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Formulation of indirect BIEs in plane elasticity using single or double layer potentials and complex variable

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ABSTRACT

This paper provides a formulation for indirect BIEs in plane elasticity using single or double layer potentials and complex variable. There are two ways to obtain two kinds of layer and the relevant indirect BIEs. In the first way, the displacement expression at domain point is directly obtained from the Somigliana identity with necessary modification. In the second way, after placing some density functions, for example, the body force or the dislocation doublet, along the layers, one can obtain the displacement expression at domain point. For both single and double layers, the continuous or discontinuous properties for the displacement and traction are studied in detail when a moving point is passing through the boundaries. Formulations of the Dirichlet and the Neumann problems are proposed. The ranges for solving the boundary value problem by using the single or double layer potentials are clearly indicated. For the case of single layer, the degenerate scale problems for the finite multiply connected region and infinite multiply connected region are studied. For the case of double layer, a hypersingular BIE for crack can be formulated by assuming that the density functions are vanishing along a portion of boundary.

1. Introduction

The boundary integral equations (BIEs) attracted many researchers in the recent fifty years. Some statistics shows that papers based on BIEs rank third among all main computational methods [1]. In old years, researchers could derive some BIEs. However, the relevant integral equations are difficult to solve numerically. Nowadays, computers can compute all formulated BIEs without any difficulties. A particular advantage of the BIE method is that the method can considerably reduce the dimensionality of unknowns in the solution, if one compares it with the finite element method (FE). Some pioneer works were proposed by some researches [2,3]. Some basic formulation of BIEs can be found in the literature [4,5]. In the meantime, the development of BIEs was summarized [1].

The direct BIEs in plane elasticity are generally formulated on Simigliana's identity [1,5]. In the direct BIEs, the involved functions are the displacements and tractions defined on the domain and the boundaries. Among direct BIEs, the dual integration formulation is significant [6–8]. The first BIE relates the displacement along the boundary to two integrals with the displacement and traction density functions along the boundary. The second BIE relates the traction along the boundary to two integrals with the displacement and traction density functions along the boundary.

In boundary element terminology, the single and double layer methods are referred to as the 'indirect methods'. The formulation of relevant potentials for the Laplace equation is compactly addressed [1]. For the indirect BIEs in plane elasticity, some source density is placed along the layer. An elasticity solution for displacement is defined on the entire plane by an expression of integration operator. Depending on the structure of the integration operator, there are two kinds of layers, or the single layer and the double layer.

An interpretation was carried out for the layers of elastic potentials, which are used to solve elastic boundary value problems of bodies [9]. The solution of a Dirichlet boundary value problem of plane isotropic elasticity by the boundary integral equation (BIE) of the first kind obtained from the Somigliana identity is considered. The logarithmic function appearing in the integral kernel leads to the possibility of this operator being non-invertible, the solution of the BIE either being non-unique or not existing [10,11].

Some researchers formulated the direct and indirect BIEs using complex variable [12–17]. Some researchers derived the complex BIE using the holomorphity theorem [16]. The theorem for a holomorphic analytic function $\phi(z)$ takes the form $\varphi(t_o) = (1/\pi i)\int_{\Gamma}(\varphi(t)dt/t - t_o)$, where Γ is the boundary of a finite region or an infinite region. Clearly, this theorem is only valid for holomorphic analytic function $\phi(z)$. In the boundary value

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problem of finite region, the relevant complex potentials must be some holomorphic analytic functions. However, in the infinite region bounded by many contours, the relevant complex potentials may contain a logarithmic function if the boundary tractions applied on contours are not in equilibrium. This is a reason why the holomorphity theorem is difficult to use in the exterior boundary value problem for two cases: (a) the boundary tractions applied on contours are not in equilibrium, (b) the Dirichlet problems with arbitrary displacements assumed on the boundary. For the problem (a), one can introduce a particular complex potential and the original problem can be reduced to a complementary problem that the tractions on contours are in equilibrium [18]. However, this procedure will make the solution more complicated.

This paper provides a formulation of indirect BIEs in plane elasticity using single or double layer potentials and complex variable. There are two ways of obtaining two kinds of layers and the relevant indirect BIEs.

In the first way, after using the Somigliana identity and the complex variable, the displacement expression at domain point is expressed by two integrals. In addition, one deletes one integral on the right hand side of the displacement expression, and renames the density function in the preserved integral. Thus, the displacement expression at domain point is expressed by one integral. The next step is to prove the displacement expression at domain point is an elasticity solution. In this way, the single and double layers can be obtained in the form of complex potential.

In the second way, based on a solution for the concentrated force in an infinite plate [18,19], the single layer potentials using complex variable can be formulated. Secondly, based on a solution for the dislocation doublet in an infinite plate [19], the double-layer potentials using complex variable can also be formulated.

For both single and double layers, the continuous or discontinuous properties for the displacement and traction are studied in detail when a moving point is passing through the boundaries. The simple-valued condition of displacement is examined, and the properties for the tractions applied on the contours are studied in detail. Formulations of the Dirichlet and the Neumann problems are studied. The ranges for solving the boundary value problem by using the single or double layer potentials are clearly indicated.

For the case of single layer, the degenerate scale problems for the finite multiply connected region and infinite multiply connected region are studied. From the double layer, a hypersingular BIE for the curved crack can be formulated by assuming that the density functions are vanishing along a portion of the boundary.

2. Preliminary knowledge

2.1. Some preliminary knowledge in complex variable method of plane elasticity

The complex variable function method plays an important role in plane elasticity. Fundamental of this method is introduced. In the method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of complex potentials $\phi(z)$ and $\psi(z)$ such that [18]

$$\sigma_x+\sigma_y=4\mathrm{Re}\Phi(z)$$

$$\begin{gathered}\sigma_y-\sigma_x+2i\sigma_{xy}=2[\bar{z}\Phi'(z)+\Psi(z)]\\ \text{or}\\ \sigma_y-\sigma_x-2i\sigma_{xy}=2[z\overline{\Phi'(z)}+\overline{\Psi(z)}]\end{gathered} \tag{1}$$

$$f=-Y+iX=\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)} \tag{2}$$

$$2G(u+iv)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)} \tag{3}$$

where $\Phi(z)=\varphi'(z)$, $\Psi(z)=\psi'(z)$, a bar over a function denotes the conjugated value for the function, G is the shear modulus of elasticity, $\kappa=(3-\nu)/(1+\nu)$ in the plane stress problem, $\kappa=3-4\nu$ in the plane strain problem, and ν is Poisson's ratio. Sometimes, the displacements u and v are denoted by u_1 and u_2, the stresses σ_x, σ_y and σ_{xy} by σ_1, σ_2 and σ_{12}, the coordinates x and y by x_1 and x_2.

For the sake of the following derivations, an equation for finding derivative is introduced as follows:

$$\frac{d}{dz}\{f(z)\overline{g(z)}\}=f'(z)\,\overline{g(z)}+\frac{d\bar{z}}{dz}(f(z)\overline{g'(z)}) \tag{4}$$

In Eqs. (4) and (5), $f(z)$ and $g(z)$ denote some analytic function. The derivative in Eq. (4) is named the derivative in a specified direction (abbreviated as DISD).

Except for the physical quantities mentioned above, from Eqs. (2) and (3) two derivatives in specified direction (abbreviated as DISD) are introduced as follows [19,20]:

$$J_1(z)=\frac{d}{dz}\{-Y+iX\}=\Phi(z)+\overline{\Phi(z)}+\frac{d\bar{z}}{dz}(z\overline{\Phi'(z)}+\overline{\Psi(z)})=\sigma_N+i\sigma_{NT} \tag{5}$$

$$\begin{aligned}J_2(z)=2G\frac{d}{dz}\{u+iv\}&=\kappa\Phi(z)-\overline{\Phi(z)}-\frac{d\bar{z}}{dz}(z\overline{\Phi'(z)}+\overline{\Psi(z)})\\ &=(\kappa+1)\Phi(z)-J_1\end{aligned} \tag{6}$$

It is easy to verify that $J_1=\sigma_N+i\sigma_{NT}$ denotes the normal and shear tractions along the segment $\overline{z,\ z+dz}$. Secondly, the J_1 and J_2 values depend not only on the position of a point z but also on the direction of the segment $d\bar{z}/dz$.

In plane elasticity, the following integrals are useful [18–20]:

$$F(z)=\frac{1}{2\pi i}\int_L\frac{f(t)dt}{t-z} \tag{7}$$

$$G(z)=\frac{1}{2\pi i}\int_L\frac{g(t)d\bar{t}}{t-z} \tag{8}$$

$$H(z,\bar{z})=\frac{1}{2\pi i}\int_L\frac{\bar{t}-\bar{z}}{(t-z)^2}h(t)dt \tag{9}$$

where L is a smooth curve or a closed contour Γ. Also, we assume that the functions $f(t)$, $g(t)$ and $h(t)$ satisfy the *Hölder* condition [18]. Sometimes, the functions $f(t)$, $g(t)$ and $h(t)$ are called the density functions hereafter. Clearly, the two integrals defined by Eqs. (7) and (8) are analytic functions, and one defined by Eq. (9) is not. The integral (7) is precisely the well-known Cauchy type integral.

Generally speaking, these integrals take different values when $z\to t_0^+$ and $z\to t_0^-$, $(t_0\in L)$. The limit values of these functions from the upper and lower sides of the curve L are found to be [18–20]

$$F^{\pm}(t_o)=\pm\frac{f(t_o)}{2}+\frac{1}{2\pi i}\int_L\frac{f(t)dt}{t-t_o} \tag{10}$$

$$G^{\pm}(t_o)=\pm\frac{g(t_o)}{2}\frac{d\bar{t}_o}{dt_o}+\frac{1}{2\pi i}\int_L\frac{g(t)d\bar{t}}{t-t_o} \tag{11}$$

$$H^{\pm}(t_o,\bar{t}_o)=\pm\frac{h(t_o)}{2}\frac{d\bar{t}_o}{dt_o}+\frac{1}{2\pi i}\int_L\frac{\bar{t}-\bar{t}_o}{(t-t_o)^2}h(t)dt \tag{12}$$

In Eqs. (10)–(19), all the integrals should be understood in the sense of principal value of the integral. Note that the notations of $f(t)$, $g(t)$, $h(t)$, $F(z)$, $G(z)$ and $H(z,\bar{z})$ used in Eqs. (10)–(12) have no relation with those mentioned in other places.

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2.2. Formulation of BIE using real variable or complex variable

In the following analysis, the α-field shown by Fig. 1(a) relates to the fundamental field caused by concentrated force at the point $z=\tau$. After using Betti's reciprocal theorem, or the Somigliana identity, between the fundamental field (or the α-field shown by Fig. 1(a)) and the physical field (or the β-field shown by Fig. 1(b)), for the plan strain case we have

$$u_i(\tau) = -\int_B P^*_{ij}(\tau,x)u_j(x)\,ds(x) + \int_B U^{*1}_{ij}(\tau,x)p_j(x)\,ds(x),$$
$$(i=1,2, \quad \tau \in S^+, \quad B=B_0+B_1+\cdots+B_N) \tag{13}$$

where region S^+ is a finite multiplied region bounded by contours B_j (j=0,1,2, …, N).

Letting the domain point τ ($\tau \in S^+$) approach a boundary point $\xi(\xi \in \Gamma)$, the following BIE is formulated [5]:

$$\frac{1}{2}u_i(\xi) = -\int_B P^*_{ij}(\xi,x)u_j(x)\,\mathrm{d}s(x) + \int_B U^{*1}_{ij}(\xi,x)p_j(x)\mathrm{d}s(x),$$
$$(i=1,2, \quad \xi \in B, \quad B=B_0+B_1+\cdots+B_N) \tag{14}$$

In the case of $\tau \to \xi$ ($\tau \in S^+$, $\xi \in \Gamma$), a jump value $u_i(\xi)/2$ is found on the right hand side of Eq. (13).

In Eq. (14), the kernel $P^*_{ij}(\xi,x)$ is defined by [5]

$$P^*_{ij}(\xi,x) = -\frac{2H_1}{r}\{(r_{,1}n_1+r_{,2}n_2)((1-2\nu)\delta_{ij}+2r_{,i}r_{,j}) + (1-2\nu)(n_ir_{,j}-n_jr_{,i})\} \tag{15}$$

$$H_1 = \frac{1}{8\pi(1-\nu)} = \frac{1}{2\pi(\kappa+1)} \quad \text{(in plane strain case)} \tag{16}$$

where Kronecker delta δ_{ij} is defined as δ_{ij}=1 for $i=j$, δ_{ij}=0 for $i \neq j$, and

$$r_{,1} = \frac{x_1-\xi_1}{r} = \cos\alpha, \quad r_{,2} = \frac{x_2-\xi_2}{r} = \sin\alpha \tag{17}$$

and $n(n_1,n_2)$ is a unit normal to the boundary at the field point $x(x_1,x_2)$.

In Eq. (14), the kernel $U^{*1}_{ij}(\xi,x)$ is defined by [21]

$$U^{*1}_{ij}(\xi,x) = \frac{H_1}{G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}-0.5\delta_{ij}\} \tag{18}$$

In addition, the following kernel was suggested in the literature [5]:

$$U^{*2}_{ij}(\xi,x) = \frac{H_1}{G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}\} \tag{19}$$

It is proved that the kernel shown by Eq. (19) cannot be used to the exterior problem in case the tractions on the contour are not in equilibrium [21].

Alternatively, the BIE can be derived by using the complex variable. In the following analysis, the α-field shown by Fig. 1(a) relates to the fundamental field caused by concentrated force at the point $z=\tau$. The relevant complex potentials are as follows [18–20]:

$$\varphi(z) = F\ln(z-\tau), \quad \varphi'(z) = \Phi(z) = \frac{F}{z-\tau}, \quad \varphi''(z) = -\frac{F}{(z-\tau)^2} \tag{20}$$

$$\psi(z) = -\kappa\bar{F}\ln(z-\tau) - \frac{F\bar{\tau}}{z-\tau}, \quad \psi'(z) = -\frac{\kappa\bar{F}}{z-\tau} + \frac{F\bar{\tau}}{(z-\tau)^2} \tag{21}$$

where

$$F = -\frac{P_x+iP_y}{2\pi(\kappa+1)} \tag{22}$$

In Eq. (22), P_x+iP_y is the concentrated force applied at the point $z=\tau$ in Fig. 1(a). Note that the complex potentials shown by Eqs. (20) and (21) are expressed in a pure deformable form [21].

The complex potentials shown by Eqs. (20) and (21) are defined in a fully infinite plane. From Eqs. (3), (20) and (21), we can evaluate the relevant displacement at the point t as follows (Fig. 1):

$$2G(u+iv)_* = 2\kappa F\ln|t-\tau| - \bar{F}\frac{t-\tau}{\bar{t}-\bar{\tau}} \tag{23}$$

Similarly, from Eqs. (4), (20) and (21) we can evaluate the relevant boundary traction at the point t as follows (Fig. 1):

$$(\sigma_N+i\sigma_{NT})_* = \frac{F}{t-\tau} + \frac{\bar{F}}{\bar{t}-\bar{\tau}} + \frac{d\bar{t}}{dt}\left(-\frac{\kappa F}{\bar{t}-\bar{\tau}} - \frac{\bar{F}(t-\tau)}{(\bar{t}-\bar{\tau})^2}\right) \tag{24}$$

In Eqs. (23) and (24), the subscript "$*$" denotes that the arguments are derived from the fundamental solution.

After using Betti's reciprocal theorem, or the Somigliana identity, between the fundamental field (or the α-field in Fig. 1(a)) and the physical field (or the β-field in Fig. 1(b)), we have

$$P_xu(\tau)+P_yv(\tau)+\mathrm{Re}\left(\int_B(u-iv)(dX+idY)_*\right) = \mathrm{Re}\left(\int_B(u-iv)_*(dX+idY)\right),$$
$$(\tau \in S^+, \quad B=B_0+B_1+\cdots+B_N) \tag{25}$$

where the left hand term represents the work done by traction in the fundamental field (the α-field) to the displacement of the physical field (the β-field). In addition, the right hand term represents the work done by traction in the physical field (the β-field) to the displacement of the fundamental field (the α-field).

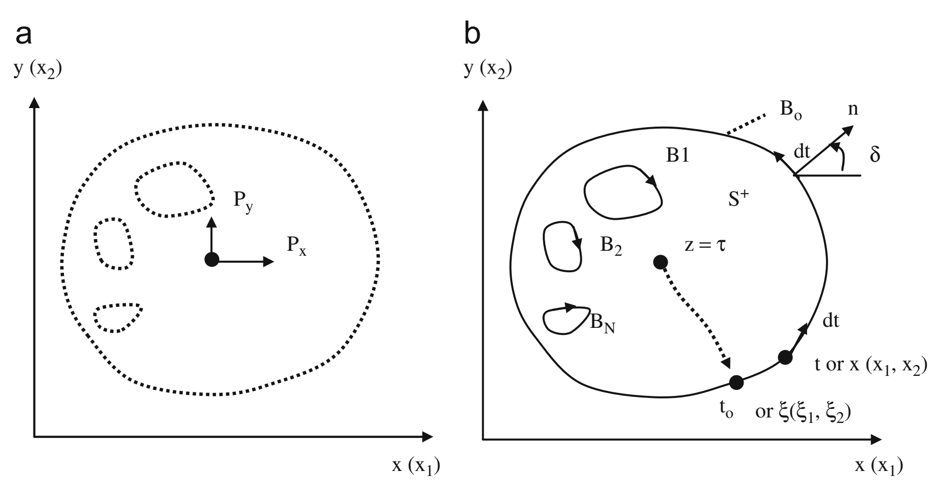

Fig. 1. (a) the α-field caused by concentrated forces, (b) the β-field defined by a physical field.

In Eq. (25), $dX+idY$ denotes the force applied on the segment dt (Fig. 1). From Eq. (2) and Fig. 1, we can find

$$dX+idY=(\sigma_N+i\sigma_{NT})e^{i\delta}ds, \quad dt=ie^{i\delta}ds, \quad dX+idY=-i(\sigma_N+i\sigma_{NT})dt \tag{26}$$

where δ denotes an inclined angle for the normal at a boundary point (Fig. 1).

Thus, Eq. (25) can be rewritten as

$$P_x u(\tau)+P_y v(\tau)+\mathrm{Re}\int_B(-i)(\sigma_N+i\sigma_{NT})_*(u-iv)\,dt = \mathrm{Re}\int_B(-i)(u-iv)_*(\sigma_N+i\sigma_{NT})\,dt \quad (\tau\in S^+) \tag{27}$$

Substituting the explicit form for $(\sigma_N+i\sigma_{NT})_*$ and $(u\text{–}iv)_*$ into Eq. (27) yields

$$P_x u(\tau)+P_y v(\tau)+\mathrm{Re}\int_B(-i)\left(\frac{F}{t-\tau}+\frac{\overline{F}}{\overline{t}-\overline{\tau}}+\frac{d\overline{t}}{dt}\left(-\frac{\kappa F}{\overline{t}-\overline{\tau}}-\frac{\overline{F}(t-\tau)}{(\overline{t}-\overline{\tau})^2}\right)\right)(u-iv)dt = \frac{1}{2G}\mathrm{Re}\int_B(-i)\left(2\kappa\overline{F}\ln|t-\tau|-F\frac{\overline{t}-\overline{\tau}}{t-\tau}\right)(\sigma_N+i\sigma_{NT})\,dt \quad (\tau\in S^+) \tag{28}$$

In the following analysis, one can let

$$U(t)=u(t)+iv(t), \quad Q(t)=d(-Y(t)+iX(t))/dt=\sigma_N(t)+i\sigma_{NT}(t) \quad (t\in B,\ B=B_0+B_1+\cdots+B_N) \tag{29}$$

In Eq. (28), if we let P_x=1, P_y=0 and $F=-1/2\pi(\kappa+1)$, we can find an equation for $u(\tau)$. Similarly, if we let P_x=0, P_y=1 and $F=-i/2\pi(\kappa+1)$, we can find an equation for $v(\tau)$. Thus, we will find

$$\begin{aligned} u(\tau)+iv(\tau)&=H_1 i\int_B\left(-\frac{1}{\overline{t}-\overline{\tau}}+\frac{d\overline{t}}{dt}\frac{(t-\tau)}{(\overline{t}-\overline{\tau})^2}\right)\overline{U(t)}\,dt \\ &+H_1 i\int_B\left(\frac{1}{\overline{t}-\overline{\tau}}-\frac{dt}{d\overline{t}}\frac{\kappa}{t-\tau}\right)U(t)\,d\overline{t} \\ &+\frac{H_1 i}{2G}\int_B\kappa[\ln(t-\tau)+\ln(\overline{t}-\overline{\tau})]Q(t)\,dt \\ &+\frac{H_1 i}{2G}\int_B\frac{t-\tau}{\overline{t}-\overline{\tau}}\overline{Q(t)}\,d\overline{t} \ (\tau\in S^+) \end{aligned} \tag{30}$$

Eq. (30) can be rewritten as

$$\begin{aligned} U(\tau)&=u(\tau)+iv(\tau) \\ &=H_1 i\int_B\left(-\frac{(\kappa-1)}{t-\tau}U(t)\,dt+L_1(t,\tau)U(t)\,dt-L_2(t,\tau)\overline{U(t)}\,dt\right) \\ &+\frac{H_1 i}{2G}\int_B\kappa[\ln(t-\tau)+\ln(\overline{t}-\overline{\tau})]Q(t)\,dt \\ &+\frac{H_1 i}{2G}\int_\Gamma\frac{t-\tau}{\overline{t}-\overline{\tau}}\overline{Q(t)}\,d\overline{t} \quad (\tau\in S^+) \end{aligned} \tag{31}$$

where

$$L_1(t,\tau)=-\frac{d}{dt}\left\{\ln\frac{t-\tau}{\overline{t}-\overline{\tau}}\right\}=-\frac{1}{t-\tau}+\frac{1}{\overline{t}-\overline{\tau}}\frac{d\overline{t}}{dt} \tag{32}$$

$$L_2(t,\tau)=\frac{d}{dt}\left\{\frac{t-\tau}{\overline{t}-\overline{\tau}}\right\}=\frac{1}{\overline{t}-\overline{\tau}}-\frac{t-\tau}{(\overline{t}-\overline{\tau})^2}\frac{d\overline{t}}{dt} \tag{33}$$

An expression similar to Eq. (31) was suggested in [14, Eq. (14)], where the term for traction $Q(t)$ is replaced by the resultant force function $f=-Y+iX$. It will be discussed in Appendix A that the formulation suggested in [14, Eq. (14)] has an inconvenient point, which can only be used in case the tractions on contour are in equilibrium in the exterior boundary value problem (BVP).

The kernel functions in Eq. (31) are expressed in an explicit form. Thus, the expression is more convenient in derivation when a domain point τ approaches the boundary point t_o, or $\tau\to t_o$ ($\tau\in S^+$, $t_o\in B$ in Fig. 1).

In Eq. (31), letting $\tau\to t_o$ ($\tau\in S^+$, $t_o\in B$) and using the generalized Sokhotski–Plemelj formulae shown by Eqs. (10)–(12) and some results in Appendix B, yields

$$\begin{aligned} \frac{U(t_o)}{2}&=H_1 i\int_B\left(-\frac{\kappa-1}{t-t_o}U(t)\,dt+L_1(t,t_o)U(t)\,dt-L_2(t,t_o)\overline{U(t)}\,dt\right) \\ &+\frac{H_1 i}{2G}\int_B\left(2\kappa\ln|t-t_o|Q(t)\,dt+\frac{t-t_o}{\overline{t}-\overline{t_o}}\overline{Q(t)}\,d\overline{t}\right) \quad (t_o\in\Gamma_B) \end{aligned} \tag{34}$$

It is noted here that, when taking the limit process $\tau\to t_o$ for the first integral on the right hand side of Eq. (31), an additional term $U(t_o)/2$ was found. Thus, the left hand term in Eq. (34) becomes $U(t_o)/2$ (note that $U(t_o)/2=U(t_o)-(U(t_o)/2)$). In the real variable BIE, this property has been obtained previously [5]. However, the property is obtained in a more explicit way in this paper, by using the generalized Sokhotski–Plemelj formulae shown by Eqs. (10)–(12) and some results in Appendix B. It is easy to prove that the complex BIE shown by (34) is equivalent to its counterpart in a real variable form shown by Eq. (14).

3. Indirect BIEs based on single layer potential using complex variable

3.1. Formulation of indirect BIEs based on single layer potential using complex variable

After taking the following steps: (1) deleting one term containing the function $U(t)$ on the right hand side of Eq. (31), (2) the function $Q(t)$ is replaced by $q(t)$ and (3) τ is rewritten as z, we can propose a displacement expression:

$$u(z)+iv(z)=\sum_{j=0}^{N}(u(z)+iv(z))_j \tag{35}$$

where

$$(u(z)+iv(z))_j=\frac{H_1 i}{2G}\int_{B_j}\kappa[\ln(t-z)+\ln(\overline{t}-\overline{z})]q_j(t)\,dt+\frac{H_1 i}{2G}\int_{B_j}\frac{t-z}{\overline{t}-\overline{z}}\overline{q_j(t)}\,d\overline{t} \tag{36}$$

Now the function $q_j(t)$ (j=0,1,2, …, N) in Eq. (36) is the body force density function rather than the traction on the boundary (Fig. 2).

It is easy to see that the displacement expression shown by Eq. (35) corresponds to the following complex potentials:

$$\varphi(z)=\sum_{j=0}^{N}\varphi_j(z),\ \psi(z)=\sum_{j=0}^{N}\psi_j(z) \tag{37}$$

where

$$\varphi_j(z)=H_1 i\int_{B_j}\ln(t-z)q_j(t)\,dt \quad (j=0,1,2,\ldots,N) \tag{38}$$

$$\psi_j(z)=H_1 i\int_{B_j}\kappa\ln(t-z)\overline{q_j(t)}\,d\overline{t}+H_1 i\int_{B_j}\frac{\overline{t}}{t-z}q_j(t)\,dt \quad (j=0,1,2,\ldots,N) \tag{39}$$

If one substitutes the complex potentials Eqs. (38) and (39) into Eq. (3), the displacement $(u(z)+iv(z))_j$ (j=0,1,2, …, N) is exactly expressed by Eq. (36). Thus, the displacement expression shown by Eq. (36) is an elasticity solution.

It can be proved that the complex potentials shown by Eqs. (38) and (39) correspond to a body force layer placed along the contour B_j (j=0, 1, 2, …, N). In fact, if a concentrated body force with the intensity "F" is applied at the point z=t (Fig. 3), the relevant complex potentials are as follows [18–20]:

$$\varphi(z)=F\ln(t-z),\ \psi(z)=-\kappa\overline{F}\ln(t-z)+\frac{F\overline{t}}{t-z} \tag{40}$$

where

$$F=F_x+iF_y=\frac{P_y-iP_x}{2\pi(\kappa+1)i} \tag{41}$$

We make a substitution $F=H_1 iq_j(t)$ and $\overline{F}=-H_1 i\overline{q_j(t)}$ in Eqs. (40) and (41), and an integration with respect to dt, and the complex potentials shown by Eqs. (38) and (39) are obtained. Thus, the assertion is proved.

Assuming $z\to t_o^+$ and $z\to t_o^-$ ($t_o\in B_j$), and using the generalized Sokhotski–Plemelj formulae Eqs. (10)–(12) and some results in Appendix B, from Eq. (36) we will find

$$\begin{aligned}[(u(t_o)+iv(t_o))_j]_{B_j}^{\pm}&=\frac{H_1 i}{2G}\int_{B_j}\kappa[\ln(t-t_o)+\ln(\overline{t}-\overline{t}_o)]q_j(t)\,dt\\&+\frac{H_1 i}{2G}\int_{B_j}\frac{t-t_o}{\overline{t}-\overline{t}_o}\overline{q_j(t)}\,d\overline{t}\quad(j=0,1,2,\ldots,N)\end{aligned} \tag{42}$$

$$[(u(t_o)+iv(t_o))_j]_{B_j}^{+}-[(u(t_o)+iv(t_o))_j]_{B_j}^{-}=0,\quad(j=0,1,2,\ldots,N) \tag{43}$$

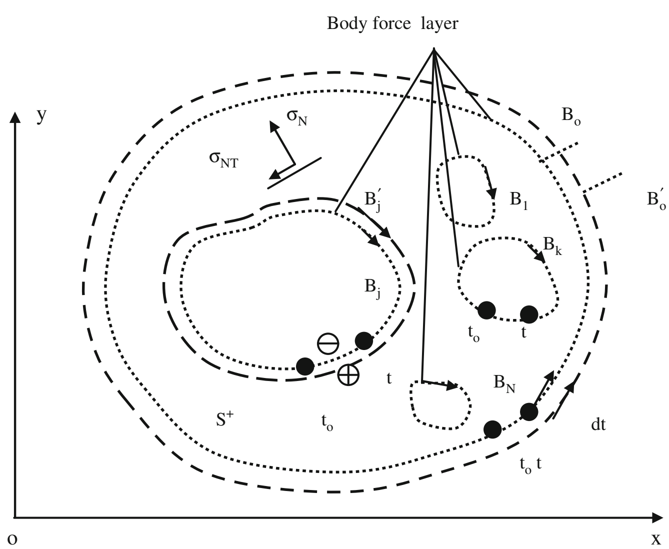

Fig. 2. Multiply connected region with body force layers along $B=B_0+B_1+\cdots+B_N$.

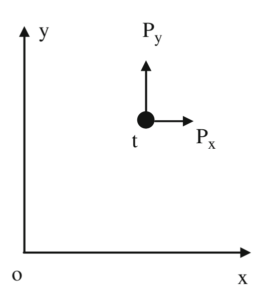

Fig. 3. Concentrated forces in an infinite plate.

From Eqs. (42) and (43), we see that the displacements are continuous in case of a moving point t_o across the boundary B_j (j=0,1,2,…N).

Substituting Eqs. (38) and (39) into (2) yields the resultant force function

$$-Y(z)+iX(z)=\sum_{j=0}^{N}(-Y(z)+iX(z))_j \tag{44}$$

where

$$\begin{aligned}&(-Y(z)+iX(z))_j\\&=-H_1 i\int_{B_j}(\kappa-1)\ln(t-z)q_j(t)\,dt\\&+H_1 i\int_{B_j}\kappa[\ln(t-z)-\ln(\overline{t}-\overline{z})]q_j(t)\,dt\\&-H_1 i\int_{B_j}\frac{t-z}{\overline{t}-\overline{z}}\overline{q_j(t)}\,d\overline{t},\quad(j=0,1,2,\ldots,N)\end{aligned} \tag{45}$$

or

$$\begin{aligned}(-Y(z)+iX(z))_j&=H_1 i\int_{B_j}(\kappa+1)\ln(t-z)q_j(t)\,dt\\&-H_1 i\int_{B_j}\kappa[\ln(t-z)+\ln(\overline{t}-\overline{z})]q_j(t)\,dt\\&-H_1 i\int_{B_j}\frac{t-z}{\overline{t}-\overline{z}}\overline{q_j(t)}\,d\overline{t},\quad(j=0,1,2,\ldots,N)\end{aligned} \tag{46}$$

From Eqs. (5), (44) and (45), the traction $\sigma_N(z)+i\sigma_{NT}(z)$ at a domain point can be evaluated as follows:

$$\sigma_N(z)+i\sigma_{NT}(z)=\sum_{j=0}^{N}(\sigma_N(z)+i\sigma_{NT}(z))_j \tag{47}$$

where

$$\begin{aligned}(\sigma_N(z)+i\sigma_{NT}(z))_j&=\frac{d}{dz}((-Y(z)+iX(z))_j)\\&=H_1 i\int_{B_j}\frac{\kappa-1}{t-z}q_j(t)\,dt+H_1 i\int_{B_j}\kappa K_1(t,z)q_j(t)\,dt\\&+H_1 i\int_{B_j}K_2(t,z)\overline{q_j(t)}\,d\overline{t}\end{aligned} \tag{48}$$

In Eq. (48), two kernels are defined by

$$\begin{aligned}K_1(t,z)&=\frac{d}{dz}\left\{\ln\frac{t-z}{\overline{t}-\overline{z}}\right\}=-\frac{1}{t-z}+\frac{1}{\overline{t}-\overline{z}}\frac{d\overline{z}}{dz},\\K_2(t,z)&=-\frac{d}{dz}\left\{\frac{t-z}{\overline{t}-\overline{z}}\right\}=\frac{1}{\overline{t}-\overline{z}}-\frac{t-z}{(\overline{t}-\overline{z})^2}\frac{d\overline{z}}{dz}\end{aligned} \tag{49}$$

Similarly, assuming $z\to t_o^+$ and $z\to t_o^-$ ($t_o\in B_j$), and using the generalized Sokhotski–Plemelj formulae Eqs. (10)–(12) and some results in Appendix B, from Eq. (48) we will find

$$\begin{aligned}&[(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]_{B_i}^{\pm}\\&=\pm\frac{q_j(t_o)}{2}+H_1 i\int_{B_j}\frac{\kappa-1}{t-t_o}q_j(t)\,dt\\&+H_1 i\int_{B_j}\kappa K_1(t,t_o)q_j(t)\,dt\\&+H_1 i\int_{B_j}K_2(t,t_o)\overline{q_j(t)}\,d\overline{t},\quad(t_o\in B_j)\end{aligned} \tag{50}$$

$$[(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]_{B_j}^{+}-[(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]_{B_j}^{-}=q_j(t_o),(t_o\in B_j) \tag{51}$$

From Eq. (51) we find that the tractions are discontinuous when a moving point t_o across the boundary B_j (j=0,1,2,…,N).

In addition, when a moving point "z" goes forward in the clockwise direction along the contour B_j' (j=1,2, …, N), the increase for a function $f(z)$ is denoted by $f(z)_{B_j',inc}$. Similarly, when a moving point z goes forward in the anti-clockwise direction along the contour B_o', the increase for a function $f(z)$ is denoted by $f(z)_{B_o',inc}$.

From Eq. (36), we have

$$\{(u(z)+iv(z))_k\}_{B'_j,\,inc}=0 \quad (k,j=0,1,2,\ldots,N) \tag{52}$$

From Eq. (46), properties for the resultant force functions can be obtained as follows:

$$\{(-Y(z)+iX(z))_0\}_{B'_0,\,inc}=-\int_{B_0} q_0(t)\,dt \tag{53a}$$

$$\{(-Y(z)+iX(z))_j\}_{B'_j,\,inc}=\int_{B_j} q_j(t)\,dt \quad (j=1,2,\ldots,N) \tag{53b}$$

$$\begin{aligned}\{(-Y(z)+iX(z))_j\}_{B'_0,\,inc} &= -\{(-Y(z)+iX(z))_j\}_{B'_j,\,inc}\\ &= -\textstyle\int_{B_j} q_j(t)dt, \quad (j=1,2,\ldots,N)\end{aligned} \tag{53c}$$

$$\{(-Y(z)+iX(z))_0\}_{B'_j,\,inc}=0 \quad (j=1,2,\ldots,N) \tag{53d}$$

$$\{(-Y(z)+iX(z))_j\}_{B'_k,\,inc}=0 \quad (j,\ k=1,2,\ldots,N,\ \ j\neq k) \tag{53e}$$

Finally, we have

$$\{(-Y(z)+iX(z))\}_{B'_0,\,inc}=-\sum_{j=0}^{N}\int_{B_j} q_j(t)\,dt \tag{54}$$

$$\{(-Y(z)+iX(z))\}_{B'_j,\,inc}=\int_{B_j} q_j(t)\,dt \quad (j=1,2,\ldots,N) \tag{55}$$

Eqs. (53a) and (53b) reveal that the applied tractions along the contours B'_j (j=0,1,2, …, N) may not be in equilibrium in forces.

In addition, a moment function is defined as follows [18]:

$$m(z)=\sum_{j=0}^{N} m_j(z) \tag{56}$$

where

$$m_j(z)=\mathrm{Re}\{\chi_j(z)-z\psi_j(z)-z\bar{z}\varphi'_j(z)\}, \quad \text{with } \chi_j(z)=\int_{z_o}^{z}\psi_j(z)dz \tag{57}$$

where z_o is a fixed point, and z is variable.

From Eq. (39), we have

$$\begin{aligned}\chi_j(z) &= \int_{z_o}^{z}\psi_j(z)dz\\ &= -H_1 i\int_{B_j}\kappa[(t-z)\ln(t-z)+z]\overline{q_j(t)}\,d\bar{t}\\ &\quad -H_1 i\int_{B_j}\ln(t-z)\bar{t}q_j(t)\,dt, \quad (j=0,1,2,\ldots,N)\end{aligned} \tag{58}$$

$$\begin{aligned}\chi_j(z)-z\psi_j(z) &= -H_1 i\int_{B_j}\kappa[t\ln(t-z)+z]\overline{q_j(t)}d\bar{t}\\ &\quad -H_1 i\int_{B_j}\ln(t-z)\bar{t}q_j(t)\mathrm{d}t\\ &\quad -H_1 i\int_{B_j}\frac{\bar{t}z}{t-z}q_j(t)\,\mathrm{d}t, \quad (j=0,1,2,\ldots N)\end{aligned} \tag{59}$$

Thus, from Eqs. (57) and (59), we will find

$$\begin{aligned}\{m_0(z)\}_{B'_o\ inc} &= -2\pi H_1\mathrm{Re}\textstyle\int_{B_o}(\kappa t\overline{q_0(t)}\,d\bar{t}+\bar{t}q_0(t)\,dt)\\ &= -\mathrm{Re}\textstyle\int_{B_o}\bar{t}q_0(t)\,dt\end{aligned} \tag{60}$$

$$\{m_j(z)\}_{B'_j\ \mathrm{inc}}=2\pi H_1\ \mathrm{Re}\int_{B_j}(\kappa t\overline{q_j(t)}d\bar{t}+\bar{t}q_j(t)\mathrm{d}t)=\mathrm{Re}\int_{B_j}\bar{t}q_j(t)\mathrm{d}t \tag{61}$$

where $\{m_k(z)\}_{B'_j,\,inc}$ represents the moment applied on contour B'_j caused by the kth density distribution along the contour B_k.

From Eqs. (53a), (53b), (60) and (61), we see that the boundary loadings applied on the contours $B_j(t)$ (j=0,1,2, …, N) may not be in equilibrium in forces and moment.

3.2. Formulations of boundary value problems

The Dirichlet problem is formulated first. It is assumed that the displacements along the contours $t_o\in B_k$ (k=0,1,2, …, N) have been given beforehand

$$(u(t_o)+iv(t_o))^+_{B_k}=(\tilde{u}(t_o)+i\tilde{v}(t_o))_k, \quad (t_o\in B_k,\ k=0,1,2,\ N) \tag{62}$$

In Eq. (62), $(\tilde{u}(t_o)+i\tilde{v}(t_o))_k$ (k=0,1,2, …, N) are given functions.

Assuming $z\to t_o^+$ ($t_o\in B_k$, k=0,1,2, …, N) in Eqs. (35), (36), using Eqs. (42) and (62) and the principle of superposition, we will find the following integral equation:

$$\begin{aligned}&\frac{H_1 i}{2G}\sum_{j=0}^{N}\int_{B_j}\left\{\kappa[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)]q_j(t)dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_j(t)}\,d\bar{t}\right\}\\ &\qquad=(\tilde{u}(t_o)+i\tilde{v}(t_o))_k \quad (t_o\in B_k,\ k=0,1,2,\ldots,N)\end{aligned} \tag{63}$$

Physically, along the contours B_k (k=0,1,2, …, N) we can assume any deformation expressed by $(\tilde{u}(t_o)+i\tilde{v}(t_o))_k$ ($t_o\in B_k$, k=0,1,2, …, N). Clearly, regardless of the assumed boundary deformation $(\tilde{u}(t_o)+i\tilde{v}(t_o))_k$ ($t_o\in B_k$, k=0,1,2, …, N), the Dirichlet problem governed by Eq. (63) has a definite solution, when the degenerate scale has not been reached.

In Eq. (63), when the integration (dt) is performed along the contour B_k (k=0,1,2, …, N), and the observation point t_o is also on the contour B_k, a weaker kernel $\ln(t-t_o)$ is presented. It is known that Eq. (63) may have non-unique solution when the degenerate scale is reached. The degenerate scale problem arising from Eq. (63) will be discussed below.

The Neumann problem is formulated secondly. It is assumed that the tractions along the contours $t_o\in B_k$ (k=0,1,2, …, N) have been given beforehand

$$(\sigma_N(t_0)+i\sigma_{NT}(t_0))^+_{B_k}=(\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k, \quad (t_0\in B_k,\ k=0,1,2,\ N), \tag{64}$$

In Eq. (64), $(\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0)))_k$ (k=0,1,2, …, N) are given functions.

Similarly, letting $z\to t_o^+$ ($t_o\in B_k$, k=0,1,2, …, N) in Eqs. (47), (48), using Eqs. (50), (64) and the principle of superposition, we will find the following integral equation:

$$\begin{aligned}&\frac{q_k(t_o)}{2}+H_1 i\int_{B_k}\left\{\left(\frac{\kappa-1}{\bar{t}-\bar{t}_o}\frac{d\bar{t}_o}{dt_o}+K_1(t,t_o)\right)q_k(t)\,\mathrm{d}t+K_2(t,t_o)\overline{q_k(t)}\,d\bar{t}\right\}\\ &\quad+H_1 i{\sum_{j=0}^{N}}'\int_{B_j}\left\{\left(\frac{\kappa-1}{\bar{t}-\bar{t}_o}\frac{d\bar{t}_o}{dt_o}+K_1(t,t_o)\right)q_j(t)\,\mathrm{d}t+K_2(t,t_o)\overline{q_j(t)}\,d\bar{t}\right\}\\ &\quad=(\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k \quad (t_o\in B_k,\ k=0,1,2,\ldots N)\ (t_o\in B_k,\\ &\quad k=0,1,2,\ldots N)\end{aligned} \tag{65}$$

In Eq. (65), the prime in ${\sum_{j=0}^{N}}'$ means that the term j=k should be excluded in the summation.

In the formulation, the individual traction $\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k$ $(t_o\in B_k,\ k=0,1,2,\ldots,N)$ applied on kth contour may not be in equilibrium. However, all the tractions $\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k$ $(t_o\in B_k,\ k=0,1,2,\ldots,N)$ must be in equilibrium. In the meantime, if there is no outer boundary B_0, the individual traction $\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k$ $(t_o\in B_k,\ k=0,1,2,\ldots,N)$ applied on kth contour may not be in equilibrium.

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3.3. Formulation of the degenerate scale problem for finite multiply connected region

The degenerate scale problem is a particular problem arising from the Dirichlet problem defined by Eq. (63). Many degenerate scale problems were studied and solved [10,11,22–25]. A physical explanation for the existence of a degenerate scale can be referred to [25]. It is necessary to consider the problem in two cases: (1) with the outer boundary B_o and (2) without the outer boundary B_o (Fig. 2).

For the first case, the degenerate scale problem is formulated for the finite multiply connected region (Fig. 4). After letting the right hand term in Eq. (63) to be zero, we will obtain the following homogenous BIE:

$$\sum_{j=0}^{N}\int_{B_j}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_j(t)\,dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_j(t)}\,d\bar{t}\right\}=0 \qquad (t_o\in B_k,\ k=0,1,2,\ldots,N) \tag{66}$$

Clearly, for any configurations of B_k (k=0,1,2, …, N), Eq. (66) has a trivial solution ($q_k(t)$=0, $t\in B_k$, k=0,1,2, …, N). This trivial solution may not be interesting.

In this case, one may propose normal demand for the degenerate scale problem. In the normal demand, one wants to find some configurations for B_k (k=0,1,2, …, N), Eq. (66) has a non-trivial solution ($q_k(t)\neq 0$, $t\in B_k$, for all k=0,1,2, …, N). In this case, the homogenous BIE (66) may not have such a non-trivial solution.

Alternatively, one may propose a modified demand for the degenerate scale problem. The modified demand was suggested by some researchers [24,26]. In the modified demand, one wants to find some configurations for B_k (k=0,1,2, …, N), Eq. (66) has a non-trivial solution ($q_o(t)\neq 0$, $t\in B_o$, $q_k(t)$=0, $t\in B_k$ for k=1,2, …, N). In this case, because of $q_k(t)$=0 (for $t\in B_k$, k=1,2, …, N), the homogenous BIE (66) can be reduced to

$$\int_{B_o}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_o(t)\,dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_o(t)}\,d\bar{t}\right\}=0,\quad (t_o\in B_k,\ \ k=0,1,2,\ldots,N) \tag{67}$$

We prefer writing Eq. (67) in an alternative form:

$$\int_{B_o}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_o(t)\,dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_o(t)}\,d\bar{t}\right\}=0,\ (t_o\in B_o) \tag{68a}$$

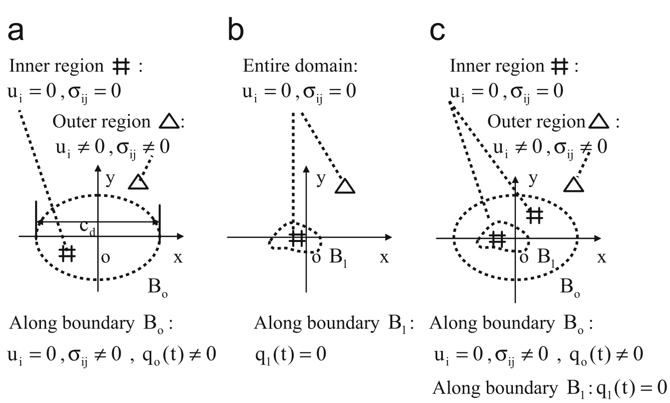

Fig. 4. (a) The stress field caused by density function $q_o(t)\neq 0$ ($t\in B_o$) in the degenerate scale, (b) The stress field caused by density function $q_1(t)$=0 ($t\in B_1$) in the degenerate scale, (c) The stress field from a superposition of two cases (a) and (b).

$$\int_{B_o}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_o(t)\,dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_o(t)}\,d\bar{t}\right\}=0 \qquad (t_o\in B_k,\ k=1,2\ldots N) \tag{68b}$$

After preserving the first equation (or Eq. (68a)) from Eqs. (68a) and (68b), we get a homogenous BIE for the single outer contour B_o:

$$\int_{B_o}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_o(t)\,dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_o(t)}\,d\bar{t}\right\}=0,\quad (t_o\in B_o) \tag{69}$$

It is known that for the homogenous BIE we have two degenerate scales such that $q_o(t)$ has a non-trivial solution, or $q_o(t)\neq 0$ ($t\in B_o$) [10,11] .

We will prove that for two problems: (1) the homogenous BIEs defined by (68a) and (68b) for finite multiply connected region and (2) the homogenous BIE defined by Eq. (69) for one contour B_o, they have the same degenerate scale. Clearly, it is sufficient to prove the statement for case of two contours B_o and B_1 (Fig. 4).

It is assumed that the degenerate scale for the homogenous equation (69) for the single outer contour B_o has been obtained beforehand, and the scale is denoted by c_d and the relevant non-trivial solution is named $q_o(t)$. Let us consider the properties of the relevant complex potentials:

$$\varphi_o(z)=H_1 i\int_{B_0}\ln(t-z)q_o(t)\,dt \tag{70}$$

$$\psi_o(z)=H_1 i\int_{B_o}\kappa\ln(t-z)\overline{q_o(t)}\,d\bar{t}+H_1 i\int_{B_o}\frac{\bar{t}}{t-z}q_o(t)\,dt \tag{71}$$

In Eqs. (70) and (71), the variable z is defined in the entire plane. When z is moving in the inner region (Fig. 4(a)), the function $\ln(t-z)$ (or $\ln(z-t)$) is a single-valued analytic function. This means that the displacement and stress fields derived from Eqs. (70) and (71) are continuous in the inner region. From degenerate scale solution we have u_i=0 along the boundary B_o. Therefore, the displacements must be u_i=0 in the inner region from the unique theorem of elasticity. Further, the stresses must also be σ_{ij}=0 in the inner region. The structures of the displacement and stress field are indicated in Fig. 4(a).

In the meantime, when the vanishing body force $q_1(t)$=0 ($t\in B_1$) is placed along the contour B_1, the following solution u_i=0, σ_{ij}=0 is obtained in entire domain (Fig. 4(b)). For this case, the structures of the displacement and stress field are indicated in Fig. 4(b).

The superposition of Fig. 4(a) and (b) will result in the stress field shown by Fig. 4(c). Clearly, the stress field shown by Fig. 4(c) represents a non-trivial solution for finite multiply connected region.

From the above-mentioned results, we see the following facts. For two problems: (1) the BIEs defined by Eqs. (68a) and (68b) and for finite multiply connected region and (2) the BIE defined by Eq. (69) for one contour, they have the same degenerate scale. Secondly, the configurations of the inner contours B_k (k=1,2,3, …, N) do not play any role in the derivation, or they can be arbitrary. This point was pointed out by many researchers [10,11,26].

3.4. Formulation of the degenerate scale problem for infinite multiply connected region

For the second case, the degenerate scale problem is formulated for infinite multiply connected region. After deleting the term for B_o and letting the right hand term in Eq. (63) to be zero, we will obtain the following homogenous BIE:

$$\sum_{j=1}^{N}\int_{B_j}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_j(t)\,dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_j(t)}\,d\bar{t}\right\}=0 \qquad (t_o\in B_k,\ k=1,2\ldots N) \tag{72}$$

Clearly, it is sufficient to make a statement for the case of two contours B_1 and B_2 with no boundary B_o (Fig. 5).

In this case, one may propose normal demand for the degenerate scale problem. In the normal demand, one wants to find some configurations for B_k (k=1,2, …, N), Eq. (72) has a non-trivial solution ($q_k(t)\neq 0$, ($t\in B_k$), for all k=1,2, …, N).

Contrary to the normal demand in the first case (the finite multiply connected region), the degenerate scale problem for Eq. (72) has a solution. It is sufficient to make a statement for the case of two contours B_1 and B_2. First, contours B_1 and B_2 are in an equivalent position. Secondly, the body force density $q_k(t)$ ($t\in B_k$, k=1,2) causes continuous displacement and stress in the entire plane with the exception at the contours B_k (k=1,2). Thus, once the boundary values of displacements from inner regions are equal to zero along the boundaries B_k (k=1,2), the u_i and σ_{ij} components must be vanishing in the inner regions bounded by the boundaries B_k (k=1,2). Clearly, the obtained solution has a property $q_k(t)\neq 0$ (k=1,2). For this case, the structures of the displacement and stress field are indicated in Fig. 5. Thirdly, the degenerate scale problem for case of two contours B_1 and B_2 is simply an extension of the same problem for one contour case, say only one contour B_1. Using the coordinate transformation technique [4,27], the degenerate scale problem for the case of two contours B_1 and B_2 was solved recently [28].

Alternatively, one may propose a modified demand for the degenerate scale problem. In the demand, we need to find some configurations for B_k (k=1,2), Eq. (72) has a non-trivial solution ($q_1(t)\neq 0$, $t\in B_1$, $q_2(t)$=0, $t\in B_2$). In this case, because of $q_2(t)$=0, $t\in B_2$, the homogenous BIE (72) can be reduced to

$$\int_{B_1}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_1(t)dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_1(t)}\,d\bar{t}\right\}=0,\quad (t_o\in B_k,\ k=1,2) \tag{73}$$

We prefer writing Eq. (73) in an alternative form

$$\int_{B_1}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_1(t)dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_1(t)}\,d\bar{t}\right\}=0,\quad (t_o\in B_1) \tag{74a}$$

$$\int_{B_1}\left\{\kappa\left[\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o)\right]q_1(t)dt+\frac{t-t_o}{\bar{t}-\bar{t}_o}\overline{q_1(t)}\,d\bar{t}\right\}=0,\quad (t_o\in B_2) \tag{74b}$$

It is possible to obtain a solution for degenerate scale problem defined by Eq. (74a). However, after substituting the obtained solution $q_1(t)$ ($t\in B_1$) into the left hand side of (74b), Eq. (74b) may not be satisfied. Thus, the modified demand (assume $q_1(t)\neq 0$, $t\in B_1$, $q_2(t)$=0, $t\in B_2$) is not a reasonable assumption in the formulation. This situation is quite different to his counterpart in the finite multiply connected case (see some statements after Eq. (67)).

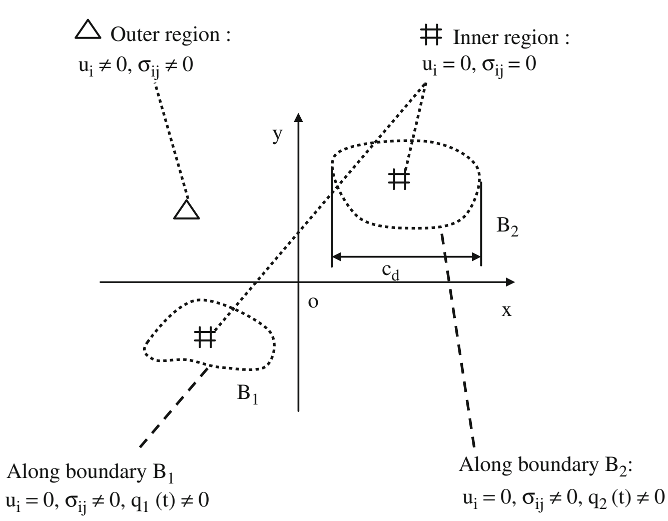

Fig. 5. Stress field for infinite multiply connected region in degenerate scale case.

4. Indirect BIEs based on double layer potential using complex variable

4.1. Formulation of indirect BIEs based on double layer potential using complex variable

After taking the following steps: (1) deleting one term containing the function $Q(t)$ on the right hand side of Eq. (31), (2) the function $U(t)$ is replaced by $g(t)$ and (3) τ is written as z, we can propose a displacement expression:

$$u(z)+iv(z)=\sum_{j=0}^{N}(u(z)+iv(z))_j \tag{75}$$

where

$$(u(z)+iv(z))_j=H_1 i\int_{B_j}\left(-\frac{\kappa-1}{t-z}g_j(t)dt+L_1(t,z)g_j(t)dt-L_2(t,z)\overline{g_j(t)}\,dt\right) \tag{76}$$

$$L_1(t,z)=-\frac{d}{dt}\left\{\ln\frac{t-z}{\bar{t}-\bar{z}}\right\}=-\frac{1}{t-z}+\frac{1}{\bar{t}-\bar{z}}\frac{d\bar{t}}{dt} \tag{77}$$

$$L_2(t,z)=\frac{d}{dt}\left\{\frac{t-z}{\bar{t}-\bar{z}}\right\}=\frac{1}{\bar{t}-\bar{z}}-\frac{t-z}{(\bar{t}-\bar{z})^2}\frac{d\bar{t}}{dt} \tag{78}$$

Note that the functions $g_j(t)$ (j=0,1,2, …, N) in Eq. (76) are the density functions assumed along the contours B_j (j=0,1,2, …, N), rather than the displacement on the boundaries (Fig. 6).

It is easy to see that the displacement expression shown by Eq. (75) corresponds to the following complex potentials:

$$\varphi(z)=\sum_{j=1}^{N}\varphi_j(z),\ \psi(z)=\sum_{j=1}^{N}\psi_j(z) \tag{79}$$

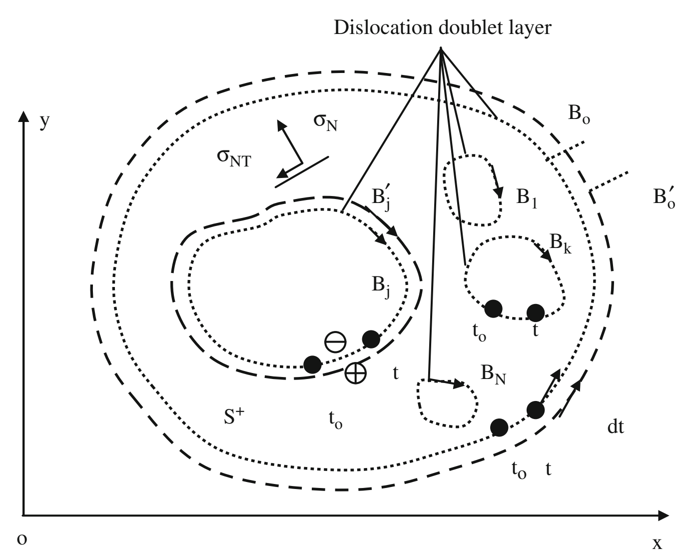

Fig. 6. Multiply connected region with dislocation doublet layers along $B=B_0+B_1+\cdots+B_N$.

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where

$$\varphi_j(z) = -2GH_1 i \int_{B_j} \frac{1}{t-z} g_j(t)\,dt, \quad (j=0,1,2,\ldots,N) \tag{80}$$

$$\begin{aligned}\psi_j(z) &= 2GH_1 i \int_{B_j} \left(-\frac{1}{t-z} + \frac{dt}{d\bar{t}} \frac{\bar{t}}{(t-z)^2} \right) g_j(t)\,d\bar{t} \\ &+ 2GH_1 i \int_{B_j} \frac{1}{t-z} \overline{g_j(t)}\,dt \quad (j=0,1,2,\ldots,N)\end{aligned} \tag{81}$$

If one substitutes the complex potentials (80) and (81) into Eq. (3), the displacement $(u(z)+iv(z))_j$ (j=0,1,2, …, N) is exactly expressed by Eq. (76). Thus, the displacement expression shown by Eq. (76) is an elasticity solution.

At this stage, one may find some differences between two formulations based on the single layer and the double layer. In fact, in the single layer formulation, the complex potential $\phi_j(z)$ shown by Eq. (38) has a term $\ln(t-z)$ ($\ln(t-z) = \ln d + i\theta$) in the integrand. The distance between two points t and z is denoted by d. Thus, the influence at point z has a factor proportional to $\ln(d)$ in the single layer formulation. However, in the double layer formulation, the complex potential $\phi_j(z)$ shown by Eq. (80) has a term $1/(t-z)$ in the integrand. Therefore, the relevant influence at point z is proportional to $1/d$ in the double layer formulation. This situation also exists in the formulation of the 2D Laplace equation based on the single layer or double layer.

It can be proved that the complex potentials shown by Eqs. (80) and (81) correspond to a dislocation doublet layer placed along the contour B_j (j=0,1,2, …, N). In fact, if a dislocation doublet with the intensity $D = D_x + iD_y$ $(-D)$ is applied at the point z=t (z=t+dt) (Fig. 7), the complex potentials for the dislocation doublet are available [19] as

$$\varphi(z) = D\left[\log(z-t) - \log(z-t-dt)\right] = -D\frac{dt}{t-z} \tag{82}$$

$$\begin{aligned}\psi(z) &= \bar{D}\left[\log(z-t) - \log(z-t-dt)\right] - D\left[\frac{\bar{t}}{z-t} - \frac{\bar{t}+d\bar{t}}{z-t-dt}\right] \\ &= -\bar{D}\frac{dt}{t-z} - D\frac{d\bar{t}}{t-z} + D\frac{\bar{t}dt}{(t-z)^2}\end{aligned} \tag{83}$$

In fact, if we make the substitutions $D = 2GH_1 i g_j(t)$ and $\bar{D} = -2GH_1 i\overline{g_j(t)}$ in Eqs. (82) and (83), and perform an integration with respect to dt, the complex potentials shown by Eqs. (80) and (81) are obtained. Thus, the mentioned assertion is proved.

Assuming $z \to t_o^+$ and $z \to t_o^-$ ($t_o \in B_j$) and using the generalized Sokhotski–Plemelj formulae and some results in Appendix B, from Eq. (76) we will find

$$\begin{aligned}&[(u(t_o)+iv(t_o))_j]^{\pm}_{B_j} \\ &= \pm\frac{g_j(t_o)}{2} + H_1 i \int_{B_j} \left(-\frac{(\kappa-1)}{t-t_o} g_j(t)\,dt + L_1(t,t_o) g_j(t)\,dt - L_2(t,t_o)\overline{g_j(t)}\,dt \right) \\ &(t_o \in B_j,\ j=0,1,2,\ldots,N)\end{aligned} \tag{84}$$

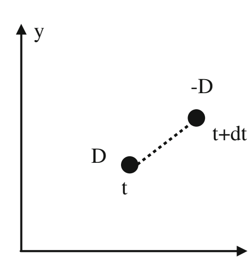

Fig. 7. Dislocation doublet with the intensity $D = D_x + iD_y$ $(-D)$ applied at the point z=t (z=t+dt).

$$\begin{aligned}&[(u(t_o)+\mathrm{i}v(t_o))_j]^{+}_{B_j} \\ &- [(u(t_o)+\mathrm{i}v(t_o))_j]^{-}_{B_j} = g_j(t_o) \quad (t_o \in B_j,\ j=0,1,2\ldots,N)\end{aligned} \tag{85}$$

From Eqs. (84) and (85), we see that the displacements are discontinuous in case of a moving point t_o across the boundary B_j (j=0,1,2, …, N).

Substituting Eqs. (80) and (81) into Eq. (2) yields the resultant force function

$$-Y(z) + iX(z) = \sum_{j=0}^{N} (-Y(z)+iX(z))_j \tag{86}$$

where

$$\begin{aligned}&(-Y(z)+iX(z))_j = 2GH_1 i \\ &\int_{B_j} \left(-\frac{2}{t-z} g_j(t)\,dt - L_1(t,z) g_j(t)\,dt + L_2(t,z)\overline{g_j(t)}\,dt \right)\end{aligned} \tag{87}$$

Similarly, letting $z \to t_o^+$ and $z \to t_o^-$ ($t_o \in B_j$) and using the generalized Sokhotski–Plemeljs formulae and some results in Appendix B, from Eq. (87) we will find

$$\begin{aligned}&[(-Y(t_o)+iX(t_o))_j]^{+}_{B_j} = [(-Y(t_o)+iX(t_o))_j]^{-}_{B_j} \\ &= 2GH_1 i \int_{B_j} \left(-\frac{2}{t-t_o} g_j(t)\,dt - L_1(t,t_o) g_j(t)\,dt + L_2(t,t_o)\overline{g_j(t)}\,dt \right) \\ &(t_o \in B_j,\ j=0,1,2,\ldots,N)\end{aligned} \tag{88}$$

$$\begin{aligned}&[(-Y(t_o)+iX(t_o))_j]^{+}_{B_j} - [(-Y(t_o)+iX(t_o))_j]^{-}_{B_j} = 0 \\ &(t_o \in B_j,\ j=0,1,2,\ldots,N)\end{aligned} \tag{89}$$

From Eqs. (88) and (89) we see that the resultant force function is continuous in case of a moving point t_o across the boundary B_j.

From Eqs. (5), (80) and (81), the traction $\sigma_N(z) + i\sigma_{NT}(z)$ at a domain point can be evaluated as follows:

$$\sigma_N(z) + i\sigma_{NT}(z) = \sum_{j=0}^{N} (\sigma_N(z) + i\sigma_{NT}(z))_j \tag{90}$$

where

$$\begin{aligned}&(\sigma_N(z) + i\sigma_{\mathrm{NT}}(z))_j \\ &= \frac{d}{\mathrm{d}z}((-Y(z)+\mathrm{i}X(z))_j) \\ &= H_1 i \int_{B_j} \left(-\frac{2}{(t-z)^2} g_j(t)\,\mathrm{d}t - M_1(t,z) g_j(t)\,\mathrm{d}t + M_2(t,z)\overline{g_j(t)}\,\mathrm{d}t \right)\end{aligned} \tag{91}$$

In Eq. (91), two kernels are defined by

$$\begin{aligned}M_1(t,z) &= \frac{d}{dz}\left\{L_1(t,z)\right\} = -\frac{d}{dz}\left\{\frac{d}{dt}\left\{\ln\frac{t-z}{\bar{t}-\bar{z}}\right\}\right\} \\ &= -\frac{1}{(t-z)^2} + \frac{1}{(\bar{t}-\bar{z})^2}\frac{d\bar{t}}{dt}\frac{d\bar{z}}{dz}\end{aligned} \tag{92}$$

$$\begin{aligned}M_2(t,z) &= \frac{d}{dz}\left\{L_2(t,z)\right\} = \frac{d}{dz}\left\{\frac{d}{dt}\left\{\frac{t-z}{\bar{t}-\bar{z}}\right\}\right\} \\ &= \frac{1}{(\bar{t}-\bar{z})^2}\left(\frac{d\bar{t}}{dt} + \frac{d\bar{z}}{dz}\right) - \frac{2(t-z)}{(\bar{t}-\bar{z})^3}\frac{d\bar{t}}{dt}\frac{d\bar{z}}{dz}\end{aligned} \tag{93}$$

Similarly, assuming $z \to t_o^+$ and $z \to t_o^-$ ($t_o \in B_j$) and using results for jump values for three integrals in Eq. (91) (see Appendix B), from Eq. (91) we will find

$$[(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]^+_{B_j} = [(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]^-_{B_j} = H_1 i\int_{B_j}\left(-\frac{2}{(t-t_o)^2}g_j(t)dt - M_1(t,t_o)g_j(t)dt + M_2(t,t_o)\overline{g_j(t)}dt\right) \quad (t_o\in B_j) \tag{94}$$

$$[(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]^+_{B_j} - [(\sigma_N(t_0)+i\sigma_{NT}(t_0))_j]^-_{B_j} = 0 \quad (t_o\in B_j) \tag{95}$$

In the derivation of Eq. (94), the jump values from the first integral and second integral on the right hand side of Eq. (91) are just compensated, and the third integral has no contribution to the jump value (see Appendix B). From Eq. (95) we see that the tractions are continuous in case of a moving point t_o across the boundary B_j. In addition, the first term on the right hand side of Eq. (94) is a hypersingular integral.

Thus, from Eqs. (76) and (87) we have

$$\{(u(z)+iv(z))_k\}_{B'_j,\,inc} = 0 \quad (k,j=0,1,2,\ldots,N) \tag{96}$$

$$\{(-Y(z)+iX(z))_k\}_{B'_j,inc} = 0 \quad (k,j=0,1,2,\ldots,N) \tag{97}$$

Eq. (96) represents the single-valued condition of displacements, which has been satisfied in the present formulation. Eq. (97) shows that the tractions applied on contour B_j must be in equilibrium in resultant forces. This is a disadvantage in the formulation based on dislocation doublet layer.

From Eq. (81) we have

$$\chi_j(z) = \int_{z_o}^{z}\psi_j(z)dz = 2GH_1 i\int_{B_j}\left(\ln(z-t)+\frac{dt}{d\bar{t}}\frac{\bar{t}}{t-z}\right)g_j(t)d\bar{t} - 2GH_1 i\int_{B_j}\ln(z-t)\overline{g_j(t)}\,dt \quad (j=0,1,2,\ldots,N) \tag{98}$$

Therefore, substituting Eq. (98) into (57) yields

$$\{m_0(z)\}_{B'_0,\,inc} = 4\pi GH_1\mathrm{Re}\int_{B_0}(\overline{g_0(t)}\,dt - g_0(t)\,d\bar{t}) = 0 \tag{99}$$

$$\{m_j(z)\}_{B'_j,\,inc} = -4\pi GH_1\mathrm{Re}\int_{B_j}(\overline{g_j(t)}\,dt - g_j(t)d\bar{t}) = 0, \quad (j=1,2,\ldots,N) \tag{100}$$

$$\{m_k(z)\}_{B'_j,\,inc} = 0, \quad (k,j=0,1,2,\ldots,N) \tag{101}$$

where $\{m_k(z)\}_{B'_j,\,inc}$ represents the moment applied on contour B'_j caused by the kth density distribution along the contour B_k. Eqs. (97) and (101) reveal that the tractions applied on the contour B_j (j=0,1, …, N) must be in equilibrium in forces and moment. This is a disadvantage in the formulation based on dislocation doublet layer.

4.2. Formulations of boundary value problems

The Dirichlet problem is formulated first. It is assumed that the displacement along the contours $t_o\in B_k$ (k=0,1,2, …, N) has been given beforehand, which is as follows:

$$(u(t_o)+iv(t_o))^+_{B_k} = (\tilde{u}(t_o)+i\tilde{v}(t_o))_k \quad (t_o\in B_k, k=0,1,2,\ldots,N) \tag{102}$$

In Eq. (102), $(\tilde{u}(t_o)+i\tilde{v}(t_o))_k$ (k=0,1,2, …, N) are given functions.

Using Eqs. (75), (76), (84) and (102), and the principle of superposition, we will find the following integral equation:

$$\frac{g_k(t_o)}{2} + H_1 i\int_{B_k}\left(-\frac{(\kappa-1)}{t-t_o}g_k(t)\,dt + L_1(t,t_o)g_k(t)\,dt - L_2(t,t_o)\overline{g_k(t)}\,dt\right) + H_1 i{\sum_{j=0}^{N}}'\int_{B_j}\left(-\frac{(\kappa-1)}{t-t_o}g_j(t)\,dt + L_1(t,t_o)g_j(t)\,dt - L_2(t,t_o)\overline{g_j(t)}\,dt\right) = (\tilde{u}(t_o)+i\tilde{v}(t_o))_k \quad (t_o\in B_k,\ k=0,1,2,\ldots,N) \tag{103}$$

In Eq. (103), the prime in $\sum_{j=0}^{N}{}'$ means that the term j=k should be excluded in the summation.

As claimed previously, the tractions applied on the contour B_j (j=0,1,2, …, N) must be in equilibrium in forces and moment. However, an arbitrary assumed boundary condition for $(\tilde{u}(t_o)+i\tilde{v}(t_o))_k$ (k=1,2, …, N) may not satisfy this condition. Thus, the singular integral equation (103) cannot be used in the general case. It can be only used in some particular case, for example, a symmetric elliptic ring region with symmetric deformation as indicated in Fig. 8(a).

Clearly, if (1) the inner contour is fixed, or u+iv=0 for inner contour, and (2) the outer contour has a translation in x-direction, or u+iv=c (c-real), there must be a resultant force applied on the contour (Fig. 8(b)). Thus, the integral equation must have no solution in this case (Fig. 8(b)). Clearly, the formulation based on dislocation doublet potentials cannot be used in the Dirichlet problem in general.

The Neumann problem is formulated secondly. It is assumed that the tractions along the contours $t_o\in B_k$ (k=0,1,2, …, N) have been given beforehand

$$(\sigma_N(t_0)+i\sigma_{NT}(t_0))^+_{B_k} = (\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k \quad (t_o\in B_k,\ k=0,1,2,\ldots,N) \tag{104}$$

In Eq. (104), $(\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k$, ($k$=0,1,2, …, N) are given functions.

Using Eqs. (90), (91), (94), (95) and (104), and the principle of superposition, we will find the following hypersingular integral equation:

$$H_1 i\int_{B_k}\left(-\frac{2}{(t-t_o)^2}g_k(t)\,dt - M_1(t,t_o)g_k(t)\,dt + M_2(t,t_o)\overline{g_k(t)}\,dt\right) + H_1 i{\sum_{j=0}^{N}}'\int_{B_j}\left(-\frac{2}{(t-t_o)^2}g_j(t)\,dt - M_1(t,t_o)g_j(t)\,dt + M_2(t,t_o)\overline{g_j(t)}\,dt\right) = (\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k \quad (t_o\in B_k,\ k=0,1,2,\ldots,N) \tag{105}$$

In Eq. (105), the prime in $\sum_{j=0}^{N}{}'$ means that the term j=k should be excluded in the summation.

As claimed previously, the tractions applied on the contour B_j (j=0,1,2, …, N) must be in equilibrium in forces and moment. However, the arbitrary assumed boundary tractions for $(\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_k$ (k=0,1,2, …, N) may not satisfy this condition. Thus, the presented hypersingular integral equation cannot be used in the general case. It can be only used to some particular case, for example, a ring region with tractions on contours in equilibrium in forces and moment (Fig. 9(a)). This is a disadvantage in the formulation based on dislocation doublet layer.

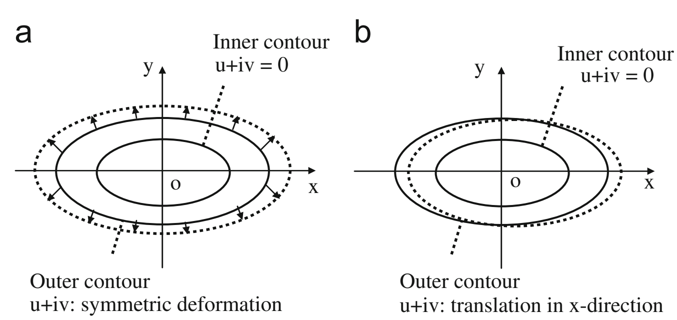

Fig. 8. (a) An elliptic ring with symmetric deformation along boundaries. (b) An elliptic ring with non-symmetric deformation along boundaries.

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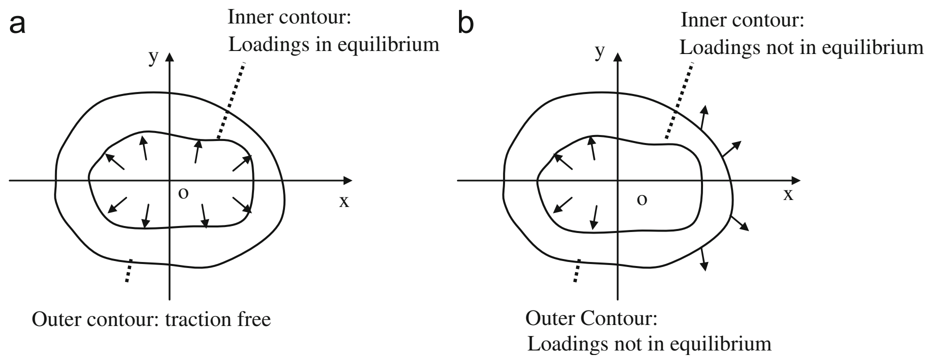

Fig. 9. Loadings on individual contour in (a) equilibrium and (b) not in equilibrium.

Clearly, Eq. (105) cannot be used in the loading condition shown by Fig. 9(b). Previously, a particular complex potential was introduced [18]. After introducing this complex potential, the problem shown by Fig. 9(b) can be reduced to a complementary problem that the tractions on contours are in equilibrium. Later, the integral equation (105) is used to solve the complementary problem. However, this procedure will make the solution more complicated.

Clearly, the above-mentioned boundary value problems are defined for multiply connected region for a finite region. If there is no outer boundary B_0, we simply delete the relevant term in the formulation. In this case, the boundary value problem is defined for multiply connected region for an infinite region.

As stated in Ref. [22], no general proof has been done for the degenerate scale problem. A physical background for the problem is introduced below. Typically, the degenerate scale problem arises from the exterior BVP when the vanishing displacement is imposed on the boundary. For a single contour case of the exterior BVP (only k=1 in Eq. (103)), one may assume vanishing displacement on the right hand side of Eq. (103). In this case, one may obtain a non-zero solution for $g_1(t)$ [26]. However, this non-trivial solution represents a rigid rotation of the inner portion to the exterior portion, and no stress was initiated in the exterior region. Note that the complex potentials can be defined in the interior and exterior regions with respect to contour B_1. That is to say, we have a unique solution for the stresses for the BIE defined by Eq. (103). Thus, no degenerate scale problem can be found from the double layer formulation.

4.3. *Formulation of hypersingular integral equation for a curved crack*

A particular case is introduced below. Assume that there is only one contour B_o, which is composed of curves L and L_1, or B_o=L+L_1 (Fig. 10). The stress field is defined on the entire plane with the exception of the boundary B_o. In this case, the density function is denoted by $g_o(t)$. In addition, we assume that

$$g_o(t) \neq 0, \ (t \in L) \quad \text{and} \quad g_o(t) = 0, (t \in L_1) \tag{106}$$

In this case, Eq. (105) is reduced to

$$H_1 i \int_L \left(-\frac{2}{(t-t_o)^2} g_0(t)\,dt - M_1(t,t_o) g_0(t)\,dt + M_2(t,t_o)\overline{g_0(t)\,dt} \right) = (\tilde{\sigma}_N(t_0) + i\tilde{\sigma}_{NT}(t_0))_0 \quad (t_o \in L) \tag{107}$$

where $(\tilde{\sigma}_N(t_0)+i\tilde{\sigma}_{NT}(t_0))_0$ is the traction applied on the crack face. From (85) we see that the density function represents the following COD (crack opening displacement) function:

$$g_0(t_o) = (u(t_o)+iv(t_o))_L^+ - (u(t_o)+iv(t_o))_L^- \quad (t_o \in L) \tag{108}$$

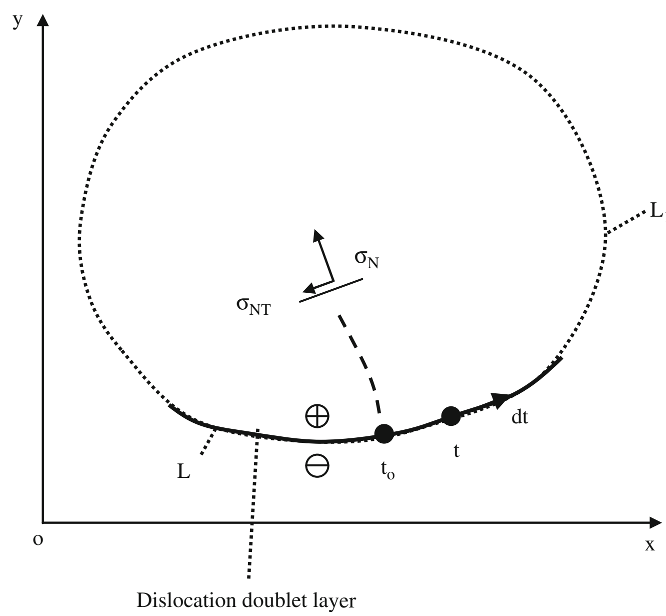

Fig. 10. A dislocation doublet layer placed along curve "L".

Eq. (108) represents a hypersingular integral equation for the curved crack problem. The hypersingular integral equation for the curved crack was suggested by many researchers [6,14,29–31].

At this stage, we can summarize the particular behaviors of the single and double layers in the formulation of the crack problem. In fact, in the single layer formulation, a relation is shown by Eq. (43): $[(u(t_o)+iv(t_o))_j]_{B_j}^+ - [(u(t_o)+iv(t_o))_j]_{B_j}^- = 0$, ($j$=0.1,2, …, N). This means that the formulation cannot provide the COD behavior in the crack problem. On the contrary, in the double layer formulation, a relation is shown by Eq. (85): $[(u(t_o)+iv(t_o))_j]_{B_j}^+ - [(u(t_o)+iv(t_o))_j]_{B_j}^- = g_j(t_o)$ ($t_o \in B_j$, j=0,1,2, …, N). This means that the formulation can provide the COD behavior in the crack problem. As stated previously, only assuming $g_j(t_o)$=0 on some portion of contour, and also assuming $g_j(t_o) \neq 0$ on the other portion of contour, the BIE for the curved crack problem is naturally formulated.

As stated in Ref. [22], no general proof has been done for the degenerate scale problem. A physical background for the problem is introduced below. Physically, the degenerate scale problem arises from the logarithmic kernel involved in the integrand. In the BIE shown by Eq. (107), one kernel involved on the left hand side is a hypersingular integral with the form $\int_L (g_0(t)dt/(t-t_o)^2)$. We may propose a coordinates transformation x_2=hx_1, y_2=hy_1 (h—a positive value) between (x_1,y_1) and (x_2,y_2). In this case, we can obtain the relevant matrix after discretization for the integral.

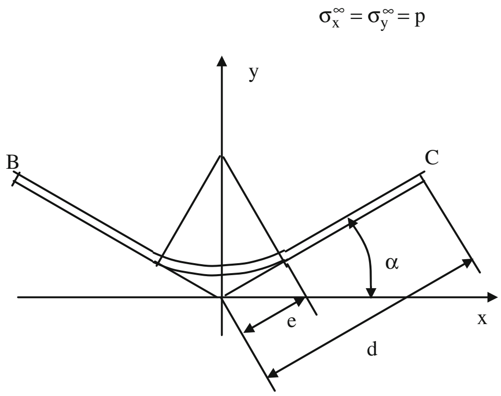

Fig. 11. A V-shaped crack with rounded corner.

Clearly, all elements of matrix in the coordinates (x_2,y_2) will be proportional to those in the coordinates (x_1,y_1). That is to say the adopted different scales have no essential influence on the resulting algebraic equation, and only influence with a constant multiplied factor h. Thus, there is no degenerate scale problem for BIE (107). Clearly, this is a physical statement for the degenerate scale problem of the curved crack.

4.4. A numerical example for a curved crack using hypersingular integral equation

A numerical example is carried out for a curved crack problem using the hypersingular equation shown by Eq. (107). The curved crack has a V-shaped configuration with a rounded corner (Fig. 11). The arm length is denoted by d, and the length of the rounded portion is denoted by e. The arm of V-shape has an inclined angle α with respect to the x-axis. The remote loading is $\sigma_x^\infty = \sigma_y^\infty = p$. A quadrature rule for the hypersingular integral equation proposed in [32] is used. In addition, the curve length coordinate technique is suggested in the solution [33]. Therefore, the hypersingular integral equation (107) can be solved numerically. The computed results for SIFs at the right tip are expressed as

$$K_1 = f_1(\alpha, e/d)p\sqrt{\pi a}, \quad K_2 = f_2(\alpha, e/d)p\sqrt{\pi a} \tag{109}$$

where

$$a = d + e\left(\frac{\alpha}{\tan\alpha} - 1\right) \tag{110}$$

The $2a$ is equal to the length of the curved crack after expanding.

The computed results for $f_1(\alpha,e/d)$ and $f_2(\alpha,e/d)$, under conditions $\alpha=\pi/12, \ldots, 5\pi/12$, and $e/d=0.1, 0.2, \ldots, 1.0$, are plotted in Figs. 12 and 13, respectively. From the plotted results we see that, the influence of the rounded corner (or the ratio e/d) is not significant. However, the influence caused by the inclined angle α is significant. For example, in the case of $e/d=0.5$, we have $f_1(\alpha,e/d)|_{\alpha=\pi/12} = 0.9645$, $f_1(\alpha,e/d)|_{\alpha=5\pi/12} = 0.5380$, $f_2(\alpha,e/d)|_{\alpha=\pi/12} = 0.1253$, $f_1(\alpha,e/d)|_{\alpha=5\pi/12} = 0.2392$, respectively.

5. Conclusions

For two kinds of potentials, it is important to study the properties of potentials in detail. The continuous or discontinuous properties for the displacements and tractions for a moving point across the boundary play an important role in the study. For this

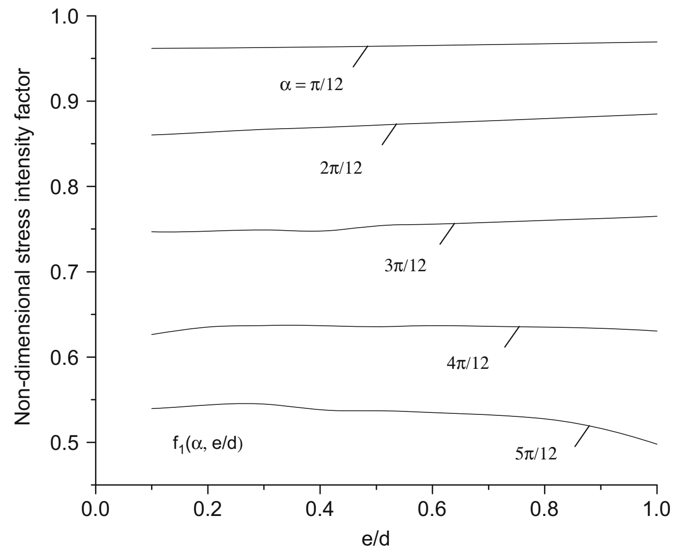

Fig. 12. Non-dimensional mode I SIFs for a V-shaped crack (see Eq. (109) and Fig. 11).

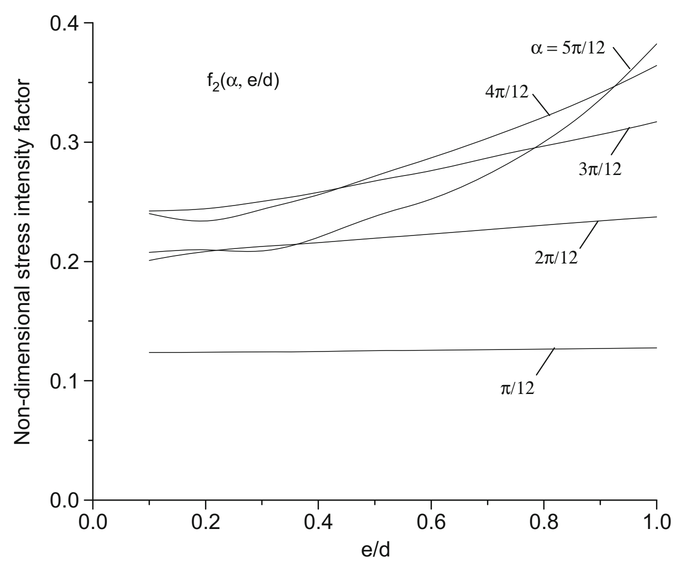

Fig. 13. Non-dimensional mode II SIFs for a V-shaped crack (see Eq. (109) and Fig. 11).

point, in addition to the generalized Sokhotski–Plemelj formulae shown by Eqs. (10)–(12), this paper provides some results in Appendix B. Those results represent some extension of the generalized Sokhotski–Plemelj formulae. Clearly, the results shown in Appendix B are difficult to obtain using real variable analysis.

Previously, when a moving point z goes forward in the clockwise direction along the contour B_j' ($j=1,2,\ldots,N$), the increase for a function $f(z)$ is defined by $f(z)_{B_j', inc}$. The analysis for behaviors of $f(z)_{B_j', inc}$ also plays an important role in the study. Clearly, only after studying those behaviors, we can know the range of the solution for the formulated BIE. It can be seen from analysis in the fourth section that the BIEs based on double layer for exterior problem (shown by Eqs. (103) or (105)) can only be used in the cases where the tractions along the individual contour are in equilibrium. Particularly, it is not easy to overcome the inconvenient points in the formulation, e.g., for the boundary value problem shown by Fig. 8(b).

In addition, the limitation for the indirect BIEs based on the single layer is minor. Only one demand in the single layer formulation is that the total tractions applied on the all boundaries for finite multiply connected region must be in equilibrium. When using the indirect BIEs based on single layer, one may meet the degenerate scale problem. However, people have sufficient knowledge of the degenerate scale. Thus, it is easy to avoid meeting unsatisfying conditions arising from the degenerate scale.

A significant feature in the direct BIE is that all functions involved are the tractions or the displacements on the boundary. However, in the indirect BIE, the free terms are the tractions or the displacements on the boundary, and all involved functions in the integrals are not the tractions or the displacements.

A variety of indirect BIEs in plane elasticity could be suggested. At least, we can point out six of them.

(1) Previously, a fundamental solution based on a point dislocation was suggested [19, Eqs. (1.51) and (1.52)]. Based on this fundamental solution, an indirect BIE was formulated [20,26]. The detailed computations are presented in [26]. The particular feature of this BIE is that the applied tractions on contour should be equilibrated in forces and moment.
(2) Secondly, a fundamental solution based on a dislocation doublet was suggested [19, Eqs. (1.58) and (1.59)]. Based on this fundamental solution, an indirect BIE can be formulated. In fact, the BIEs shown by Eqs. (103) and (105) represent this type of BIE, which is based on the dislocation doublet distribution along the contours. In fact, the mentioned dislocation doublet distribution is equivalent to a COD distribution. Since the BIE of second type can be obtained from the first type from integration by part, the demand that the applied tractions on contour should be equilibrated in forces and moment remains.
(3) Thirdly, a fundamental solution based on a point concentrated force was suggested [19, Eqs. (1.54) and (1.55)]. Based on this fundamental solution, an indirect BIE can be formulated. In fact, the BIEs shown by Eqs. (63) and (65) represent this type of BIE, which is based on the body force distribution long the contours. The particular feature of this BIE is that the applied tractions on the individual contours may not be equilibrated in forces and moment. However, the applied tractions on all contours for finite multiply connected region must be equilibrated in forces and moment. Therefore, the limitation for this BIE (type (3)) is minor.
(4) Fourthly, a fundamental solution based on a force doublet was suggested [19, Eqs. (1.60) and (1.61)]. Based on this fundamental solution, an indirect BIE can be formulated. The BIE of fourth type can be obtained from the third type from integration by part.
(5) Previously, the boundary value of an analytic function, or $\phi(t)$, is taken as the density function; Muskhelishvili proposed an indirect BIE for plan elasticity [18,&98].
(6) Alternatively, an intermediate function, or $\omega(t)$, is taken as the density function; an indirect BIE for plan elasticity was suggested [18,&101].

Obviously, varieties of formulation of indirect BIEs can provide many ways to solve BVP. However, it is not easy to determine which one is better among them.

Appendix A

About different displacement expressions at domain point in exterior BVP

The exterior BVP is considered in the following analysis. It is sufficient to consider an infinite plate bounded by contour B_1 (Fig. 14). If τ is rewritten as z, from Eq. (31), the displacement expression at a domain point z ($z \in S^+$, exterior to B_1) can be expressed as

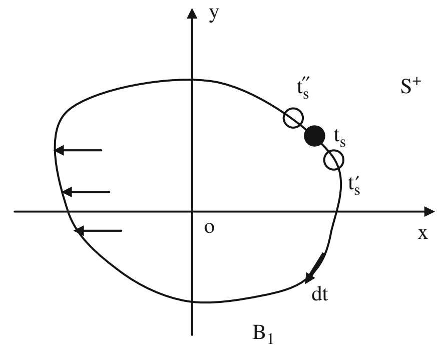

Fig. 14. An exterior BVP with tractions on contour not in equilibrium.

$$U(z) = u(z) + iv(z) = (u(z) + iv(z))_1 + (u(z) + iv(z))_2 \tag{a1}$$

where

$$(u(z)+iv(z))_1 = H_1 i \int_{B_1} \left(-\frac{(\kappa-1)}{t-z} U(t)dt + L_1(t,z)U(t)dt - L_2(t,z)\overline{U(t)}\,d\bar{t} \right) \quad (z \in S^+) \tag{a2}$$

$$(u(z)+iv(z))_2 = \frac{H_1 i}{2G} \int_{B_1} \kappa[\ln(t-z) + \ln(\bar{t}-\bar{z})]Q(t)dt + \frac{H_1 i}{2G} \int_{B_1} \frac{t-z}{\bar{t}-\bar{z}} \overline{Q(t)}\,d\bar{t} \quad (z \in S^+) \tag{a3}$$

In the following derivation, from Eq. (2) we have

$$df(t) = d(-Y(t) + iX(t)) = (\sigma_N(t) + i\sigma_{NT}(t))dt = Q(t)dt$$
$$d\overline{f(t)} = d(-Y(t) - iX(t)) = \overline{Q(t)}\,d\bar{t} \tag{a4}$$

We can perform the integration from t'_s to t''_s along the contour B_1 (Fig. 14), where t'_s to t''_s are two points very near the point t_s. Thus, an integration by part for Eq. (a3) yields

$$(u(z)+iv(z))_2 = \frac{H_1 i}{2G}\left(\kappa[\ln(t_s - z) + \ln(\bar{t}_s - \bar{z})] f_{inc} + \frac{t_s - z}{\bar{t}_s - \bar{z}} \overline{f}_{inc} \right) + \frac{H_1 i}{2G} \int_{B_1} \left(-\overline{f(t)} \frac{d}{dt} \left\{ \frac{t-z}{\bar{t}-\bar{z}} \right\} dt - \frac{\kappa}{t-z} f(t)dt - \frac{\kappa}{\bar{t}-\bar{z}} f(t)d\bar{t} \right) \quad (z \in S^+) \tag{a5}$$

where the value f_{inc} is defined by

$$f_{inc} = f(t)|_{t=t''_s} - f(t)|_{t=t'_s} \tag{a6}$$

Thus, if and only if f_{inc}=0, or the tractions applied along contour B_1 are in equilibrium in forces, Eq. (a5) can be reduced to

$$(u(z)+iv(z))_2 = \frac{H_1 i}{2G} \int_{B_1} \left(-\overline{f(t)} \frac{d}{dt} \left\{ \frac{t-z}{\bar{t}-\bar{z}} \right\} dt - \frac{\kappa}{t-z} f(t)dt - \frac{\kappa}{\bar{t}-\bar{z}} f(t)d\bar{t} \right) \quad (z \in S^+) \tag{a7}$$

Eq. (a7) is exactly one term in Eq. (14) of [14].

The above-mentioned derivation reveals that Eq. (14) in [14] cannot be used in the exterior BVP with tractions on contour B_1 not in equilibrium. This is a disadvantage for Eq. (14) in [14].

Appendix B

Properties for some integrals with kernel functions $L_1(t,z)$, $L_2(t,z)$, $K_1(t,z)$, $K_2(t,z)$, $M_1(t,z)$ and $M_2(t,z)$ defined by Eqs. (77), (78), (49), (92) and (93)

Two integrals with the kernel functions $L_1(t,z)$, $L_2(t,z)$ shown by Eqs. (77) and (78) are defined as follows:

$$W_1(z) = \frac{1}{2\pi i}\int_\Gamma L_1(t,z)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b1}$$

$$W_2(z) = \frac{1}{2\pi i}\int_\Gamma L_2(t,z)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b2}$$

where

$$L_1(t,z) = -\frac{d}{dt}\left\{\ln\frac{t-z}{\bar t-\bar z}\right\} = -\frac{1}{t-z}+\frac{1}{\bar t-\bar z}\frac{d\bar t}{dt} \tag{b3}$$

$$L_2(t,z) = \frac{d}{dt}\left\{\frac{t-z}{\bar t-\bar z}\right\} = \frac{1}{\bar t-\bar z}-\frac{t-z}{(\bar t-\bar z)^2}\frac{d\bar t}{dt} \tag{b4}$$

In Eqs. (b1) and (b2), Γ denotes a closed contour and $f(t)$ is an arbitrary function. If dt goes forward in an anti-clockwise direction, S^+ and S^- are the inside finite region and the outside infinite region, respectively. In addition, if dt goes forward in a clockwise direction, S^+ and S^- are the outside infinite region and the inside finite region, respectively (refer to Fig. 2).

In Eqs. (b1) and (b2), assuming $z\to t_o$ ($z\in S^+$, $t_o\in\Gamma$) and assuming $z\to t_o$ ($z\in S^-$, $t_o\in\Gamma$), and using the generalized Sokhotski–Plemelj formulae shown by Eqs. (10)–(12), we will find

$$W_1^\pm(t_o) = \mp f(t_o) + \frac{1}{2\pi i}\int_\Gamma L_1(t,t_o)f(t)\,dt \quad (t_o\in\Gamma) \tag{b5}$$

$$W_2^\pm(t_o) = \frac{1}{2\pi i}\int_\Gamma L_2(t,t_o)f(t)\,dt \quad (t_o\in\Gamma) \tag{b6}$$

We can prove the assertion shown by Eq. (b5) as follows. In fact, we can rewrite $W_1(z)$ as

$$W_1(z) = I_1 + I_2 \quad (z\in S^+ \text{ or } z\in S^-) \tag{b7}$$

where

$$I_1(z) = \frac{1}{2\pi i}\int_\Gamma\left(-\frac{1}{t-z}\right)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b8}$$

$$I_2(z) = \frac{1}{2\pi i}\int_\Gamma\left(\frac{1}{\bar t-\bar z}\frac{d\bar t}{dt}\right)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b9}$$

For convenience in derivation, we can define

$$I_3(z) = -\overline{I_2(z)} = \frac{1}{2\pi i}\int_\Gamma\frac{1}{t-z}\overline{f(t)}\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b10}$$

In Eqs. (b8) and (b10), assuming $z\to t_o$ ($z\in S^+$, $t_o\in\Gamma$) and assuming $z\to t_o$ ($z\in S^-$, $t_o\in\Gamma$) , and using the generalized Sokhotski–Plemelj formulae shown by Eqs. (10)–(12), we will find

$$I_1^\pm(t_o) = \mp\frac{f(t_o)}{2} + \frac{1}{2\pi i}\int_\Gamma\left(-\frac{1}{t-z}\right)f(t)\,dt \quad (t_o\in\Gamma) \tag{b11}$$

$$I_3^\pm(t_o) = -\overline{I_2^\pm(t_o)} = \pm\frac{\overline{f(t_o)}}{2} + \frac{1}{2\pi i}\int_\Gamma\frac{1}{t-t_o}\overline{f(t)}\,dt \quad (t_o\in\Gamma) \tag{b12}$$

$$I_2^\pm(t_o) = -\overline{I_3^\pm(t_o)} = \mp\frac{f(t_o)}{2} + \frac{1}{2\pi i}\int_\Gamma\left(\frac{1}{\bar t-\bar t_o}\frac{d\bar t}{dt}\right)f(t)\,dt \quad (t_o\in\Gamma) \tag{b13}$$

From Eqs. (b7), (b11) and (b13), the validity of Eq. (b5) is proved. Similarly, we can prove the validity of Eq. (b6).

In addition, other two integrals with the kernel functions $K_1(t,z)$, $K_2(t,z)$ shown by Eq. (49) are defined as follows:

$$W_3(z) = \frac{1}{2\pi i}\int_\Gamma K_1(t,z)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b14}$$

$$W_4(z) = \frac{1}{2\pi i}\int_\Gamma K_2(t,z)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b15}$$

where

$$K_1(t,z) = \frac{d}{dz}\left\{\ln\frac{t-z}{\bar t-\bar z}\right\} = -\frac{1}{t-z}+\frac{1}{\bar t-\bar z}\frac{d\bar z}{dz} \tag{b16}$$

$$K_2(t,z) = -\frac{d}{dz}\left\{\frac{t-z}{\bar t-\bar z}\right\} = \frac{1}{\bar t-\bar z}-\frac{(t-z)}{(\bar t-\bar z)^2}\frac{d\bar z}{dz} \tag{b17}$$

As before, we can prove the following equalities:

$$W_3^\pm(t_o) = \mp f(t_o) + \frac{1}{2\pi i}\int_\Gamma K_1(t,t_o)f(t)\,dt \quad (t_o\in\Gamma) \tag{b18}$$

$$W_4^\pm(t_o) = \frac{1}{2\pi i}\int_\Gamma K_2(t,t_o)f(t)\,dt \quad (t_o\in\Gamma) \tag{b19}$$

If in all above-mentioned integrals, the integration is performed along the curve L (Fig. 10), all equations (b5), (b6), (b18) and (b19) are still valid.

In the following, some formulae relating to the hypersingular integrals are introduced. We can define a Cauchy type integral as follows:

$$F(z) = \frac{1}{2\pi i}\int_\Gamma\frac{f(t)dt}{t-z} \quad (z\in S^+ \text{ or } z\in S^-) \tag{b20}$$

From Eq. (b20), we can define

$$F'(z) = \frac{dF}{dz} = \frac{1}{2\pi i}\int_\Gamma\frac{f(t)dt}{(t-z)^2} \quad (z\in S^+ \text{ or } z\in S^-) \tag{b21}$$

In Eq. (b21), assuming $z\to t_o$ ($z\in S^+$, $t_o\in\Gamma$) and assuming $z\to t_o$ ($z\in S^-$, $t_o\in\Gamma$), $dz=dt_o$ and using the generalized Sokhotski–Plemelj formulae, we will find

$$\begin{aligned} F'^\pm(t_o) &= \frac{d}{dt_0}\left\{\pm\frac{f(t_o)}{2}+\frac{1}{2\pi i}\int_\Gamma\frac{f(t)dt}{t-t_o}\right\} \\ &= \pm\frac{f'(t_o)}{2}+\frac{1}{2\pi i}\int_\Gamma\frac{f(t)dt}{(t-t_o)^2} \quad (t_o\in\Gamma) \end{aligned} \tag{b22}$$

where the second term in the right hand of Eq. (b22) represents a hypersingular integral. This equation was obtained in an alternative way [30].

From Eqs. (b1) and (b2), we can define the following integrals:

$$V_1(z) = \frac{d}{dz}\{W_1(z)\} = \frac{1}{2\pi i}\int_\Gamma M_1(t,z)f(t)dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b23}$$

$$V_2(z) = \frac{d}{dz}\{W_2(z)\} = \frac{1}{2\pi i}\int_\Gamma M_2(t,z)f(t)\,dt \quad (z\in S^+ \text{ or } z\in S^-) \tag{b24}$$

where

$$\begin{aligned} M_1(t,z) &= \frac{d}{dz}\{L_1(t,z)\} = -\frac{d}{dz}\left\{\frac{d}{dt}\left\{\ln\frac{t-z}{\bar t-\bar z}\right\}\right\} = \\ &-\frac{1}{(t-z)^2}+\frac{1}{(\bar t-\bar z)^2}\frac{d\bar t}{dt}\frac{d\bar z}{dz} \end{aligned} \tag{b25}$$

$$\begin{aligned} M_2(t,z) &= \frac{d}{dz}\{L_2(t,z)\} = \frac{d}{dz}\left\{\frac{d}{dt}\left\{\frac{t-z}{\bar t-\bar z}\right\}\right\} = \frac{1}{(\bar t-\bar z)^2}\left(\frac{d\bar t}{dt}+\frac{d\bar z}{dz}\right) \\ &-\frac{2(t-z)}{(\bar t-\bar z)^3}\frac{d\bar t}{dt}\frac{d\bar z}{dz} \end{aligned} \tag{b26}$$

In Eq. (b23), letting $z \to t_o$ ($z \in S^+$, $t_o \in \Gamma$) and letting $z \to t_o$ ($z \in S^-$, $t_o \in \Gamma$), $dz = dt_o$ and using Eq. (b5), we will find

$$V_1^{\pm}(t_o) = \frac{d}{dt_o}\{W_1^{\pm}(t_o)\} = \frac{d}{dt_o}\left\{ \mp f(t_o) + \frac{1}{2\pi i}\int_\Gamma L_1(t,t_o)f(t)dt \right\}$$
$$= \mp f'(t_o) + \frac{1}{2\pi i}\int_\Gamma M_1(t,t_o)f(t)\,dt \quad (t_o \in \Gamma) \tag{b27}$$

Similarly, from Eqs. (b6), (b24), we can obtain

$$V_2^{\pm}(t_o) = \frac{1}{2\pi i}\int_\Gamma M_2(t,t_o)f(t)\,dt \quad (t_o \in \Gamma) \tag{b28}$$

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