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Numerical solution for degenerate scale problem for exterior multiply connected region

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ABSTRACT

Based on some previous publications, this paper investigates the numerical solution for degenerate scale problem for exterior multiply connected region. In the present study, the first step is to formulate a homogenous boundary integral equation (BIE) in the degenerate scale. The coordinate transform with a magnified factor, or a reduced factor h is performed in the next step. Using the property $\ln(hx) = \ln(x)+\lg(h)$, the new obtained BIE equation can be considered as a non-homogenous one defined in the transformed coordinates. The relevant scale in the transformed coordinates is a normal scale. Therefore, the new obtained BIE equation is solvable. Fundamental solutions are introduced. For evaluating the fundamental solutions, the right-hand terms in the non-homogenous equation, or a BIE, generally take the value of unit or zero. By using the obtained fundamental solutions, an equation for evaluating the magnified factor "h" is obtained. Finally, the degenerate scale is obtainable. Several numerical examples with two ellipses in an infinite plate are presented. Numerical solutions prove that the degenerate scale does not depend on the normal scale used in the process for evaluating the fundamental solutions.

1. Introduction

The boundary integral equation (BIE) was widely used in elasticity, and the fundamental for BIE could be found from [1–3]. Heritage and early history of the boundary element method was summarized more recently [4].

The degenerate scale problem in BIE is a particular boundary value problem in plane elasticity as well as in Laplace equation. The problem typically arises in the exterior Dirichlet problem of plane elasticity. For example, an infinite plate contains an elliptic notch with assumption of vanishing displacements on the elliptic contour. In this case, if the size reaches a critical size, there is a stress field existing in the notched infinite region. The critical size is called the degenerate scale. This situation means that the homogenous BIE has been shown to yield a non-trivial solution for boundary tractions, (or $\sigma_{ij} \neq 0$ at the boundary points) when the adopted configuration is equal to a degenerate scale. From the viewpoint of engineering, the non-trivial solution of the homogenous BIE is an illogical one. Clearly, the relevant non-homogenous integral equation must have non-unique, or many solutions. Therefore, one must avoid meeting illogical solution caused by occurrence of the degenerate scale.

For the Laplace equation, the non-unique solution of a circle with a unit radius has been noted [5]. In fact, this is the degenerate scale problem. Some researchers define the degenerate scale problem such that the degenerate scale results in a zero eigenvalue for the integral operator.

Since the displacement and stress condition at infinity may have the following properties, $u_i = O(\ln R)$, $\sigma_{ij} = O(1/R)$, where R is the radius of a sufficient large circle, it is expected to have a non-trivial solution from the viewpoint of mathematics and mechanics [6].

Many researchers using a variety of methods studied the degenerate scale problems. Some researchers investigated the problem from mathematical theory of BIE. For example, it was pointed out that an explicit equation for evaluation of critical scales for a given boundary, when the single-layer operator fails to be invertible, is deduced. It is proved that there are either two simple critical scales or one double critical scale for any domain boundary [7]. The logarithmic function appearing in the integral kernel leads to the possibility of this operator being non-invertible, the solution of the BIE either being non-unique or not existing [8].

The boundary integral equation is used for a ring region with the vanishing displacements along the boundary. In some particular scale of the configuration, the corresponding homogenous equation has non-trivial solution for the boundary tractions [9,10]. Numerical technique for evaluating the degenerate scale was also suggested by using two sets of some particular solution based on a modified BIE [8]. Mathematical

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analysis of the degenerate scale problems for an elliptic-domain problem in elasticity was presented [11].

Numerical procedure was developed to evaluate the degenerate scale directly from the zero value of a determinant. In the procedure, the influence matrix is denoted by $U(a)$, where "a" is a size. In the computation, the size "a" is changed gradually. If at some particular value $a=\lambda$, the following conditions, det $U(a)|_{a=\lambda-\varepsilon}>0$ and det $U(a)|_{a=\lambda+\varepsilon}>0$ ($\varepsilon=0.00002$ a small value) are satisfied, the obtained $a=\lambda$ is the degenerate scale. In the time of evaluating the degenerate scale, the relevant non-trivial solution was also obtained [12,13].

It was pointed out that the influence matrix of the weakly singular kernel (logarithmic kernel) might be singular for the Dirichlet problem of Laplace equation when the geometry is reaching some special scale [14]. In this case, one may obtain a non-unique solution. The non-unique solution is not physically realizable. From the viewpoint of linear algebra, the problem also originates from rank deficiency in the influence matrices [14,15]. By using a particular solution in the normal scale, the degenerate scale can be evaluated numerically [14]. The degenerate scale problem in BIE for the two-dimensional Laplace equation was studied by using degenerate kernels and Fourier Series [16]. In the circular domain case, the degenerate scale problem in BIE for the two-dimensional Laplace equation was studied by using degenerate kernels and circulants, where the circulants mean that the influence matrices for the discrete system has a particular character [17].

Degenerate scale for multiply connected Laplace problems was studied [18]. It is concluded from the paper that: "no matter how many inner holes are randomly distributed inside the outer boundary, the Dirichlet problem with radius ($a_1=1.0$) of the outer boundary is not solvable due to the rank-deficiency matrix of $[U]$ in Eq. (13)".

The coordinate transform technique for evaluating the degenerate scale in for Laplace equation was proposed [5,14]. Similar derivation for the case of plane elasticity was studied [8]. Based on the mentioned references, this paper investigates the numerical solution for degenerate scale problem for exterior multiply connected region.

In the present study, the first step is to formulate a homogenous equation in the degenerate scale. The coordinate transform with a magnified factor, or a reduced factor "h" is performed in the next step. Using the property $\ln(hx)=\ln(x)+\lg(h)$, the new obtained BIE equation is non-homogenous defined in the transformed coordinates. The relevant scale in the transformed coordinates is a normal scale. Therefore, the new obtained BIE equation is solvable. Fundamental solutions are introduced. For evaluating the fundamental solutions, the right-hand terms in the non-homogenous equation, or a BIE, generally take the value of unit or zero. By using the obtained fundamental solutions, an equation for evaluating the magnified factor "h" is obtained. Finally, the degenerate scale is obtainable. Several numerical examples with two ellipses in an infinite plate are presented.

2. Formulation of the degenerate scale problem for exterior multiply connected region

The concept of using the coordinate transform technique to evaluate the degenerate scale was introduced [5]. This technique was used in the case of the Laplace equation [14]. Similar derivation for the case of plane elasticity was studied [8]. Based on the mentioned references, this paper investigates the numerical solution for degenerate scale problem for exterior multiply connected region.

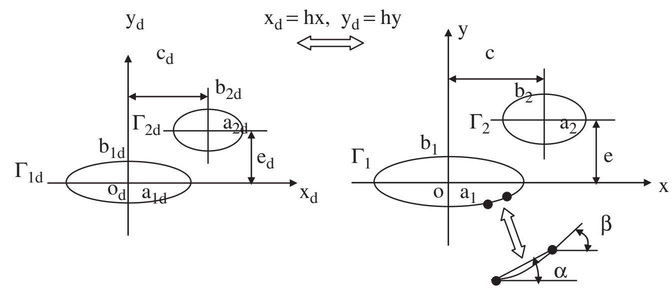

Fig. 1. Formulation of the degenerate scale problem, (a) the degenerate scale and (b) the normal scale with the relation to the degenerate scale by $x_d=hx$, $y_d=hy$, $a_{1d}=ha_1$, $b_{1d}=hb_1$, $a_{2d}=\delta a_2$, $b_{2d}=hb_2$, $c_d=hc$ and $e_d=he$.

In the first step, the degenerate scale problem for two holes is reduced to find a non-trivial solution for the following homogenous equation in the degenerate scale or in the coordinates $o_dx_dy_d$ (Fig. 1)

$$\int_{\Gamma_{1d}} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_{2d}} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=0,\ (\xi\in\Gamma_{1d},\ i=1,2) \tag{1}$$

$$\int_{\Gamma_{1d}} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_{2d}} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=0,\ (\xi\in\Gamma_{2d},\ i=1,2) \tag{2}$$

where Γ_{1d} and Γ_{2d} denotes the boundary of two ellipses, and the kernel $U_{ij}{}^*(\xi,x)$ is defined by

$$U^*_{ij}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}-0.5\delta_{ij}\} \tag{3}$$

where Kronecker deltas δ_{ij} is defined as, $\delta_{ij}=1$ for $i=j$, $\delta_{ij}=0$ for $i\neq j$, and

$$r_{,1}=\frac{x_1-\xi_1}{r}=\cos\alpha,\ r_{,2}=\frac{x_2-\xi_2}{r}=\sin\alpha$$
$$n_1=-\sin\beta,\ n_2=\cos\beta \tag{4}$$

where the angles α and β are indicated in Fig. 1. In Eq. (3), G denotes the shear modulus of elasticity and ν the Poisson's ratio. It was proved previously that the following kernel $U_{ij}{}^*(\xi,x)$ satisfies the regularity condition at infinity in BIE [12]. Therefore, this kernel is used in the formulation of the degenerate scale problem.

The equation may be solved in the normal scale with coordinates oxy. The two coordinates have the following relations (Fig. 1)

$$x_d=hx,\ y_d=hy,\ a_{1d}=ha_1,\ b_{1d}=hb_1,\ a_{2d}=\delta a_2,\ b_{2d}=hb_2,$$
$$c_d=hc,\ e_d=he \tag{5}$$

where "h" denotes a magnified factor or a reduced factor.

Therefore, Eqs. (1) and (2) can be converted to the following non-homogenous equations in the normal oxy coordinates

$$\int_{\Gamma_1} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_2} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=\frac{1}{\gamma}q_i,\ (\xi\in\Gamma_1,\ i=1,2) \tag{6}$$

$$\int_{\Gamma_1} U^*_{ij}(\xi,x)p_{1j}(x)\mathrm{d}s(x)+\int_{\Gamma_2} U^*_{ij}(\xi,x)p_{2j}(x)\mathrm{d}s(x)=\frac{1}{\gamma}q_i,\ (\xi\in\Gamma_2,\ i=1,2) \tag{7}$$

where Γ_1, Γ_2 denote relevant configuration in the normal scale and

$$\frac{1}{\gamma}=\frac{3-4\nu}{8\pi(1-\nu)G}\ln h \tag{8}$$

$$q_i = Q_{1i} + Q_{2i},\ Q_{1i} = \int_{\Gamma_1} p_{1i}(x)\mathrm{d}s(x),\ Q_{2i} = \int_{\Gamma_2} p_{2i}(x)\mathrm{d}s(x)\ (i = 1, 2) \tag{9}$$

Q_{1i} (Q_{2i}) ($i = 1$, 2) represents the resultant force applied on the holes Γ_1, Γ_2, respectively. Note that the right-hand terms in Eqs. (6) and (7) or q_i/γ represent some constants.

In this case, one may propose following integral equation:

$$\int_{\Gamma_1} U^*_{ij}(\xi, x)p_{1j}(x)\mathrm{d}s(x) + \int_{\Gamma_2} U^*_{ij}(\xi, x)p_{2j}(x)\mathrm{d}s(x) = f_i,\ (\xi \in \Gamma_1,\ i = 1, 2) \tag{10}$$

$$\int_{\Gamma_1} U^*_{ij}(\xi, x)p_{1j}(x)\mathrm{d}s(x) + \int_{\Gamma_2} U^*_{ij}(\xi, x)p_{2j}(x)\mathrm{d}s(x) = g_i,\ (\xi \in \Gamma_2,\ i = 1, 2) \tag{11}$$

From Eqs. (10) and (11), we can define two fundamental solutions by

$$f_1 = 1,\ f_2 = 0,\ g_1 = 1,\ g_2 = 0 \text{ (for the first fundamental solution)} \tag{12}$$

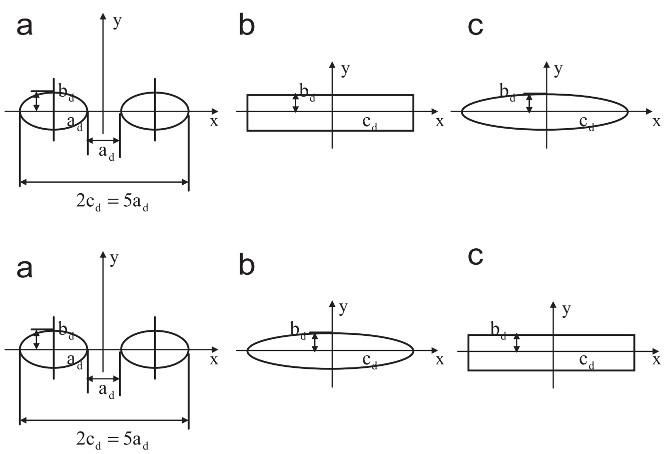

Fig. 2. (a) Two ellipses in series in an infinite plate, (b) an ellipse in an infinite plate and (c) a rectangle in an infinite plate.

$$f_1 = 0,\ f_2 = 1,\ g_1 = 0,\ g_2 = 1 \text{ (for the second fundamental solution)} \tag{13}$$

The obtained fundamental solutions for tractions are denoted by $p_{1i}{}^{(k)}(x)$ ($i = 1$, 2, $x \in \Gamma_1$), $p_{2i}{}^{(k)}(x)$ ($i = 1$, 2, $x \in \Gamma_2$), for $k = 1$, 2. In the notion for $p_{mi}{}^{(k)}(x)$, the superscript (k) denotes the kind of solution, the first subscript "m" denotes the number of hole, and the second subscript "i" denotes the direction of the force. For example, $p^{(1)}_{12}(x)$ denotes the force in "2" (or y) direction at "1" hole (or the first hole) in the first fundamental solution defined by Eqs. (10)–(12).

From obtained fundamental solutions, we can define

$$t^{(k)}_{1i} = \int_{\Gamma_1} p^{(k)}_{1i}(x)ds(x),\ t^{(k)}_{2i} = \int_{\Gamma_2} p^{(k)}_{2i}(x)ds(x) \tag{14}$$

Clearly, $t_{1i}{}^{(k)}$ ($t_{2i}{}^{(k)}$) ($i = 1$, 2) represents the resultant force applied on the holes Γ_1, Γ_2 from k-th fundamental solution, respectively. In the notion for $t_{mi}{}^{(k)}$, the superscript (k) denotes the kind of solution, the first subscript "m" denotes the number of hole, and the second subscript "i" denotes the direction of the resultant force. For example, $t^{(1)}_{12}$ denotes the resultant force in "2" (or y) direction at "1" hole (or the first hole) in the first fundamental solution defined by Eqs. (10)–(12).

From the definition of the fundamental solutions, the solution for Eqs. (6) and (7) is obtained as follows:

$$p_{1i}(x) = \frac{1}{\gamma}\{q_1 p^{(1)}_{1i}(x) + q_2 p^{(2)}_{1i}(x)\},\ (i = 1, 2,\ x \in \Gamma_1) \tag{15}$$

$$p_{2i}(x) = \frac{1}{\gamma}\{q_1 p^{(1)}_{2i}(x) + q_2 p^{(2)}_{2i}(x)\},\ (i = 1, 2,\ x \in \Gamma_2) \tag{16}$$

One operates the following integration $\int_{\Gamma_1}[\ldots]ds(x)$ to Eq. (15), we will obtain

$$\gamma Q_{1i} = \{q_1 t^{(1)}_{1i} + q_2 t^{(2)}_{1i}\},\ (i = 1, 2) \tag{17}$$

Similarly, one operates the following integration $\int_{\Gamma_2}[\ldots]ds(x)$ to Eq. (16), we will obtain

$$\gamma Q_{2i} = \{q_1 t^{(1)}_{2i} + q_2 t^{(2)}_{2i}\},\ (i = 1, 2) \tag{18}$$

Table 1
Computed degenerate scales $c_{d,1} = f_1(b_d/c_d)$ and $c_{d,2} = f_2(b_d/c_d)$ (see Fig. 2(a) and Eq. (24)).

Taking normal size $c = 10$										
$b_d/c_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c_{d,1}$	2.2453	1.9317	1.6973	1.5147	1.3683	1.2482	1.1478	1.0627	0.9896	0.9404
$c_{d,2}$	1.4573	1.3742	1.2993	1.2319	1.1712	1.1162	1.0663	1.0206	0.9788	0.9260
Taking normal size $c = 100$										
$b_d/c_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c_{d,1}$	2.2453	1.9317	1.6973	1.5147	1.3683	1.2482	1.1478	1.0627	0.9896	0.9404
$c_{d,2}$	1.4573	1.3742	1.2993	1.2319	1.1712	1.1162	1.0663	1.0206	0.9788	0.9260

Table 2
Comparison results for the degenerate scales $c_{d,1} = f_1(b_d/c_d)$ and $c_{d,2} = f_2(b_d/c_d)$ (see Fig. 2(a)–(c) and Eq. (24)).

For the case of two ellipses shown by Fig. 2(a)										
$b_d/c_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c_{d,1}$	2.2453	1.9317	1.6973	1.5147	1.3683	1.2482	1.1478	1.0627	0.9896	0.9404
$c_{d,2}$	1.4573	1.3742	1.2993	1.2319	1.1712	1.1162	1.0663	1.0206	0.9788	0.9260
For the case of an ellipse shown in Fig. 2(b)										
$b_d/c_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c_{d,1}$	2.2828	2.0063	1.7872	1.6096	1.4631	1.3403	1.2359	1.1463	1.0684	1.0003
$c_{d,2}$	1.4490	1.3853	1.3251	1.2686	1.2158	1.1665	1.1205	1.0776	1.0376	1.0003
For the case of a rectangle shown in Fig. 2(c)										
$b_d/c_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c_{d,1}$	2.0720	1.7488	1.5266	1.3607	1.2309	1.1255	1.0382	1.9643	0.9009	0.8457
$c_{d,2}$	1.3797	1.2755	1.1913	1.1207	1.0603	1.0071	0.9601	0.9181	0.8801	0.8457

By adding Eqs. (17) and (18), we will obtain

$$\gamma q_i = \{(t_{1i}^{(1)} + t_{2i}^{(1)})q_1 + (t_{1i}^{(2)} + t_{2i}^{(2)})q_2\},\ (i=1,2) \tag{19}$$

Eq. (19) may be written in an explicit form

$$(t_{11}^{(1)} + t_{21}^{(1)} - \gamma)q_1 + (t_{11}^{(2)} + t_{21}^{(2)})q_2 = 0,$$
$$(t_{12}^{(1)} + t_{22}^{(1)})q_1 + (t_{12}^{(2)} + t_{22}^{(2)} - \gamma)q_2 = 0 \tag{20}$$

In order that Eq. (20) has a non-trivial solution for q_1, q_2, we have

$$\begin{vmatrix} t_{11}^{(1)} + t_{21}^{(1)} - \gamma & t_{11}^{(2)} + t_{21}^{(2)} \\ t_{12}^{(1)} + t_{22}^{(1)} & t_{12}^{(2)} + t_{22}^{(2)} - \gamma \end{vmatrix} = 0 \tag{21}$$

Generally, we have two roots $\gamma_i (i = 1, 2)$ from Eq. (21). Further, from Eq. (8), the following magnification factors can be found

$$h_i = \exp\left(\frac{8\pi(1-\nu)G}{\gamma_i(3-4\nu)}\right)\ (i=1,2) \tag{22}$$

Finally, two sets for the degenerate scales of for ellipses can be obtained

$$a_{1d,i} = h_i a_1,\ b_{1d,i} = h_i b_1,\ a_{2d,i} = h_i a_2,\ b_{2d,i} = h_i b_2,\ c_{d,i} = h_i c,$$
$$e_{d,i} = h_i e,\ (i=1,2) \tag{23}$$

Since the fundamental solutions defined by Eqs. (10)–(13) are solved on the normal size, one must choose the normal size in advance. A detailed description for this problem will be introduced in the section “discussions”.

3. Numerical examples

Several numerical examples are presented below. The present results can prove the efficiency of suggested technique.

3.1. Example 1

In the first example, two ellipses are in series (Fig. 2(a)). In computation, 60 divisions are used in for the integral Eqs. (10) and (11) and $\nu = 0.3$ is assumed. In addition, we choose $c_d = 2.5a_d$, and b_d/c_d changes from 0.1, 0.2,… to 1.0. Since all sizes are proportional each other, one can choose the c_d value as the degenerate scale. The computed degenerate scale can be expressed as

$$c_{d,1} = f_1(b_d/c_d) \text{ and } c_{d,2} = f_2(b_d/c_d) \tag{24}$$

The counterpart of the degenerate size c_d is the size “c” with the relation $c_d = hc$. If we choose the normal size $c = 10$, the computed results for $c_{d,1}$ and $c_{d,2}$ are listed in Table 1. Similarly, if we choose the normal size $c = 100$, the computed results for $c_{d,1}$ and $c_{d,2}$ are also listed in Table 1. From computed results, we see that the degenerate scale do not depend on the normal size used.

For comparison, the computed results from different geometries of notches are carried out. For three cases, (a) the results in the present example shown by Fig. 2(a) and (b) the results from a simple ellipse shown by Fig. 2(b) and (c) the results from a rectangular notch shown by Fig. 2(c), are listed in Table 2. The computed results for the second and third cases are obtained from [12]. It is seen that, if b_d/c_d keeps on the same value, the differences for $c_{d,1}$ and $c_{d,2}$ for first two cases are not significant.

3.2. Example 2

In the second example, two ellipses are in stacking position (Fig. 3). In computation, 60 divisions are used in for the integral Eqs. (10) and (11) and $\nu = 0.3$ is assumed. In addition, we choose $c_d = 2.5b_d$, and c_d/a_d changes from 0.1, 0.2,… to 1.0. Since all sizes are proportional to each other, one can choose the a_d value as the degenerate scale. The computed degenerate scale can be expressed as

$$a_{d,1} = f_1(c_d/a_d) \text{ and } a_{d,2} = f_2(c_d/a_d) \tag{25}$$

The counterpart of the degenerate size a_d is the size “a” with the relation $a_d = ha$. If we choose the normal size $a = 4$, the computed results for $a_{d,1}$ and $a_{d,2}$ are listed in Table 3. Similarly, if we choose the normal size $a = 40$, the computed results for $a_{d,1}$ and $a_{d,2}$ are also listed in Table 3. From computed results, we see that the degenerate scale do not depend on the normal size used.

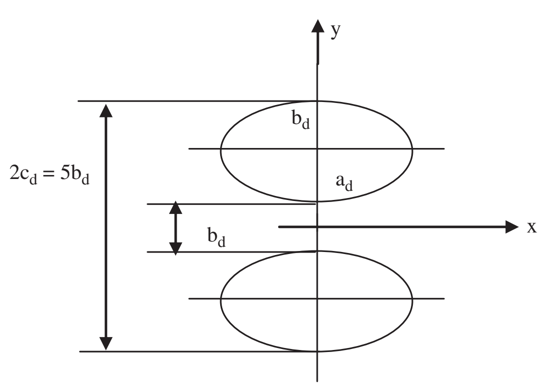

Fig. 3. Two ellipses in sacking position in an infinite plate.

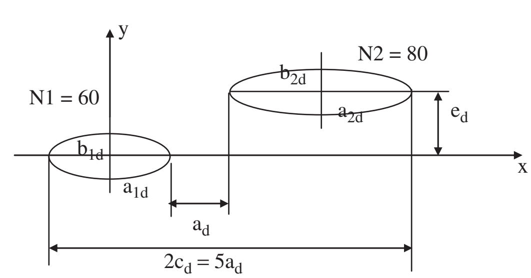

Fig. 4. Two ellipses with unequal sizes in an infinite plate with $a_{1d} = 0.8a_d$, $b_{1d} = 0.5a_d$, $a_{2d} = 1.2a_d$, $b_{2d} = 0.5a_d$ and $2c_d = 5a_d$.

Table 3
Computed results for the degenerate scales $a_{d,1} = f_1(c_d/a_d)$ and $a_{d,2} = f_2(c_d/a_d)$ (see Fig. 3 and Eq. (25)).

Taking normal size $a = 4$										
$c_d/a_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$a_{d,1}$	2.1578	1.8666	1.6540	1.4885	1.3549	1.2443	1.1508	1.0708	1.0013	0.9260
$a_{d,2}$	1.4047	1.3207	1.2497	1.1878	1.1329	1.0836	1.0389	0.9980	0.9606	0.9404
Taking normal size $a = 40$										
$c_d/a_d =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$a_{d,1}$	2.1578	1.8666	1.6540	1.4885	1.3549	1.2443	1.1508	1.0708	1.0013	0.9260
$a_{d,2}$	1.4047	1.3207	1.2497	1.1878	1.1329	1.0836	1.0389	0.9980	0.9606	0.9404

Table 4
Computed results for the degenerate scales $c_{d,1}=f_1(e_d/c_d)$, and $c_{d,2}=f_2(e_d/c_d)$ (see Fig. 4 and Eq. (26)).

$e_d/c_d=$	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$c_{d,1}$	1.9304	1.9132	1.8663	1.8001	1.7248	1.6478	1.5735	1.5038	1.4396	1.3809	1.3274
$c_{d,2}$	1.3737	1.3670	1.3477	1.3187	1.2832	1.2443	1.2040	1.1640	1.1253	1.0883	1.0533

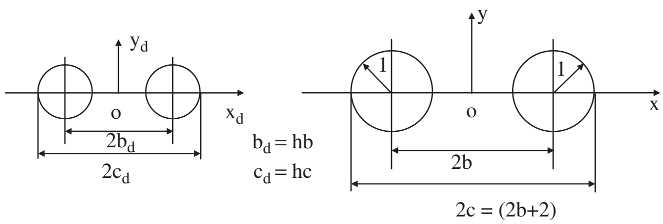

Fig. 5. Two circular holes with equal sizes.

3.3. Example 3

In the third example, two ellipses have unequal sizes in an infinite plate with $a_{1d}=0.8a_d$, $b_{1d}=0.5a_d$, $a_{2d}=1.2a_d$, $b_{2d}=0.5a_d$ and $2c_d=5a_d$ (Fig. 4). In computation, 60 (80) divisions are used in for the first hole and the second hole, respectively. $\nu=0.3$ is assumed. In addition, we choose $c_d=2.5a_d$, and e_d/c_d changes from 0.2, 0.4,… to 2.0. Since all sizes are proportional each other, one can choose the c_d value as the degenerate scale. The computed degenerate scale can be expressed as

$$c_{d,1}=f_1(e_d/c_d) \text{ and } c_{d,2}=f_2(e_d/c_d) \tag{26}$$

The counterpart of the degenerate size a_d is the size "a" with the relation $a_d=ha$. If we choose the normal size $a=4$, the computed results for $c_{d,1}$ and $c_{d,2}$ are listed in Table 4.

From listed results, we see that if the ratio for height versus width takes $e_d/(2c_d)=1$, we have $c_{d,1}=1.3274$, $c_{d,2}=1.0533$and $c_{d,1}/c_{d,2}=1.2602$. However, in the example 1, if the ratio for height versus width takes $(2b_d)/(2c_d)=1$, we have $c_{d,1}=0.9404$, $c_{d,2}=0.9260$ and $c_{d,1}/c_{d,2}=1.0156$. Those mentioned results reflect the nature of asymmetric condition in the present example.

3.4. Example 4

In the fourth example, two circular holes are in series (Fig. 5). In computation, 60 divisions are used in for the integral Eqs. (10) and (11) and $\nu=0.25$ is assumed. The present example is devoted to compare the present procedure to the same computation in [7]. The distance between two holes is "2b", and the width of the configuration is $2c=2(b+1)$ (Fig. 5). In computation, the c_d is taken as the degenerate scale. Therefore, the two generate scales can be expressed as

$$c_{d,1}=h_1c \text{ and } c_{d,2}=h_2c \tag{27}$$

Clearly, once the value for "b" is assumed, we can perform the relevant computation. The computed results are expressed as

$$h_1=h_1(b),\ h_2=h_2(b) \tag{28}$$

$$c_{d,1}=f_1(b),\ c_{d,2}=f_2(b) \tag{29}$$

The computed results for two sets: (1) "b" changes from 0.25, 0.50,… to 2.5 and (2) "b" changes from 2.5, 5.0,… to 25, are listed in Table 5.

In literature [3], the following kernel is suggested:

$$U_{ij}^{**}(\xi,x)=\frac{1}{8\pi(1-\nu)G}\{-(3-4\nu)\ln(r)\delta_{ij}+r_{,i}r_{,j}\} \tag{30}$$

By using this kernel, the computed results for two sets: (1) "b" changes from 0.25, 0.50,… to 2.5 and (2) "b" changes from 2.5, 5.0,… to 25, are also listed in Table 5. It is found from the computed results that the computed h_1 and h_2 values coincide with those obtained in Ref. [7].

The additional constant in the kernel $U_{ij}{}^*(\xi,\ x)$ will influence the computed degenerate scale. This phenomena can be found from the present example. For example, for $b=1$ case, we have $c_{d,1}=1.3675$, $c_{d,2}=1.1822$, if $U_{ij}{}^*(\xi,\ x)$ is used. However, in the same condition of $b=1$, we have $c_{d,1}=1.7559$, $c_{d,2}=1.5180$, if $U_{ij}{}^*(\xi,\ x)$ is used.

4. Discussions

The merit of suggested method is that the relevant BIE in the normal size shown by Eqs. (6) and (7) has the invertible property, or the invertibility. Similar property exists in Eqs. (10) and (11) for evaluating the fundamental solutions. This property will lead to the fact that the BIE shown by Eqs. (6) and (7) has a unique solution whatever the right-hand term is assumed. Some researchers proposed this conclusion [8,14].

After discretization, Eqs. (6) and (7) may be written in the form

$$\mathbf{Up}=\mathbf{r} \tag{31}$$

The invertible property in the normal size means the matrix $\mathbf{U}$ has its inverse matrix $\mathbf{U}^{-1}$. Therefore, the following solution is obtainable

$$\mathbf{p}=\mathbf{U}^{-1}\mathbf{r} \tag{32}$$

However, the right-hand terms of Eqs. (6) and (7) are composed are some integrals with respect to the unknown functions $p_{1i}(x)$ and $p_{2i}(x)$ $(i=1,\ 2)$. Eqs. (6) and (7) may be written in the form

$$\int_{\Gamma_1}(U_{ij}^*(\xi,x)-\frac{1}{\gamma}p_{1j}(x))ds(x)+\int_{\Gamma_2}(U_{ij}^*(\xi,x)-\frac{1}{\gamma})p_{2j}(x)ds(x)=0,\quad (\xi\in\Gamma_1,\ i=1,2) \tag{33}$$

$$\int_{\Gamma_1}(U_{ij}^*(\xi,x)-\frac{1}{\gamma}p_{1j}(x))ds(x)+\int_{\Gamma_2}(U_{ij}^*(\xi,x)-\frac{1}{\gamma})p_{2j}(x)ds(x)=0,\quad (\xi\in\Gamma_2,\ i=1,2) \tag{34}$$

Clearly, Eqs. (33) and (34) are represented in the form of finding a particular value γ. Those equations mean that one needs to find some particular values of γ such that Eqs. (33) and (34) have non-trivial solution for $p_{1i}(x)$ and $p_{2i}(x)$ $(i=1,\ 2)$. Here, $p_{1i}(x)=0$ and $p_{2i}(x)=0$ $(i=1,\ 2)$ denote the trivial solution. The particular features of those equations are as follows. First, Eqs. (33) and (34) are formulated in the normal scale. Second, the value γ is clearly indicated in equations. After introducing the two fundamental solutions defined by Eqs. (10)–(13), the problem for finding γ can be exactly formulated, which is shown by Eq. (21).

Since the computation is performed on the normal size, one must choose the normal size in advance. In fact, the normal size is easy to choose. It is assumed that we have an elliptic notch with the major half-axis "a" in x-direction and minor half-axis "b" in y-direction. For b/a changes from 0.1, 0.2,… to 1.0, the a_d values are generally less than 2.28278 [12]. Therefore, if one chooses $a>4$, it is sufficient to meet the condition of normal size in the case of $0.1<b/a<1$.

Table 5
Computed results for the reduced factors $h_1 = h_1(b)$, $h_2 = h_2(b)$ and degenerate scales $c_{d,1} = f_1(b)$ and $c_{d,2} = f_2(b)$ (see Fig. 5 and Eqs. (27) and (28)).

Using the kernel U^* defined by Eq. (3)										
$b =$	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.5
h_1	0.8883	0.8041	0.7378	0.6837	0.6386	0.6001	0.5667	0.5377	0.5121	0.4895
h_2	0.8469	0.7373	0.6551	0.5911	0.5398	0.4983	0.4643	0.4361	0.4122	0.3918
$c_{d,1}$	1.1104	1.2061	1.2911	1.3675	1.4368	1.5001	1.5585	1.6130	1.6644	1.7134
$c_{d,2}$	1.0586	1.1059	1.1464	1.1822	1.2147	1.2457	1.2768	1.3082	1.3398	1.3714
$b =$	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5	25.0
h_1	0.4895	0.3552	0.2915	0.2529	0.2264	0.2068	0.1915	0.1792	0.1689	0.1603
h_2	0.3918	0.2787	0.2278	0.1974	0.1765	0.1612	0.1492	0.1396	0.1316	0.1249
$c_{d,1}$	1.7134	2.1315	2.4781	2.7823	3.0567	3.3086	3.5427	3.7624	3.9700	4.1673
$c_{d,2}$	1.3714	1.6722	1.9363	2.1709	2.3834	2.5789	2.7608	2.9315	3.0930	3.2465
Using the kernel U^{**} defined by Eq. (30)										
$b =$	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.5
h_1	1.1406	0.9467	0.9473	0.8780	0.8199	0.7705	0.7277	0.6904	0.6576	0.6286
h_2	1.0874	1.0324	0.8412	0.7590	0.6932	0.6398	0.5962	0.5599	0.5293	0.5031
$c_{d,1}$	1.4258	1.4200	1.6578	1.7559	1.8449	1.9262	2.0012	2.0711	2.1371	2.2000
$c_{d,2}$	1.3593	1.5486	1.4721	1.5180	1.5596	1.5995	1.6395	1.6798	1.7203	1.7609
$b =$	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5	25.0
h_1	0.6286	0.4561	0.3743	0.3248	0.2907	0.2655	0.2459	0.2301	0.2169	0.2058
h_2	0.5031	0.3579	0.2925	0.2534	0.2267	0.2070	0.1916	0.1792	0.1690	0.1603
$c_{d,1}$	2.2000	2.7369	3.1819	3.5725	3.9248	4.2483	4.5490	4.8311	5.0976	5.3510
$c_{d,2}$	1.7609	2.1472	2.4862	2.7874	3.0603	3.3113	3.5449	3.7642	3.9715	4.1686

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