayed in Table 2. It can be observed that, even for high noise level $\delta = 0.1$, the proposed method produces an acceptable numerical approximation.

The condition number of the matrix A increases dramatically with respect to the total number of point sources. The RMSE under different noise level δ when $M = 25$ is given in Table 3. Again, the numerical result indicates that the proposed method gives a reasonable result even for large number of point sources.

To further investigate the sensitivity of the method with respect to the distance between the point sources and the boundary of the solution domain, we choose five point sources on the same horizontal line in the domain Ω with *x-coordinate* equals to $\{-0.9, -0.5, -0.1, 0.3, 0.7\}$, respectively, and their same *y*-coordinate denotes the distance to be investigated. The numerical computation for noise level $\delta = 0.001$ displayed in Fig. 3 shows that the accuracy of the approximation decreases with respect to the increase of the distance y, which again can be explained by the nature of ill-posed inverse source identification problem.

Table 1
Numerical comparison for $\delta = 0.01$ and $M = 10$.

(ξ_i, η_i)	$(-0.9,0.1)$	$(-0.5,0.1)$	$(-0.1,0.1)$	$(0.3,0.1)$	$(0.7,0.1)$
Exact T	1	5	3.6	2.1	1
Numerical T^*_α	1.0934	4.9901	3.6161	2.0790	1.0088
$\lvert T - T^*_\alpha \rvert$	0.0934	0.0099	0.0161	0.0210	0.0088
(ξ_i, η_i)	$(-0.9,0.3)$	$(-0.5,0.3)$	$(-0.1,0.3)$	$(0.3,0.3)$	$(0.7,0.3)$
Exact T	1.5	2.3	5	1.6	0
Numerical T^*_α	1.201	2.4235	4.8973	1.704	-0.0501
$\lvert T - T^*_\alpha \rvert$	0.2990	0.1235	0.1027	0.1040	0.0501

Table 2
RMSE under different δ for $M = 25$.

δ	10^{-5}	10^{-4}	10^{-3}	0.01	0.05	0.1
RMSE	0.0297	0.0421	0.0963	0.1112	0.1252	0.1586

Table 3
RMSE under different M for $\delta = 0.01$.

M	10	15	20	25	30	35
RMSE	0.0348	0.0819	0.0921	0.1112	0.1115	0.1497

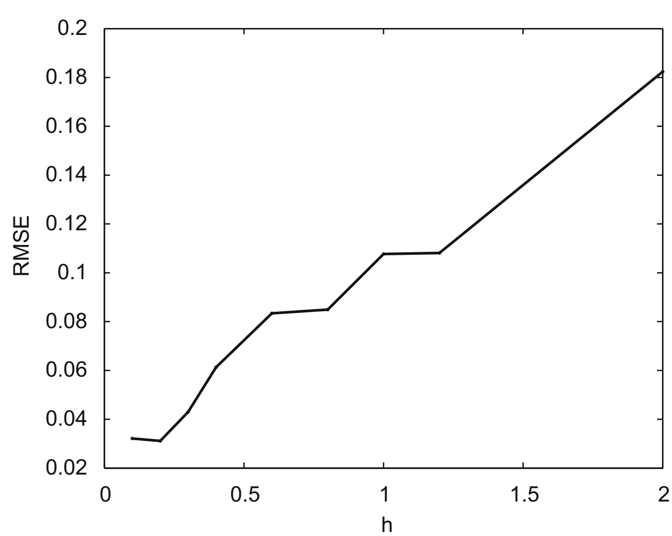

Fig. 3. RMSE with respect to the distance between the point sources and the boundary of the solution domain.

Secondly, consider the case when both the intensities and the locations of the point sources are not known. In this case, we choose three point sources located at (0.175,0.275), (0.9,0.1), (0.6,0.21) with intensities 10, 5, 1, respectively. The values of the parameters used in the computations are $\delta = 0.01$, $n = 20$, $N = 7$ and $r_j = 0.2$ for $j = 1,2,3$. The approximate point source locations are (1.766,0.27839), (0.90543,0.11667), (0.60483,0.2086) with approximate intensities 9.9894, 5.1956, 0.98663, respectively. See Fig. 4 for illustration. Furthermore, the RMSEs for different number n of additional collocation points is plotted in Fig. 5 which indicates that the accuracy of approximation increases dramatically with respect to the total number of additional collocation points. In other words, the proposed method does not

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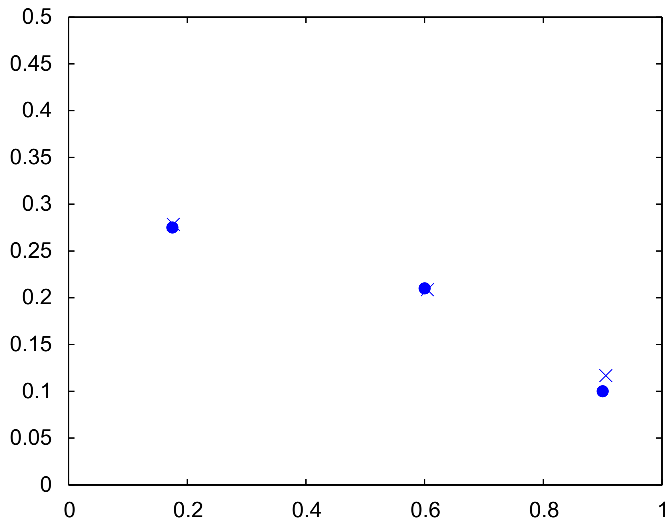

Fig. 4. • is the exact location and × is the approximate location.

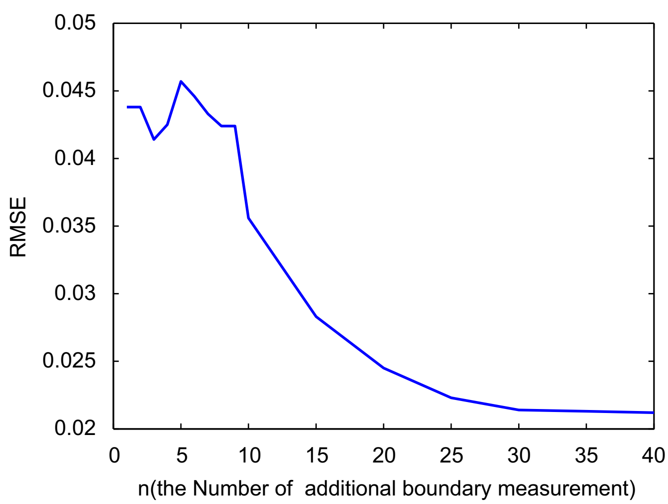

Fig. 5. RMSE with respect to different number of additional boundary measurements.

require a lot of additional collocation points for the approximation of the solution when both intensities and locations are unknown.

Example 2 In this example we consider the inverse identification problem when the domain is a disk of zero center and radius R under the Robin boundary condition. In the first case when the locations of the point sources are assumed to be known, we use the Robin boundary condition $B_1[u(r,\phi)]|_\Gamma$ as given by (3.7) and Neumann boundary measurement condition $B_2[u(R,\phi)]| = \frac{\partial u(R,\phi)}{\partial n}$. The Green's function is given by (3.8). In this computation we fix $\beta = 0.1$ and $R = 1$. By choosing M point sources (ξ_j, η_j) inside the disk with uniformly generated intensities $\{T_j\}$ in [0, 5] and $N = M$ collocation points $\{(x_i, y_i)\}$ located uniformly on the boundary, we can then compute the boundary measurements with noise level δ using the Neumann boundary condition. The comparison between the approximate intensities and the exact intensities given inTable 4 indicates that the proposed method can recover very well the intensities of all the point sources under more complicated boundary conditions.

Also, different noise levels δ have been used in the computations. The RMSEs for the approximation of the intensities $\{T_j\}$ under different noise levels δ is displayed in Table 5. It can be seen

Table 4
Numerical comparison for $\delta = 0.01$ and $M = 12$.

(ρ_i, ψ_i)	(0.8,0.2)	(0.8,1.3)	(0.8,2.4)	(0.8,3.5)	(0.8,4.6)	(0.8,5.7)
Exact T	1	1.7	3.6	2.1	1.5	2
Numerical T^*_α	0.9321	1.6199	3.5248	2.0378	1.4732	2.1174
$\lvert T\text{-}T^*_\alpha\rvert$	0.0679	0.0801	0.0752	0.0622	0.0268	0.1174
(ρ_i, ψ_i)	(0.9,0.2)	(0.9,1.3)	(0.9,2.4)	(0.9,3.5)	(0.9,4.6)	(0.9,5.7)
Exact T	1	2.3	2.6	1.6	1	2
Numerical T^*_α	1.0613	2.3777	2.6595	1.6578	1.0226	1.9166
$\lvert T\text{-}T^*_\alpha\rvert$	0.0613	0.0777	0.0595	0.0578	0.0226	0.0834

Table 5
RMSE under different δ for $M = 20$.

δ	10^{-5}	10^{-4}	10^{-3}	0.01	0.1
RMSE	3.0488×10^{-5}	3.8381×10^{-4}	0.0042	0.0616	0.5506

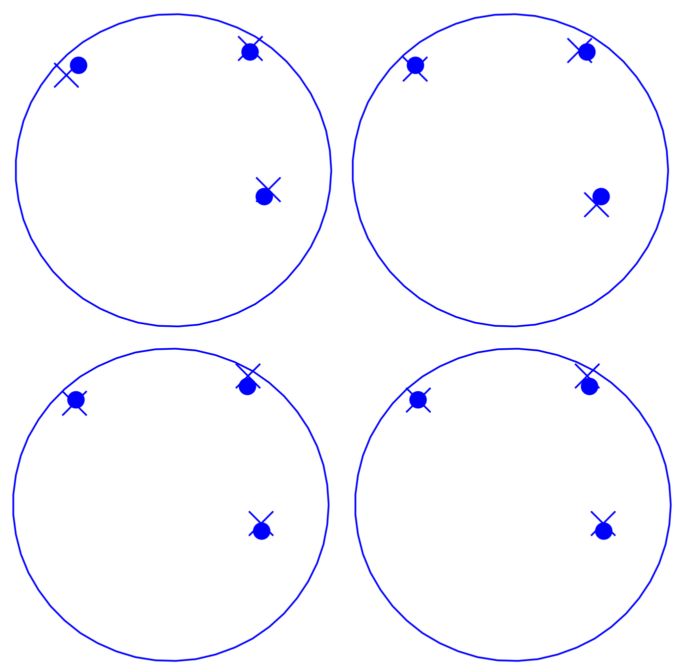

Fig. 6. Approximation under different boundary conditions where • is the exact location and × is the numerical location.

that even for high noise level $\delta = 0.1$, the proposed method produces an acceptable numerical approximation.

Secondly, we consider the case when both the intensities and the locations are not known. In this case, we choose three point sources located at (0.9,2.3), (0.9,1), (0.6,6) with intensities 9, 3, 2, respectively. The values of the parameters used in the computations are $\delta = 0.01$, $n = 20$, $N = 10$, and $r_j = 0.2$ for $j = 1, 2, 3$. The approximate locations (0.89822,2.3001), (0.94878,1.0515), (0.58423,6.0761) with approximate intensities 8.7083, 3.7884, 1.6980 show good agreement with the exact solutions. Furthermore, we compute the approximations of the unknown intensities and locations under different boundary conditions. The numerical results are displayed in Fig. 6 in which the above two figures represent the cases for Robin boundary condition with Dirichlet and Neumann boundary measurement conditions, respectively. The below two figures are the cases for Dirichlet boundary condition with Neumann and Robin boundary measurement conditions, respectively.

5. Conclusion

Based on the idea of the method of fundamental solutions and the use of Green's function, we develop in this paper a new computational method to solve inverse source identification problems. Numerical results indicate that the proposed method gives an accurate and reliable scheme for both bounded and unbounded domains under various boundary conditions. Since the proposed method enjoys both mesh-free and choice-free advantages, it is readily extendable to solve source recovery problem in higher dimension providing the Green's function is known.

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