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Comparisons of fundamental solutions and particular solutions for Trefftz methods ☆

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A B S T R A C T

In the Trefftz method (TM), the admissible functions satisfying the governing equation are chosen, then only the boundary conditions are dealt with. Both fundamental solutions (FS) and particular solutions (PS) satisfy the equation. The TM using FS leads to the method of fundamental solutions (MFS), and the TM using PS to the method of particular solutions (MPS). Since the MFS is one of TM, we may follow our recent book [20,21] to provide the algorithms and analysis. Since the MFS and the MPS are meshless, they have attracted a great attention of researchers. In this paper numerical experiments are provided to support the error analysis of MFS in Li [15] for Laplace's equation in annular shaped domains. More importantly, comparisons are made in analysis and computation for MFS and MPS. From accuracy and stability, the MPS is superior to the MFS, the same conclusion as given in Schaback [24]. The uniform FS is simpler and the algorithms of MFS are easier to carry out, so that the computational efforts using MFS are much saved. Since today, the manpower saving is the most important criterion for choosing numerical methods, the MFS is also beneficial to engineering applications. Hence, both MFS and MPS may serve as modern numerical methods for PDE.

1. Introduction

The Trefftz method (TM) as boundary methods has been fully developed in theory and computation for several decades (see [20]), where the particular solutions (PS) are used. In fact, the method of fundamental solutions (MFS) originated by Kupradze [11] is one of TM using fundamental solutions (FS). In order to distinguish easily their differences, in this paper, the TM using PS is called the method of particular solutions (MPS), as in Betcke and Trefethen [2].[2] Both the MFS and MPS belong to TM, and they can be carried out by the collocation TM (CTM) in [20]. Since both MFS and MPS are meshless, they have attracted a great attention of researchers. It is a time to explore MFS and MPS together, and to provide their comparisons. In fact, some comparisons of MFS and MPS have been addressed in Schaback [24]. In this paper, their comprehensive comparisons are explored in both analysis and computation.

Take the Laplace equation in 2D for example (see Fig. 1). If the solution is smooth, the harmonic polynomials of order n are used as the particular solutions in MPS, and if the solution has the corner singularity with $O(r^{\alpha})(\frac{1}{4} < \alpha < 2)$, the singular solutions given in [20], Chapter 11, are also used as the particular solutions in TM. Note that the harmonic particular solutions look quite differently, depending on the domain angles and the boundary conditions. On the other hand, the invariant fundamental solution, $\ln r$, is used in TM, so that the algorithms of MFS are simple and easy to carry out. More importantly, the algorithms and programs of MFS for engineering problems do not need much mathematical background, so that even students in high school may learn and use MFS easily.

For smooth solutions of Laplace's equation, the errors of MFS will not be smaller than those of MPS using harmonic polynomials. It is also proven by Schaback [24] that, when $R \to \infty$, the fundamental solutions (FS) go to the harmonic polynomials. However, the instability of MFS (even with small R) is much worse than that of MPS using harmonic polynomials, since both condition number and effective condition number grow exponen-

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2 In Chen et al. [4], the method of particular solutions (MPS) is referred only to the method to seek the special particular solutions of nonhomogeneous equations, such as Poisson's equation.

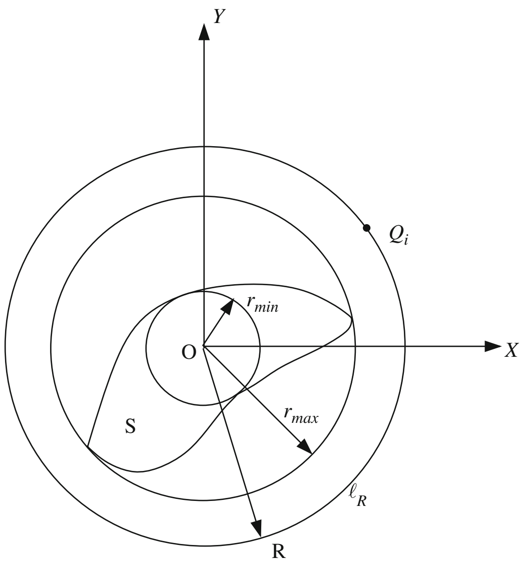

Fig. 1. The solution domain S with its circles.

tially for the MFS, see [19]. Hence, using the harmonic polynomials is superior to the FS for smooth solutions. Moreover, the MFS is inefficient for corner singularity, except for using some techniques, such as local refinements or adding singular functions (see Li [14]). For engineering problems, although MPS is more efficient and beneficial than MFS, the manpower using MFS is much saving. Since today, the computational saving is the most important criterion for choosing numerical methods, both MFS and MPS may serve as modern numerical methods for PDE.

Since *no method is perfect*, a natural strategy is to use the singular particular solutions to deal with corner singularity, and the FS for the rest of the solution domain S. Hence, the TM using both the FS and the PS should be adapted simultaneously in the TM, to provide the combination of MFS and MPS. Numerical experiments for the benchmark Motz's problem are reported in Li [13], to display the significance of such combined algorithms.

In this paper, we will focus on MFS, but briefly provide MPS, which (i.e., the TM using PS) is described in [20] systematically. This paper is organized as follows. For Laplace's equation on bounded domains, in the next section the algorithms of MFS and MPS are described, and in Section 3 the error and stability analysis is briefly provided. In Sections 4 and 5, two numerical experiments are reported to support the analysis in [15], and to make comparisons of MFS and MPS. In the last section, a few remarks are made.

2. Algorithms of MFS and MPS

Consider Laplace's equation with the mixed boundary problems of the Dirichlet and the Robin boundary conditions,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S, \tag{2.1}$$

$$u = f \quad \text{on } \Gamma_D, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = g \quad \text{on } \Gamma_R, \tag{2.3}$$

where α is a non-negative constant, S the bounded simply connected domain with the boundary $\partial S = \Gamma_D \cup \Gamma_R$, and ν the exterior normal of Γ. Denote in Fig. 1,

$$r_{\max} = \max_S r, \quad r_{\min} = \max_{S_{in}(S_{in} \subseteq S)} r, \tag{2.4}$$

where S_{in} is the maximal disc inside of S. Let the source (charge) points Q be located outside of S. The fundamental solutions

$$\phi(r,\theta) = \ln|\overline{PQ}|, \quad P \in S \cup \partial S \tag{2.5}$$

are harmonic, where

$$P = \{(x,y) | x = r\cos\theta, \quad y = r\sin\theta\}. \tag{2.6}$$

A circle surrounding S is given by

$$\ell_R = \{(r,\theta) | r = R, \quad 0 \le \theta \le 2\pi\}, \quad R > r_{\max}. \tag{2.7}$$

Based on Bogomolny [3], the source points Q_i may be simply located uniformly on ℓ_R:

$$Q_i = \{(x,y) | x = R\cos ih, \quad y = R\sin ih\}, \tag{2.8}$$

where $R > r_{\max}$ and $h = 2\pi/N$. We obtain the fundamental solutions

$$\phi_i(P) = \ln|\overline{PQ_i}|, \quad i = 1,2,\ldots,N, \tag{2.9}$$

and the numerical solution is given by the linear combination

$$u_N = \sum_{i=1}^{N} c_i \phi_i(P), \tag{2.10}$$

where c_i are the unknown coefficients to be sought. Since u_N satisfies Laplace's equation in S already, the coefficients c_i can be sought by satisfying the boundary conditions (2.2) and (2.3) only. We will follow the Trefftz method [25] (TM) in [12,20], to seek u_N (i.e., c_i). Denote the energy

$$I(u) = \int_{\Gamma_D} (u-f)^2 + w^2 \int_{\Gamma_R} \left(\frac{\partial u}{\partial \nu} + \alpha u - g \right)^2, \tag{2.11}$$

where w is a positive weight. We choose $w = 1/N$ in our computations (see [12]). Denote by V_N the finite dimensional collection of (2.10). Then the numerical solution u_N can be obtained by

$$I(u_N) = \min_{v \in V_N} I(v). \tag{2.12}$$

When the integrals in (2.11) involve approximation, denote

$$\hat{I}(v) = \hat{\int}_{\Gamma_D} (v-f)^2 + w^2 \hat{\int}_{\Gamma_R} \left(\frac{\partial v}{\partial \nu} + \alpha v - g \right)^2, \tag{2.13}$$

where $\hat{\int}_{\Gamma_D}$ and $\hat{\int}_{\Gamma_R}$ are the numerical approximations of $\int_{\Gamma_D}$ and $\int_{\Gamma_R}$ respectively, by some quadrature rules, such as the central or the Gaussian rule. Hence, the numerical solution $\tilde{u}_N \in V_N$ is obtained by

$$\hat{I}(\tilde{u}_N) = \min_{v \in V_N} \hat{I}(v). \tag{2.14}$$

We may establish the collocation equations directly from (2.2) and (2.3), to yield

$$\sum_{i=1}^{N} c_i \phi_i(P_j) = f(P_j), \quad P_j \in \Gamma_D, \tag{2.15}$$

$$\sum_{i=1}^{N} c_i \left[\frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right] = g(P_j), \quad P_j \in \Gamma_R. \tag{2.16}$$

First, let Γ_D and Γ_R be divided into small Γ_D^j and Γ_R^j with the mesh spacings Δh_j, i.e.,

$$\Gamma_D = \bigcup_{j=1}^{M_1} \Gamma_D^j, \quad \Gamma_R = \bigcup_{j=1}^{M_2} \Gamma_R^j. \tag{2.17}$$

We obtain from (2.15) and (2.16) by multiplying different weights,

$$\sqrt{\Delta h_j}\sum_{i=1}^{N} c_i\phi_i(P_j)=\sqrt{\Delta h_j}f(P_j),\quad P_j\in\Gamma_D^j,\quad j=1,2,\ldots,M_1, \tag{2.18}$$

$$w\sqrt{\Delta h_j}\sum_{i=1}^{N} c_i\left\{\frac{\partial}{\partial\nu}\phi_i(P_j)+\alpha\phi_i(P_j)\right\}=w\sqrt{\Delta h_j}g(P_j),$$
$$P_j\in\Gamma_R^j,\quad j=M_1+1,\ldots,M_1+M_2, \tag{2.19}$$

where for simplicity, P_j are the midpoints of Γ_D^j and Γ_R^j. Following [20], Eqs. (2.18) and (2.19) are just equivalent to (2.14), where the central rule is chosen for $\hat{\int}_{\Gamma_D}$ and $\hat{\int}_{\Gamma_R}$. In computation, we may choose the number of collocation points to be equal or larger than that of source points, i.e.

$$M=M_1+M_2\geq N. \tag{2.20}$$

When the Gaussian rule is chosen, the following collocation equations are obtained:

$$\beta_j\sum_{i=1}^{N} c_i\phi_i(P_j)=\beta_j f(P_j),\quad P_j\in\Gamma_D, \tag{2.21}$$

$$w\beta_j\sum_{i=1}^{N} c_i\left\{\frac{\partial}{\partial\nu}\phi_i(P_j)+\alpha\phi_i(P_j)\right\}=w\beta_j g(P_j),\quad P_j\in\Gamma_R, \tag{2.22}$$

where P_j are the Gaussian nodes, the weights $\beta_j=O(\sqrt{\Delta h})$, and $\Delta h=\max_j\Delta h_j$. Eqs. (2.21) and (2.22) (i.e., (2.14)) are called the collocation Trefftz method (CTM) in [20]. For smooth solution of (2.1)–(2.3), we choose the harmonic polynomials of order n (i.e. particular solutions),

$$u_n(r,\theta)=\frac{a_0}{2}+\sum_{i=1}^{n}\left(\frac{r}{r_0}\right)^i\{a_i\cos i\theta+b_i\sin i\theta\}, \tag{2.23}$$

where a_i and b_i are the coefficients to be sought by the MPS, and r_0 the radius parameter to be chosen for better accuracy and stability (details appear elsewhere). Let the functions u_N in (2.10) be replaced by (2.23); the algorithms of MPS are described in the same way as the MFS above.

3. Brief error and stability analysis

In this paper, we focus on the error analysis of MFS, since the analysis of MPS can be found from [12,20]. The error bounds are provided for the mixed boundary problems in bounded simply connected domains. Since the MFS can be classified into the Trefftz method (TM) using the FS, we may follow the analysis of TM in [12,20], and pay an attention only to the extra-errors between harmonic polynomials and the fundamental solutions (FS) in [3]. By our analysis, when the Laplace's solutions are infinitely smooth, the exponential convergence rates can also be achieved as in [20]. However, when $u\in H^p(S)(p>\frac{3}{2})$, only the polynomial convergence rates are obtained. Since we may extend the basic analysis of the TM to that of the MFS, more interesting results of algorithms and analysis of MFS may follow [12,20].

Choose

$$u_N=\sum_{i=1}^{N} c_i\phi_i(r,\theta),\quad (r,\theta)\in S, \tag{3.1}$$

where

$$\phi_i(r,\theta)=\ln\sqrt{R^2+r^2-2Rr\cos(\theta-\xi_i)}, \tag{3.2}$$

$$\psi_i(r,\theta)=\frac{\partial}{\partial\nu}\phi_i(r,\theta)=-\frac{a\cos(\theta-\xi_i)-1}{\rho(a^2+1-2a\cos(\theta-\xi_i))}, \tag{3.3}$$

with $\xi_i=ih, h=2\pi/N$ and $a=R/\rho>1$.

Also let V_N denote the set of the admissible functions in (3.1). Denote the boundary norm,

$$\|v\|_B=\left\{\|v\|^2_{0,\Gamma_D}+w^2\left\|\frac{\partial v}{\partial\nu}+\alpha u\right\|^2_{0,\Gamma_R}\right\}^{1/2}, \tag{3.4}$$

where $\|v\|_{0,\Gamma_D}$ is the Sobolev norm. The solution by the TM, (2.12) also satisfies

$$\|u-u_N\|_B=\min_{v\in V_N}\|u-v\|_B. \tag{3.5}$$

Denote the fundamental solutions

$$v_N=\sum_{i=1}^{N} c_i\ln|\overline{PQ_i}|, \tag{3.6}$$

and $P\in(S\cup\partial S)$, and Q_i are given in (2.8). We can obtain the following lemma without proof (also see [3,15,14,12,20]).

Lemma 3.1. *Let $u\in H^p(S)(p>\frac{3}{2})$ hold. Choose $w=1/N$, and suppose that N satisfies*

$$\left(\frac{R}{r_{\max}}\right)^{2n-N}\left(\frac{r_{\max}}{r_{\min}}\right)^n\leq n^{-p}, \tag{3.7}$$

where n is the order of harmonic polynomials in (2.23). *Then for* (2.1)–(2.3) *the solution u_N by the MFS has the error bound,*

$$\|u-u_N\|_B=O\left(\frac{1}{N^{(p-1/2)}}\right). \tag{3.8}$$

Lemma 3.2. *Suppose that there exists a positive constant μ independent of N such that*

$$|v|_{1,\Gamma_D}\leq CN^{\mu}\|v\|_{0,\Gamma_D},\quad v\in V_N. \tag{3.9}$$

For $\Delta v=0$, there exists the bound,

$$\|v\|_{1,S}\leq C\left(N^{\mu/2}+\frac{1}{w}\right)\|v\|_B. \tag{3.10}$$

Based on Lemmas 3.1 and 3.2, we obtain the following theorem (see [14]).

Theorem 3.1. *Let the conditions in Lemmas* 3.1 *and* 3.2 *hold. Then for* (2.1)–(2.3), *the numerical solution by the MFS* (*i.e., the TM using FS*) *has the error bound,*

$$\|u-u_N\|_{1,S}=O\left(\frac{1}{N^{p-t}}\right), \tag{3.11}$$

where $t=\frac{1}{2}+\max\{1,\mu/2\}$.

We use the bounds of condition number and effective condition number for stability analysis. From (2.18) and (2.19) (as well as (2.21) and (2.22)), we obtain the linear algebraic equations

$$\mathbf{Fx}=\mathbf{b}, \tag{3.12}$$

where $\mathbf{F}\in R^{m\times n}(m\geq n)$, $\mathbf{x}\in R^n$ and $\mathbf{b}\in R^m$. Assume that $rank(\mathbf{F})=n$. The condition number for (3.12) is defined by

$$\text{Cond}=\frac{\sigma_1}{\sigma_n}, \tag{3.13}$$

where σ_1 and σ_n are the maximal and the minimal singular values of matrix $\mathbf{F}$, respectively. From [16], we may define the effective condition number

$$\text{Cond\_eff}=\frac{\|\mathbf{b}\|}{\sigma_n\|\mathbf{x}\|}. \tag{3.14}$$

Since the effective condition number is smaller, or even much smaller than the Cond, the Cond_eff is a better criterion of stability. The Cond is the a priori estimate for stability; the Cond_eff is the a posteriori estimate for stability (see [16]).

In Li and Huang [17], the effective condition number is further explored. From the over-determined system $\mathbf{Fx}=\mathbf{b}$ and the perturbed system $(\mathbf{F}+\Delta\mathbf{F})(\mathbf{x}+\Delta\mathbf{x})=\mathbf{b}+\Delta\mathbf{b}$, where $\mathbf{F}\in R^{m\times n}(m\geq n)$ with $rank(\mathbf{A})=r\leq n$, the new formulas are derived as follows:

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|}\leq \mathrm{Cond}_e\mathrm{ff}\ \times\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}, \tag{3.15}$$

where $\|\mathbf{x}\|$ is the 2-norm, $\mathbf{r}=\Delta\mathbf{b}=\mathbf{b}-\mathbf{F}(\mathbf{x}+\Delta\mathbf{x})$, the effective condition number is defined in (3.14). Moreover, for the nonsingular $\mathbf{F}=\mathbf{A}\in R^{n\times n}$, it is well known that the error bounds from the perturbation of both matrix $\mathbf{F}$ and $\mathbf{b}$ are given by (see Atkinson [1])

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|}\leq\frac{1}{1-\delta}\times\left\{\mathrm{Cond}\times\frac{\|\Delta\mathbf{A}\|}{\|\mathbf{A}\|}+\mathrm{Cond}\times\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}\right\}, \tag{3.16}$$

where $\delta=\|\mathbf{A}\|/\sigma_n<1$. In [17] the following new error bounds are derived:

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|}\leq\frac{1}{1-\delta}\times\left\{\mathrm{Cond}\times\frac{\|\Delta\mathbf{A}\|}{\|\mathbf{A}\|}+\mathrm{Cond}_e\mathrm{ff}\times\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}\right\}. \tag{3.17}$$

In numerical partial differential equations (PDE), $\Delta\mathbf{b}$ is obtained not only from rounding errors, but also from discretization errors. Since the discretization errors are dominant in general, we conclude from (3.17) that the effective condition number defined in (3.14) is important. This implies that the effective condition number is the appropriate criterion for stability analysis of the numerical PDE.

The bounds of Cond of MFS for bounded simply connected domains are given by the following theorem in Li et al. [19].

Theorem 3.2. *Let $\mu\in[1,2]$ and choose $w=O(1/N)$, and for (2.1)–(2.3) the discrete Eq. (3.12) be obtained from the MFS, there exists the bound,*

$$\mathrm{Cond}(\mathbf{F})=\frac{\sigma_{\max}(\mathbf{F})}{\sigma_{\min}(\mathbf{F})}\leq CN^2\left(\frac{R}{r_{\min}}\right)^{N/2}. \tag{3.18}$$

For MFS, both Cond and Cond_eff grow exponentially. In contract, for MPS, Cond_eff may grow polynomially. Hence, the ill-conditioning of MFS is much worse than that of MPS. Fortunately, it is due to Mathematica with unlimited digits so that the severity of ill-conditioning of MFS is relaxed in some sense.

In Chen et al. [5,6], comparisons are made for MFS and MPS to solve Laplace's and biharmonic equations. Since their algorithms are similar, *the equivalence* of MFS and MPS is called. Note that their errors may be different, and the ill-conditioning of MFS is more severe. The equivalence may be interpreted as some similarity of MFS and MPS in algorithms.

In Schabak [24], several simple examples for Laplace's equation in bounded simply connected domains are discussed, and some comparisons of MFS and MPS are made. Below, we will give two advanced examples: annular problems of Laplace's equation and biharmonic problems, to explore further comprehensive comparisons of MFS and MPS.

4. Annular shaped domains

A relation of algorithms between the MFS and the MPS is also discussed in Chen et al. [6]. The error bounds of the MFS are derived in Li [15] for annular shaped domains without numerical examples. The purposes of the numerical experiments in this section are twofold: (1) to support the analysis in [15] and (2) to compare MFS and MPS.

Consider

$$\Delta u=0,\quad \text{in } S,$$

$$u|_{\Gamma_{in}}=g_{in},\quad u|_{\Gamma}=g, \tag{4.1}$$

where S is the annular shaped domain, Γ_{in} and Γ are the inner and the outer boundaries, respectively see Fig. 2. Choose the epitrochoid boundary curve as in Liu [23]

$$\rho(\theta)=\sqrt{(a+b)^2+1-2(a+b)\cos\left(\frac{a\theta}{b}\right)}. \tag{4.2}$$

Let Γ be an exterior circle with a radius $\overline{R}=6$, and Γ_{in} as (4.2) with $a=3$ and $b=1$. We can see

$$r_{\max}|_S=\overline{R},\quad R^{in}_{\min}=\min r|_{\Gamma_{in}}=3.$$

We choose the Trefftz method (TM) using fundamental solution (FS) and particular solution (PS), and provide their comparisons.

Denote

$$l_R=\{(x,y)|x=R^*\cos\theta,\quad y=R^*\sin\theta\},$$

$$R^*>\overline{R},$$

and

$$l_{R_{in}}=\{(x,y)|x=R_{in}\cos\theta,\quad y=R_{in}\sin\theta\},$$

$$R=R_{in}<r_{\min}|_{\Gamma_{in}}=3.$$

4.1. Fundamental solutions

Let $Q_i\in l_R$ and $Q_i^*\in l_{R_{in}}$, where

$$Q_i\in\{(x,y)|x=R\cos\theta,\ y=R\sin\theta\},\quad R>\overline{R}=6,$$

$$Q_i^*\in\{(x,y)|x=R_{in}\cos\theta,\quad y=R_{in}\sin\theta\},\quad R_{in}<r_{\min}|_{\Gamma_{in}}=3. \tag{4.3}$$

Choose $\phi_i=\ln|\overline{PQ_i}|$ and $\psi_i=\ln|\overline{PQ_i^*}|$, where

$$Q_i=i\frac{2\pi}{N},\ Q_j^*=j\frac{2\pi}{M},\quad P\in S\cup\partial S, \tag{4.4}$$

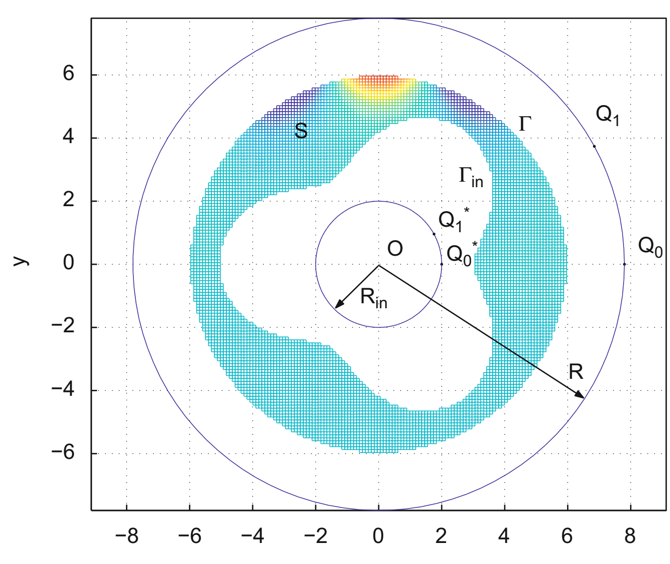

Fig. 2. An annular shaped domain.

and the linear combination:

$$u_{N,M}=c_0+\sum_{i=1}^{N}c_i\ln|\overline{PQ_i}|+\sum_{i=1}^{M}d_i\ln|\overline{PQ_i^*}|, \tag{4.5}$$

where c_i and d_i are the unknown coefficients to be determined by MFS.

Denote $V_{N,M}$ the finite dimensional collection of (4.5), and the energy by

$$I(v)=\hat{\int}_{\Gamma_{in}}(v-g_{in})^2+\hat{\int}_{\Gamma}(v-g)^2, \tag{4.6}$$

where $\hat{\int}_{\Gamma_{in}}$ and $\hat{\int}_{\Gamma}$ are the integration approximation of $\int_{\Gamma_{in}}$ and $\int_{\Gamma}$. The method of fundamental solutions (MFS) reads: to find $u_{N,M}$ such that

$$I(u_{N,M})=\min I(v),\quad v\in V_{N,M}. \tag{4.7}$$

We may easily provide the discrete collocation equations on $\Gamma_{in}\cup\Gamma$ as in Section 2.

Since the annular domain is bounded, we do not need the FS for unbounded domains. Hence we still use

$$\phi_i=\ln|\overline{PQ_i}|.$$

For the unbounded domains, however, we should use the bounded fundamental solutions:

$$u_L=c_0+\sum_{i=1}^{N}c_i\psi_i(r),$$

where

$$\psi_i(r)=\ln|\overline{PQ_i}|-\ln|\overline{OP}|$$
$$=\ln\sqrt{1+\left(\frac{R}{\rho}\right)^2-2\left(\frac{R}{\rho}\right)\cos(\theta-\varphi_i)},\quad \varphi_i=\frac{2\pi}{N}i.$$

Let S be split into two subdomains, S^+ and S^-, where S^+ and S^- may have an overlap only for an intermediate region of S. We cite the error bounds in Li [15] in the following theorem.

Theorem 4.1. *Let $u\in H^p(S^+)$ and $u\in H^\sigma(S^-)(p,\sigma\geq\frac{3}{2})$ hold. Let N and M satisfy respectively*

$$\left(\frac{R}{r_{\max}}\right)^{2n-N}\left(\frac{r_{\max}}{r_{\min}}\right)^n\leq\frac{1}{n^{(p-1/2)}}, \tag{4.8}$$

$$\left(\frac{r^{in}_{\min}}{R_{in}}\right)^{2n-M}\left(\frac{r^{in}_{\max}}{r^{in}_{\min}}\right)^n\leq\frac{1}{n^{(\sigma-1/2)}}, \tag{4.9}$$

where n is the order of harmonic polynomials in (2.23). For the Dirichlet problem on the annular shaped domain, there exists the optimal error bound,

$$\|u-u_{N,M}\|_{0,S}\leq C\left\{\frac{1}{N^p}\|u\|_{p,S^+}+\frac{1}{M^\sigma}\|u\|_{\sigma,S^-}\right\}, \tag{4.10}$$

where C is a constant independent of N and M.

4.2. Particular solutions

Let the exact solution of (4.1) be

$$u(x,y)=\exp\left(\frac{y}{x^2+y^2}\right)\cos\left(\frac{x}{x^2+y^2}\right)+\exp(y)\cos(x), \tag{4.11}$$

with two singular points $r=0$ and ∞. A solution profile of (4.11) is provided in Fig. 3. Since no symmetry exists, the particular solutions are given by

$$u_{N_1,M_1}(r,\theta)=a_0+\sum_{i=1}^{N_1}\left(\frac{r}{r_0}\right)^i(a_i\cos i\theta+b_i\sin i\theta)$$
$$+a_0^*\ln r+\sum_{i=1}^{M_1}\left(\frac{r_0^*}{r}\right)^i(a_i^*\cos i\theta+b_i^*\sin i\theta), \tag{4.12}$$

where r_0 and r_0^* are parameters, and a_i, b_i, a_i^*, and b_i^* are the coefficients to be sought by MPS.

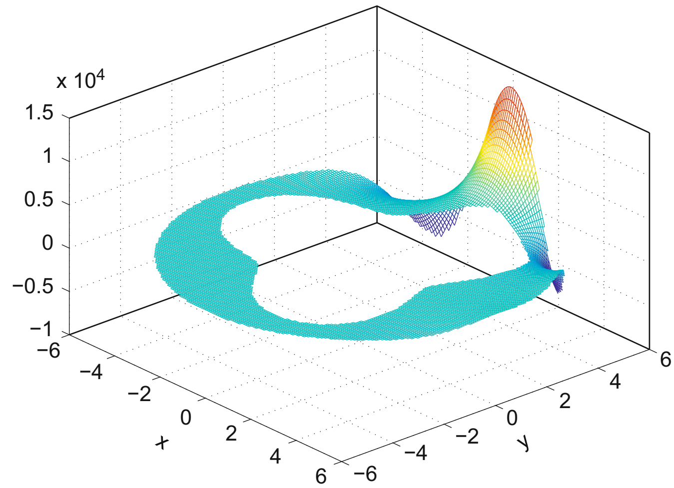

Fig. 3. Exact solution.

Table 1
Errors and condition numbers for the annular shaped domain by the MFS with $R_{in}=1.6, R=7.4$.

N_{tot}	N	M	$\overline{M}$	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\Gamma^*,\infty}$	Cond	Cond_eff	$\|\mathbf{x}\|_2$
21	16	4	25	11.79	7.66	3.38(4)	2.69(2)	4.92(2)
41	32	8	50	6.24	3.83	2.55(6)	5.21(2)	1.49(4)
61	48	12	75	1.25(−1)	9.62(−2)	1.30(8)	7.61(5)	4.27(2)
81	64	16	100	3.07(−3)	1.98(−3)	5.68(9)	3.43(7)	3.57(2)
101	80	20	125	8.26(−5)	6.37(−5)	2.26(11)	1.43(9)	3.10(2)
121	96	24	150	2.35(−6)	1.78(−6)	8.49(12)	5.29(10)	2.82(2)
141	112	28	175	6.90(−8)	5.32(−8)	3.06(14)	1.91(12)	2.61(2)
161	128	32	200	2.08(−9)	1.50(−9)	1.07(16)	5.84(13)	2.44(2)

4.3. Numerical experiments

Denote the errors

$$|\varepsilon|_B=\left\{\int_{\Gamma}\varepsilon^2+\int_{\Gamma_{in}}\varepsilon^2\right\}^{1/2}, \tag{4.13}$$

$$|\varepsilon|_{\Gamma^*,\infty}=\max_{\Gamma\cup\Gamma_{in}}|\varepsilon|, \tag{4.14}$$

where $\varepsilon=u-u_{M,N}$. For simplicity, in computation we choose the same number $\overline{M}$ of collocation nodes for Γ and Γ_{in}. Then the total number of collocation nodes is $2\overline{M}$.

We use Matlab with double precision, and the numerical results are given in Tables 1–4. Table 1 lists errors and condition numbers under the optimal matches: $N=4M$, and good choices: $R_{in}=1.6$ and $R=7.4$.[3] Based on Table 1, the curves of errors and

[3] For $N=4M$, $R_{in}=1.6$ and $R=7.4$, the details of trial computations are omitted.

Table 2
The errors and condition numbers for the annular shaped domain by the MFS with $N+M=100$ and $\overline{M}=125$.

N	M	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\Gamma^*,\infty}$	Cond	Cond_eff
0	100	568.03	360.44	2.04(17)	4.90
10	90	543.69	314.34	1.56(17)	10.25
20	80	51.67	24.37	8.01(16)	61.96
30	70	10.45	8.40	1.12(15)	4.92(4)
40	60	8.50(−1)	6.38(−1)	1.38(13)	4.80(6)
50	50	7.81(−2)	6.05(−2)	1.84(11)	6.98(8)
60	40	7.67(−3)	5.91(−3)	2.45(9)	1.37(7)
70	30	7.84(−4)	6.05(−4)	2.37(10)	1.41(8)
80	20	8.26(−5)	6.37(−5)	2.26(11)	1.40(9)
90	10	5.53(−4)	2.90(−4)	2.18(14)	1.34(10)
100	0	7.27(−1)	2.30(−1)	5.36(15)	1.36(7)

Table 3
The results for the annular shaped domain the MFS using (4.19) with outer source points only.

N	$\overline{M}$	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\Gamma^*,\infty}$	Cond	Cond_eff
20	25	53.4	25.5	4.76(5)	685.37
40	50	1.36	7.85(−1)	2.44(8)	2.10(5)
60	75	7.27(−1)	2.31(−1)	8.08(10)	6.27(7)
80	100	7.27(−1)	2.31(−1)	2.20(13)	6.36(6)
100	125	7.27(−1)	2.31(−1)	5.36(15)	1.36(7)
120	150	7.27(−1)	2.31(−1)	8.02(16)	7.83(4)
140	175	7.27(−1)	2.31(−1)	1.53(17)	5.62(4)

Table 4
The results for the annular shaped domain by the MFS using (4.20) with inner source points only.

M	$\overline{M}$	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\Gamma^*,\infty}$	Cond	Cond_eff
20	25	806.1	365	4.24(5)	6.34
40	50	603.2	352.8	7.67(9)	71.1
60	75	9.06(3)	6.10(3)	3.68(14)	74.1
80	100	569.6	358.6	1.08(17)	20.4
100	125	568	360.4	–	4.90
120	150	568	359.7	–	3.02
140	175	568	360.5	–	8.03

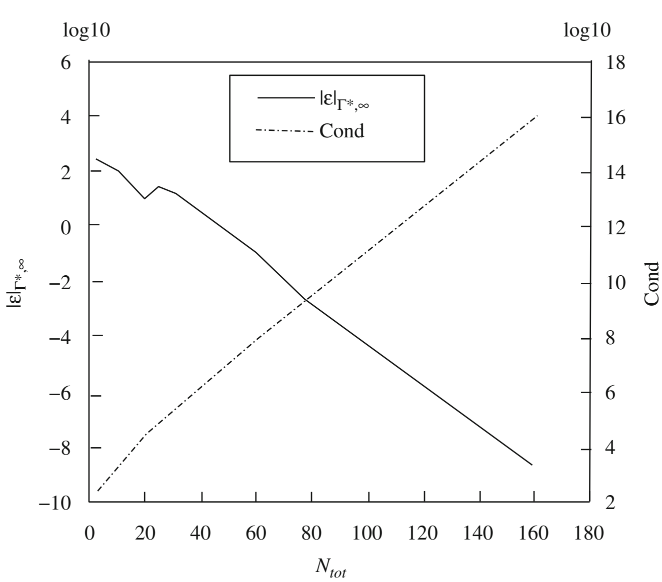

Fig. 4. The curves of errors for the annular shaped domain by the MFS, where the solid and the dashed lines are the curves of $|\varepsilon|_{\Gamma^*,\infty}$ and Cond, respectively.

condition numbers are drawn in Fig. 4. From Table 1 we can see

$$|\varepsilon|_B, |\varepsilon|_{\Gamma^*,\infty} = O(0.836^{N_{tot}}), \tag{4.15}$$

$$\text{Cond},\ \text{Cond}_e\text{ff} = O(1.19^{N_{tot}}), \tag{4.16}$$

where $N_{tot}=N+M$. The error bounds are exponential with respect to N_{tot}, since the solution (4.11) is highly smooth in the annular shaped domain S. This also coincides with Theorem 4.1, to support the analysis in [15].

On the other hand, the bounds of condition number and effective condition number are also exponential with respect to N_{tot}. Interestingly, we can see from (4.15) and (4.16) that

$$|\varepsilon|_{\Gamma^*,\infty} \times \text{Cond} = O(1). \tag{4.17}$$

We will draw the curves in Figs. 4–6 only for $|\varepsilon|_{\Gamma^*,\infty}$. From Table 2 and Fig. 5, we have discovered that the ratio

$$\frac{N}{M} = 4 \tag{4.18}$$

is optimal for $N_{tot}=M+N=100$ and $\overline{M}=125$. Then we use $N=4M$ in computation.

We also choose the following fundamental solutions:

$$u_N = c_0 + \sum_{i=1}^{N} c_i \ln|\overline{PQ_i}| \tag{4.19}$$

and

$$u_M = c_0 + \sum_{i=1}^{M} d_i \ln|\overline{PQ_i^*}|. \tag{4.20}$$

The accuracy of the numerical solutions by the MFS using (4.19) and (4.20) is very poor, see Tables 3 and 4. The reason is that the exact solution (4.11) has two singularity at $r=\infty$ and 0. However, the expansions of FS in (4.19) and (4.20) are suited for the singularity at $r=\infty$ and 0, respectively. This facts also supports the MFS algorithms and the analysis in [15]. Hence for general cases, the general fundamental solutions (4.5) are necessary for MFS.

Finally, let us cite the results of Table 1 at $N_{tot}=161$,

$$|\varepsilon|_B = 2.08(-9), \quad |\varepsilon|_{\Gamma^*,\infty} = 1.50(-9), \tag{4.21}$$

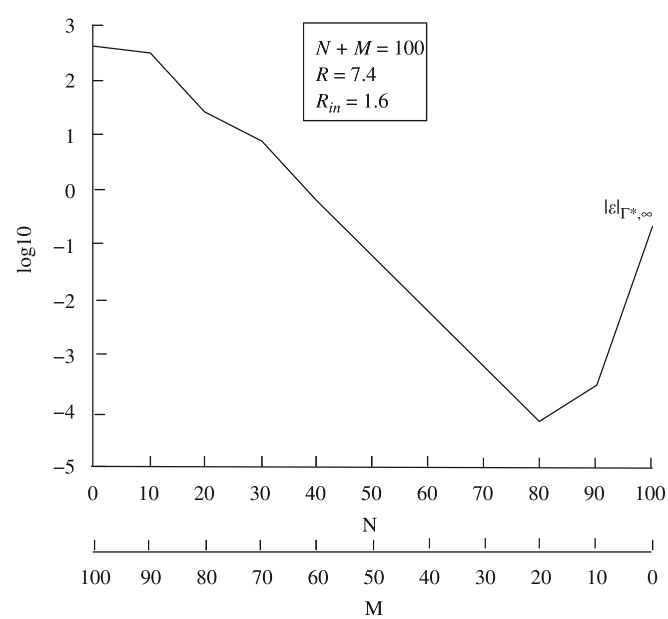

Fig. 5. The curves of errors for the annular shaped domain by the MFS with different N at $\overline{M}=125$, $R=7.4$, $R_{in}=1.6$ and $N+M=100$.

Please cite this article as: Li Z-C, et al. Comparisons of fundamental solutions and particular solutions for Trefftz methods. Eng Anal Bound Elem (2009), doi:10.1016/j.eujim.2009.08.108

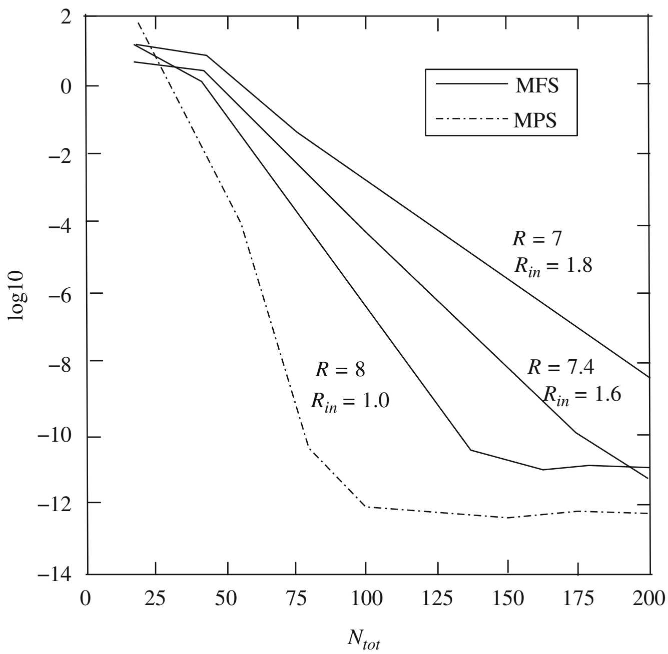

Fig. 6. The curves of errors for the annular shaped domain by the MFS and the MPS.

Table 5
Errors and condition numbers for the annular shaped domain by the MPS with $r_0=6$ and $r_0^*=5$.

$\overline{N}_{tot}$	N_1	M_1	$\overline{M}$	$\lvert\varepsilon\rvert_B$	$\lvert\varepsilon\rvert_{\Gamma^*,\infty}$	Cond	Cond_eff	$\lVert\mathbf{x}\rVert_2$
22	8	2	25	1.48(2)	6.19(1)	1.68(1)	4.89	133.32
42	16	4	50	2.20(−1)	6.58(−2)	2.46(1)	6.90	137.67
62	24	6	75	8.18(−6)	2.38(−6)	5.61(1)	7.37	137.67
82	32	8	100	1.19(−10)	3.43(−11)	1.34(2)	7.37	137.67
102	40	10	125	1.01(−12)	6.88(−13)	3.51(2)	7.51	137.67
122	48	12	120	1.03(−12)	5.37(−13)	9.36(2)	7.56	137.67
142	56	14	175	9.89(−13)	5.36(−13)	2.45(3)	7.56	137.67
162	64	16	200	1.05(−12)	7.04(−13)	6.54(3)	7.57	137.67

$$\text{Cond}=1.07(16),\quad \text{Cond}_\text{e}\text{ff}=5.84(13). \tag{4.22}$$

Although the condition number is large, the errors of numerical solutions reach $O(10^{-9})$, which may satisfy most of engineering requirements. For the comparisons of MFS and MPS we also cite the results of Table 5 at $\overline{N}_{tot}=162$,

$$|\varepsilon|_B=1.05(-12),\quad |\varepsilon|_{\Gamma^*,\infty}=7.04(-13), \tag{4.23}$$

$$\text{Cond}=6.54(3),\quad \text{Cond}_\text{e}\text{ff}=7.57. \tag{4.24}$$

Comparing (4.23) and (4.24) with (4.21) and (4.22), we may conclude that the errors of the MPS are much smaller, and both the condition number and the effective condition number are significantly smaller. Figs. 6–8 provide a clear view of superiority of the MPS over the MFS.

5. Biharmonic equations

5.1. Description of MFS

Below, consider the biharmonic equation with the clamped boundary conditions:

$$\triangle^2 u=0 \quad \text{in } S, \tag{5.1}$$

$$u=f \quad \text{in } \Gamma, \tag{5.2}$$

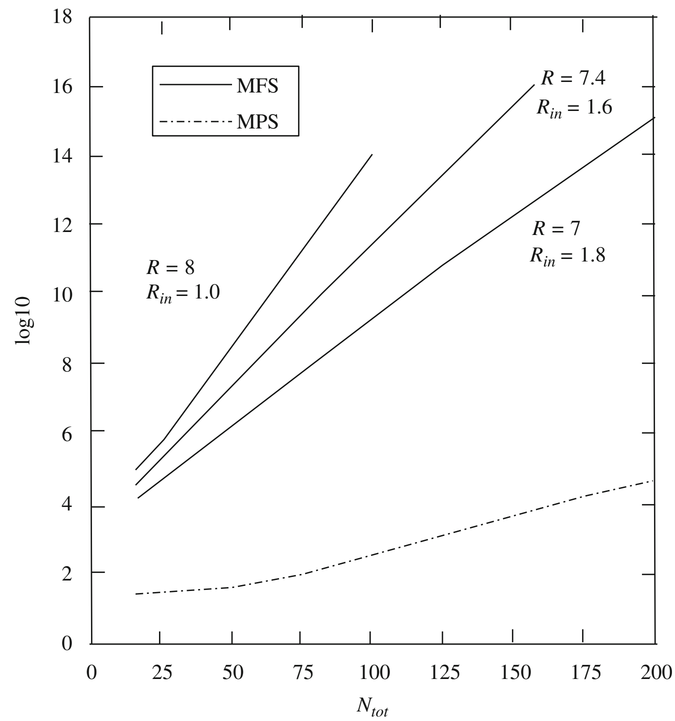

Fig. 7. The curves of condition numbers for the annular shaped domain by the MFS and the MPS.

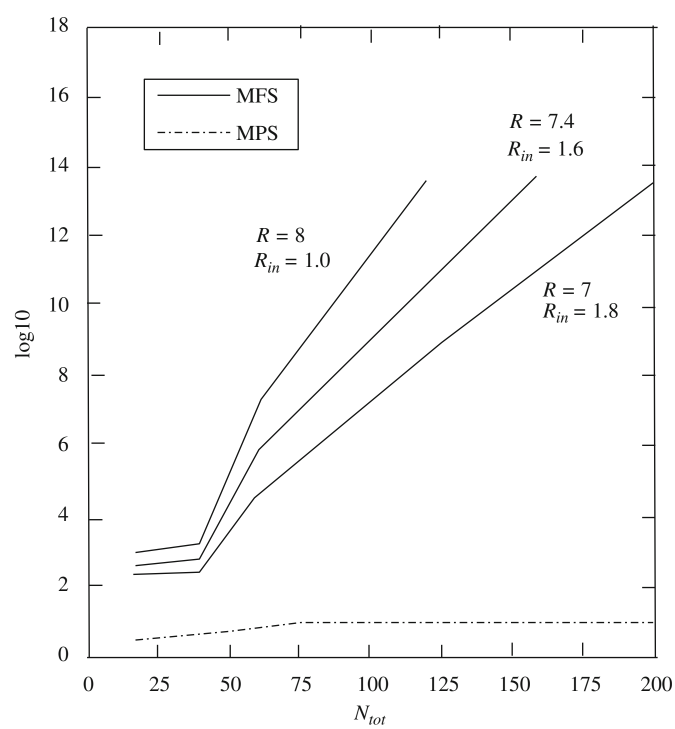

Fig. 8. The curves of effective condition numbers for the annular shaped domain by the MFS and the MPS.

$$u_\nu=\frac{\partial u}{\partial \nu}=g \quad \text{in } \Gamma, \tag{5.3}$$

where $\triangle=\partial^2/\partial x^2+\partial^2/\partial y^2$, S is the bounded simply connected domain, u_ν is the outward normal derivative to Γ, Γ is its boundary, and f and g are the functions smooth enough. The biharmonic solutions can be represented by

$$u = u(\rho,\theta) = \rho^2 v + w, \tag{5.4}$$

where v and w are the harmonic functions. Hence the fundamental solutions of biharmonic equations in 2D are found by

$$\Psi(\rho,\theta) = \bar{r}^2 \ln \bar{r} = (R^2+\rho^2-2R\rho\cos(\theta-\psi))\ln|R^2+\rho^2 - 2R\rho\cos(\theta-\psi)|, \tag{5.5}$$

where $R>\rho$, $\bar{r}=|PQ|$, $P=\rho e^{i\theta}$, $Q=Re^{i\psi}$ and $i=\sqrt{-1}$.

Denote the two kind types of fundamental solutions:

$$\Psi_j(\rho,\theta) = \bar{r}_j^2 \ln \bar{r}_j, \tag{5.6}$$

$$\Phi_j(\rho,\theta) = \ln \bar{r}_j, \tag{5.7}$$

where $\bar{r}_j = |PQ_j|$, $Q_i = Re^{i\psi_j}$ and $\psi_j = (2\pi/N)j$. Hence we may choose the linear combinations

$$v_N = c_0 + \sum_{j=1}^{N} \{c_j \Psi_j(\rho,\theta) + d_j \Phi_j(\rho,\theta)\}, \tag{5.8}$$

where c_j and d_j are the unknown coefficients to be determined by the boundary conditions (5.2) and (5.3). We may use the collocation Trefftz method (CTM). Denote V_N the set of (5.8). Then the collocation Trefftz method reads:

$$\hat{I}(u_N) = \min_{v \in V_N} \hat{I}(v), \tag{5.9}$$

where the integrals involve numerical approximation are given by

$$\hat{I}(v) = \hat{\int_\Gamma} (v-f)^2 + w^2 \hat{\int_\Gamma} (v_\nu - g)^2, \tag{5.10}$$

and w is the weight which may be chosen as $w=1/N$.

The Almansi's representation for biharmonic equations are obtained in [8] directly from (5.4). Then the Almansi's FS is given by [22]. The MFS and numerical experiments were provided in Fairweather and Karageorghis [7–10].

The analysis of error and stability is given in Li et al. [22]; here we only provide the important results for (5.1)–(5.3). Denote

$$[u,v]_H = \int_\Gamma uv + w^2 \int_\Gamma u_\nu v_\nu, \tag{5.11}$$

and the norm

$$\|v\|_H = \sqrt{[v,v]_H}. \tag{5.12}$$

The Trefftz method (5.9) using FS and PS (i.e., MFS and MPS), for biharmonic equation is rewritten as

$$\|u-u_N\|_H = \min_{v \in V_n} \|u-v\|_H. \tag{5.13}$$

Since there exists the orthogonality

$$[u-u_N, v]_H = 0, \tag{5.14}$$

where

$$[u_N, v]_H = \int_\Gamma fv + w^2 \int_\Gamma g v_\nu. \tag{5.15}$$

We cite the theorem from [22].

Theorem 5.1. *Suppose that $u \in H^p(S)(p \ge 3/2)$, and N is chosen such that*

$$\left(\frac{R}{r_{\max}}\right)^{2n-N} \left(\frac{r_{\max}}{r_{\min}}\right)^{n} \le \frac{1}{n^{(p-1/2)}}. \tag{5.16}$$

For (5.1)–(5.3) by the MFS, there exists the error bound,

$$\|u-u_N\|_H = O\left(\frac{1}{N^{(p-1/2)}}\right)\|u\|_{p,S}. \tag{5.17}$$

In computation, we also consider the mixed type of the clamped and the simply support boundary conditions on Γ. Then the boundary condition (5.3) is replaced by

$$u_\nu = g \text{ in } \Gamma_1, \quad u_{\nu\nu} = g^* \quad \text{in } \Gamma_2, \tag{5.18}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The admissible functions (5.8) retain, but $\hat{I}(v)$ in (5.10) is replaced by

$$\hat{I}^*(v) = \hat{\int_\Gamma} (v-f)^2 + w_1^2 \hat{\int_{\Gamma_1}} (v_\nu - g)^2 + w_2^2 \hat{\int_{\Gamma_2}} (v_{\nu\nu} - g^*)^2, \tag{5.19}$$

where $w_i = O(1/N^i)$. The boundary collocation equations can be established similarly, and error bounds can be obtained, similarly to Theorem 5.1.

5.2. Comparisons with MPS

Consider the rectangular domain $S=\{(x,y)|-1<x<1,-1<y<1\}$, and choose the exact solution:

$$u(x,y) = \exp(x)\cos y + (x^2+y^2)\exp(y)\cos x. \tag{5.20}$$

The boundary errors are defined as

$$\|\varepsilon\|_B = \|u_N - u\|_B = \sqrt{I(u_N)},$$

$$I(v) = \int_\Gamma (v-f)^2 + w_1^2 \int_{\Gamma_1} (v_\nu - g)^2 + w_2^2 \int_{\Gamma_2} (v_{\nu\nu} - g^*)^2,$$

where $w_i = 1/N^i$. For the plate bending problem with the clamped boundary conditions in (5.2) and (5.3), $\Gamma_1 = \Gamma$ and $\Gamma_2 = \emptyset$. We also choose the biharmonic polynomials for the MPS

$$P_n^B(\rho,\theta) = P_n^H(\rho,\theta) + P_n(\rho,\theta), \tag{5.21}$$

where

$$P_n^H(\rho,\theta) = \frac{a_0^*\rho^2}{2} + \sum_{i=1}^{n} \rho^{i+2}(a_i^* \cos i\theta + b_i^* \sin i\theta), \tag{5.22}$$

$$P_n(\rho,\theta) = \frac{a_0}{2} + \sum_{i=1}^{n} \rho^{i}(a_i \cos i\theta + b_i \sin i\theta), \tag{5.23}$$

and a_i, b_i, a_i^* and b_i^* are the coefficients. The MPS is obtained when the biharmonic polynomials (5.21) replace the fundamental solutions (5.8), and the error analysis may follow [20] directly.

The fundamental solutions in (5.8) are used for the MFS in computation. By MFS and MPS, the errors and condition numbers are listed in Table 6, where $\overline{M}$ is the total number of collocation nodes on Γ, and their curves are drawn in Figs. 9–11. From the tables and figures, the MFS is inferior to the MPS.

Next, we still choose the solution (5.20), but use the following mixed type of the clamped and simply supported boundary conditions:

$$u = f, \quad u_\nu = g \quad \text{on} \quad x = \pm 1,$$

$$u = f, \quad u_{\nu\nu} = g^* \quad \text{on} \quad y = \pm 1, \tag{5.24}$$

where ν is the exterior normal of ∂S. The errors and condition numbers are listed in Table 7, and their curves are drawn in Figs. 12–14. Moreover, the 2-norm of $\|\mathbf{x}\|_2$ by the MFS and the MPS is

Table 6
Errors and condition numbers for the biharmonic equation with the clamped boundary condition by MFS and MPS.

FS with $R=5$					PS				
N	$\overline{M}$	$\Vert\varepsilon\Vert_B$	Cond	Cond_eff	n	$\overline{M}$	$\Vert\varepsilon\Vert_B$	Cond	Cond_eff
11	20	2.86(−2)	1.81(8)	1.12(3)	5	20	8.24(−3)	3.94(1)	7.23
21	40	1.46(−6)	5.19(12)	1.73(7)	10	40	4.92(−7)	3.00(2)	13.2
31	60	3.14(−11)	1.74(17)	6.02(11)	15	60	1.98(−12)	2.07(3)	19.5
41	80	2.94(−11)	2.14(21)	7.40(15)	20	80	0.56(−14)	1.36(4)	25.8

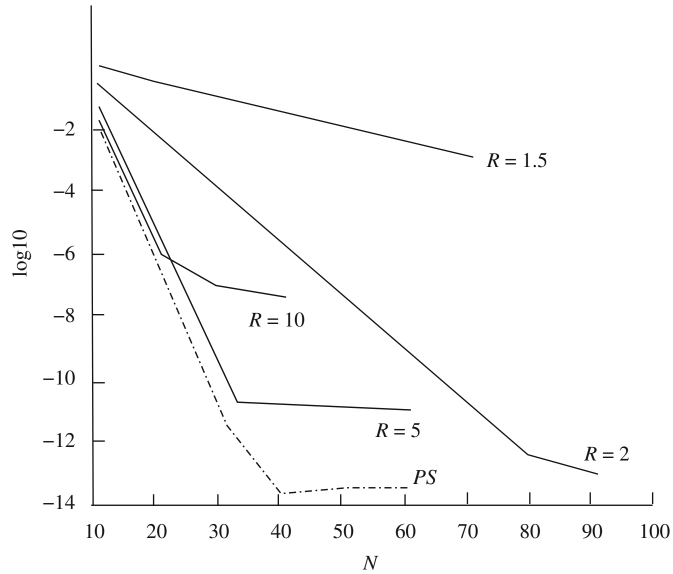

Fig. 9. The curves of errors for the biharmonic equation with the clamped boundary condition by MFS and MPS.

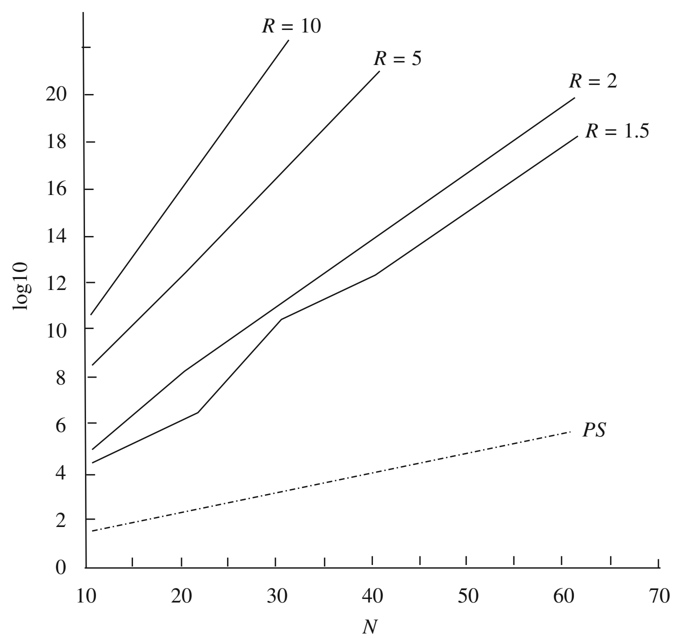

Fig. 10. The curves of condition numbers for the biharmonic equation with the clamped boundary condition by MFS and MPS.

also given in Table 8, to display that the MFS has a subtraction cancelation; but the MPS does not. The error curves of MFS in Figs. 9 and 12 will not go down for large N. The reason is that the huge Cond_eff of MFS damages the accuracy of numerical solutions.

From the above tables and figures, we may also concluded that for smooth solutions, the errors of MFS may catch up with the errors of MPS, if the huge effective condition number occurring will not deteriorate the accuracy. Evidently, it is due to better stability that MPS is superior to MFS.

6. Concluding remarks

To close this paper, let us make a few remarks.

1. The numerical experiments in Section 4 support the analysis in Li [15] of the MFS for Laplace's equation on annular shaped domains.

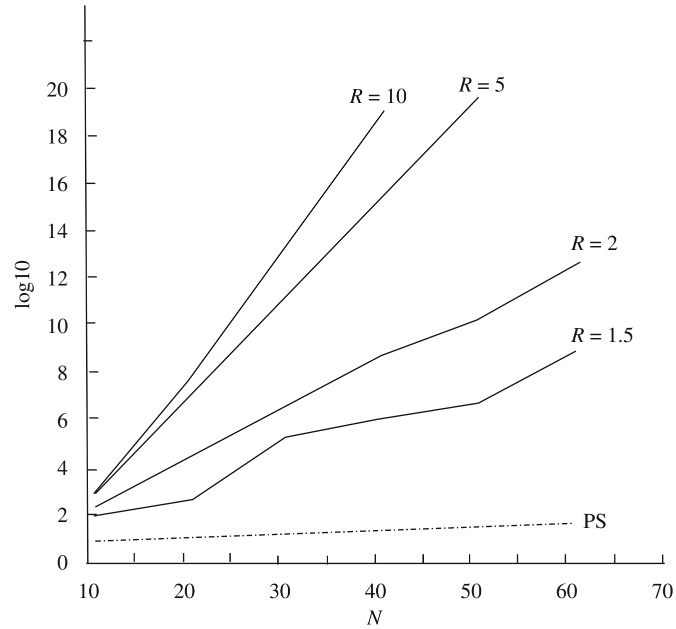

Fig. 11. The curves of effective condition numbers for the biharmonic equation with the clamped boundary condition by MFS and MPS.

Table 7
Errors and condition numbers for the biharmonic equation with the mixed type of the clamped and simply supported boundary conditions by MFS and MPS.

FS with $R=5$					PS				
N	$\overline{M}$	$\Vert\varepsilon\Vert_B$	Cond	Cond_eff	n	$\overline{M}$	$\Vert\varepsilon\Vert_B$	Cond	Cond_eff
11	20	2.48(−2)	3.32(8)	1.95(3)	5	20	1.04(−2)	4.60(1)	3.31
21	40	1.28(−6)	5.16(12)	1.73(7)	10	40	5.50(−7)	5.13(2)	4.19
31	60	4.87(−11)	1.51(17)	5.23(11)	15	60	2.32(−12)	5.48(3)	5.81
41	80	3.72(−11)	2.54(21)	8.80(15)	20	80	1.86(−14)	4.66(4)	7.53

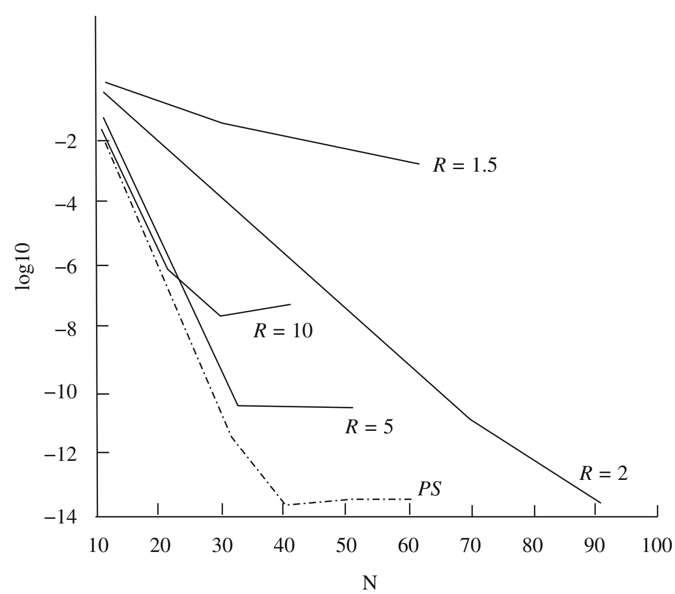

Fig. 12. The curves of errors for the biharmonic equation with the mixed type of the clamped and simply supported boundary conditions by MFS and MPS.

2. Since MFS and MPS are the TM using FS and PS, respectively, both fall into the TM family. Since the TM has been developed in algorithms and analysis in our recent book [20], the

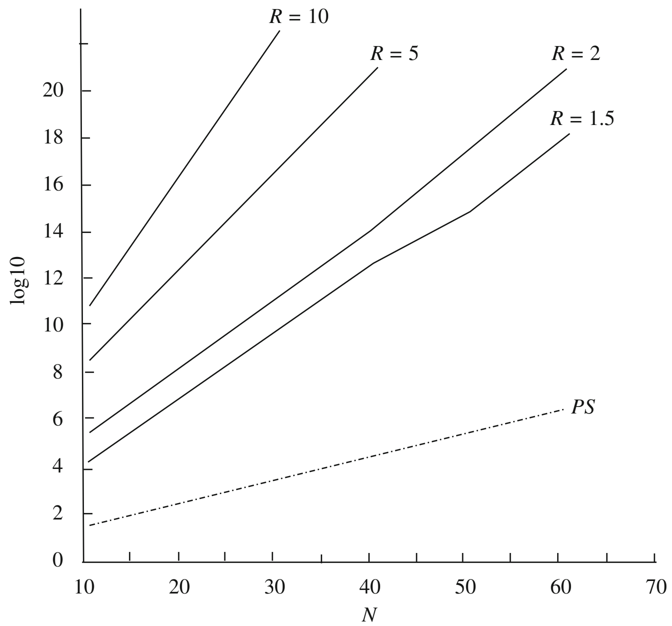

Fig. 13. The curves of condition numbers for the biharmonic equation with the mixed type of the clamped and simply supported boundary conditions by MFS and MPS.

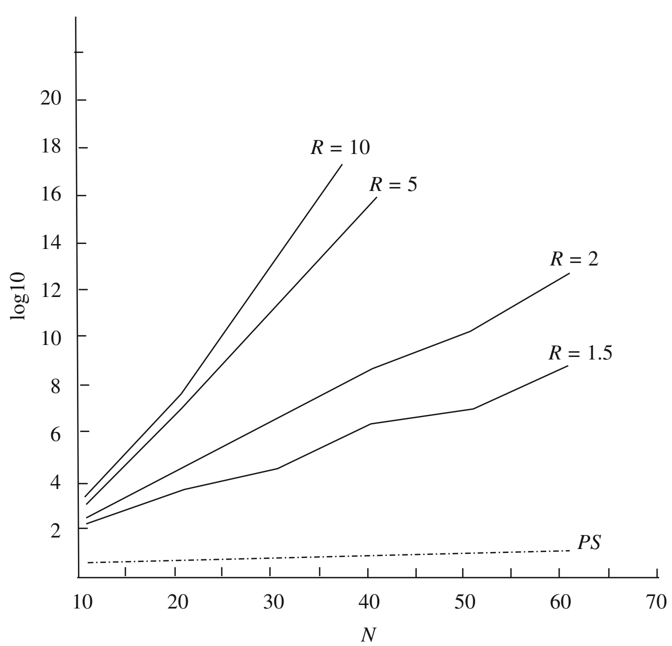

Fig. 14. The curves of effective condition numbers for the biharmonic equation with the mixed type of the clamped and simply supported boundary conditions by MFS and MPS.

Table 8
The solution norms $\|\mathbf{x}\|_2$ for the biharmonic equation with the clamped and the mixed boundary conditions by MFS and MPS.

N	Clamped type		Mixed type	
	FS	PS	FS	PS
11	2.33(3)	3.25	2.28(3)	3.25
21	3.13(3)	3.25	3.14(3)	3.25
31	2.59(3)	3.25	2.58(3)	3.25
41	2.27(3)	3.25	2.28(3)	3.25
51	5.18(3)	3.25	6.76(3)	3.25

algorithms and analysis of MFS can be easily obtained, to greatly extend its application in engineering problems.

3. Comprehensive comparisons are made for the TM using FS and PS by analysis in Section 3, and by numerical examples in Sections 4 and 5. For Laplace's equation in 2D, both FS and PS are harmonic. The ln$r$ is the fundamental solution in 2D, and the PS are obtained by means of techniques of separation, see Chapter 11 in [20]. For smooth solutions, the particular solutions are just the harmonic polynomials. For non-smooth solutions, the particular solutions are the angular particular solutions, and the mild singular solutions with r^nln$r$ may also be involved in. Hence, the PS have different formulations, depending on different solution domains and different corner boundary conditions. On the other hand, the ln$r$ is uniform and invariant, so that the algorithms of MFS is simple and easily to carry out. In philosophy, *the more general the one is, the less efficient it is*. Of course, the MFS is less efficient than the MPS.
4. The ill-conditioning is a severe issue of MFS, since both Cond and Cond_eff grow exponentially. In contrast, the Cond_eff of MPS grow polynomially. Moreover, the coefficients c_i in the MFS are large or huge, and their signs are alternatively changed to cause the other instability: subtraction cancelation [18]. Even though the MFS can be carried out by Mathematica, when the more working digits are used, the more CPU time is consumed.
5. From both accuracy and stability, if the particular solutions can be found, the MFS should be avoided. This conclusion is announced for smooth solutions by Schabck [24]; it is more true for singular solutions, see [13]. From the analysis and the numerical examples, we confirm that the MPS is better than the MPFS. Such a conclusion also supports the viewpoints on MFS by Schaback [24].
6. Although the performance of MFS is inferior to that of MPS, the MFS is still useful and developing. The reason is as follows. The uniform FS and simplicity of the MFS algorithms are the remarkable advantage of MFS, so that less computational efforts are needed. Since today the manpower saving is the most important saving, the MFS can be widely used, and very welcome by users. This proves the other philosophy: *the simplest, the best*.

Acknowledgement

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