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A method of fundamental solutions without fictitious boundary

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ABSTRACT

This paper proposes a novel meshless boundary method called the singular boundary method (SBM). This method is mathematically simple, easy-to-program, and truly meshless. Like the method of fundamental solutions (MFS), the SBM employs the singular fundamental solution of the governing equation of interest as the interpolation basis function. However, unlike the MFS, the source and collocation points of the SBM coincide on the physical boundary without the requirement of introducing fictitious boundary. In order to avoid the singularity at the origin, this method proposes an inverse interpolation technique to evaluate the singular diagonal elements of the MFS coefficient matrix. The SBM is successfully tested on a benchmark problems, which shows that the method has a rapid convergence rate and is numerically stable.

1. Introduction

Meshless methods and their applications have attracted considerable attention in recent decades, since methods of this type avoid the tedious mesh-generation required in the traditional mesh-based methods such as the finite element method. In comparison with the boundary element method, a variety of boundary-type meshless methods have been developed. For instance, the method of fundamental solutions (MFS) [1,2], boundary knot method [3], boundary collocation method [4], regularized meshless method (RMM) [5,6], and modified method of fundamental solutions [7], etc.

In this study, we propose a novel boundary-type meshless method called the singular boundary method (SBM) [8]. This method is mathematically simple, accurate, easy-to-program, and truly meshless. Like the MMFS, the SBM also directly uses the singular fundamental solution of the governing equation of interest as the interpolation basis function. Unlike the MMFS and all other boundary-type meshless methods, the SBM uses an inverse interpolation technique to evaluate the diagonal elements of the interpolation matrix to circumvent the singularity of the fundamental solutions at the origin. In the remainder of this paper, numerical experiments of the method are presented to demonstrate the rapid convergence, high accuracy and stability.

2. Formulation of singular boundary method

Without loss of generality, we consider the Laplace equation boundary value problem

$$\Delta u(\mathbf{x}) = 0 \quad \text{in } \Omega, \tag{1}$$

$$u(\mathbf{x}) = \overline{u}(\mathbf{x}) \quad \text{on } \Gamma, \tag{2}$$

where $u(\mathbf{x})$ is the potential, Ω denotes the physical solution domain in $\mathcal{R}^d$, where d denotes the dimensionality of the space, and Γ represents its boundary. For the two-dimensional Laplace equation, the fundamental solution is given by

$$u_L^*(\mathbf{x}_i, \mathbf{x}_j) = \frac{1}{2\pi} \ln\|\mathbf{x}_i - \mathbf{x}_j\|. \tag{3}$$

Like the MFS, the SBM also uses the fundamental solution as the kernel function of the approximation. Unlike the MFS, the collocation and source points of the SBM are coincident and are placed on the physical boundary without the need of using a fictitious boundary. The SBM interpolation formula is given by

$$u_N(\mathbf{x}_i) = \sum_{j=1, j\neq i}^{N} \alpha_j \ln\|\mathbf{x}_i - \mathbf{x}_j\| + \alpha_i q_{ii}, \tag{4}$$

where the α_j are the unknown coefficients, the q_{ii} are defined as the origin intensity factors. Eq. (4) manifests that the fundamental solution at the origin is replaced by q_{ii} when the collocation point $\mathbf{x}_i$ and source point $\mathbf{x}_j$ coincide ($i = j$).

The MMFS also uses the fundamental solution as the interpolation basis function while placing the source and collocation nodes on the same boundary [7]. The essential difference between the SBM and MMFS is the way of evaluating the origin intensity factors q_{ii}. The latter uses the numerical integration approach, while the SBM develops an inverse interpolation technique as detailed below.

The matrix form of Eq. (4) can be written as

$$\{q_{ij}\}\{\alpha_j\} = \{u(\mathbf{x}_i)\}, \tag{5}$$

where $q_{ij} = \ln\|\mathbf{x}_i - \mathbf{x}_j\|$. We can see that q_{ii} are the diagonal elements of matrix $Q = \{q_{ij}\}$. By collocating N source points on the physical boundary to satisfy the Dirichlet boundary condition

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(2), we obtain the following discretization algebraic equations:

$$\sum_{j=1, j\neq i}^{N} \alpha_j \ln\|\mathbf{x}_i - \mathbf{x}_j\| + \alpha_i q_{ii} = \overline{u}(\mathbf{x}_i), \quad \mathbf{x}_i \in \Gamma_1, \ \ i = 1, 2, \ldots, N_1, \tag{6}$$

where N_1 denotes the number of source points placed on the Dirichlet boundary. Obviously, we cannot simply use the fundamental solutions to compute q_{ii}. Instead we propose an inverse interpolation technique (IIT) to evaluate the diagonal elements q_{ii} of the interpolation matrix Q in the SBM.

For the boundary value problem (1)–(2), we locate source points $\mathbf{x}_j$ on the physical boundary and place computational collocation points $\mathbf{x}_k$ inside physical domain. Then we use a simple particular solution as the sample solution of Laplace equation (1), for example, $u(x,y) = x + y$. Using the interpolation formula (4), we can get

$$\{b_{kj}\}\{s_j\} = \{x_k + y_k\}, \tag{7}$$

where $b_{kj} = \ln\|\mathbf{x}_k - \mathbf{x}_j\|$. Thus, the influence coefficients s_j can be evaluated.

Replacing the computational collocation points $\mathbf{x}_k$ with the boundary source points $\mathbf{x}_j$, we have the diagonal elements

$$q_{ii} = \frac{1}{s_i}\left(x_i + y_i - \sum_{j=1, j\neq i}^{N} q_{ij} s_j \right), \quad i = 1, 2, \ldots, N \tag{8}$$

and the off-diagonal elements of the interpolation matrix $Q = \{q_{kj}\}$ can be computed by $q_{kj} = \ln\|\mathbf{x}_k - \mathbf{x}_j\|$ for the Dirichlet boundary condition. It is noted that the influence coefficients of Eq. (7) are the same as in Eq. (8). Therefore, Eq. (8) can be solved to calculate the unknown diagonal elements q_{ii} of the matrix Q.

With the calculated origin intensity factor, the SBM can be used to compute arbitrary Laplace problems with the same geometry by using interpolation formula (4).

As discussed in Section 3, the diagonal elements for the Laplace equation in a circular physical domain do not require the use of the inverse interpolation technique to be evaluated numerically. They are simply a summation of the corresponding off-diagonal elements, that is,

$$Q(i,i) = \sum_{i\neq j, j=1}^{N} q_{ij}. \tag{9}$$

However, this is an exceptional case.

3. Numerical results and discussions

Based on the above-mentioned numerical formulation, we examine a benchmark example. The known sample solutions $u(x,y) = x + y$ is chosen which yields better numerical results than the other sample solutions. The average relative error (root mean square relative error: RMSRE) is defined as follows [9]:

$$\text{RMSRE} = \sqrt{\frac{1}{K}\sum_{j=1}^{K} \text{Rerr}^2}, \tag{10}$$

where $\text{Rerr} = |(u(\overline{\mathbf{x}}_j) - \tilde{u}(\overline{\mathbf{x}}_j))/u(\overline{\mathbf{x}}_j)|$ for $|u(\overline{\mathbf{x}}_j)| \geq 10^{-3}$ and $\text{Rerr} = |u(\overline{\mathbf{x}}_j) - \tilde{u}(\overline{\mathbf{x}}_j)|$, for $|u(\overline{\mathbf{x}}_j)| < 10^{-3}$, respectively, j is the index of the inner point of interest, $u(\overline{\mathbf{x}}_j)$ and $\overline{u}(\overline{\mathbf{x}}_j)$ denote the analytical and numerical solutions at the j-th inner point, $\overline{\mathbf{x}}_j$, respectively, and K represents the total number of test points used.

For convenience, the boundary points are distributed uniformly on a unit circle. The exact solution of this case is $u(x,y) = x^2 - y^2$. To examine the resulting solution accuracy, the number of testing points scattered over the region of interest is chosen to be $K = 620$.

Here, the diagonal elements of the SBM interpolation matrix are evaluated by two different approaches: (1) By using Eq. (13), a summation of the corresponding off-diagonal elements and (2) the inverse interpolation technique introduced in Section 2.

The average relative error versus boundary point numbers for this problem is illustrated in Fig. 1. It is noted that the SBM error curves using the approach of a summation of the corresponding off-diagonal elements (called the summation approach in Fig. 1) and the inverse interpolation technique for the diagonal elements are very close. It should be stressed that the summation of the corresponding off-diagonal elements to evaluate the diagonal elements only works for Laplace problems in a circular domain. On the other hand, it is observed that the error curves of both the SBM and the RMM are decreasing with increasing boundary points, while the SBM converges faster than the RMM. When the boundary point number $N = 100$, the SBM solution accuracy is of order 10^{-5}, which is three orders of magnitude less than the RMM which is of order 10^{-2}.

For the SBM, we find that the condition numbers are smaller than the MFS and the BKM cases which is well-conditioned. Meanwhile, we find that the SBM condition number is the smallest among the known boundary-type meshless methods, which may be an attractive advantage for solving large-scale problems.

4. Conclusions

This paper introduces a novel meshless singular boundary method. Like the MFS, RMM and MMFS, the SBM uses the fundamental solution as the interpolation basis function. Unlike the MFS, the source and collocation points coincide and the fictitious boundary in the MFS is no longer required. Also, unlike the RMM and MMFS, the SBM uses a new inverse interpolation technique to remedy the singularity at origin of the fundamental solutions. The numerical solutions obtained with the SBM agree well with the analytical solutions. From the presented figures of the average relative error versus the increasing number of boundary points, we can see that numerical results for both the SBM and RMM exhibit a stable convergence trend in all tested cases, while the SBM converges faster than the RMM. However, it is also observed that the RMM condition number is in general much smaller than the SBM one.

Compared to the MFS, a disadvantage of the SBM is the fact that one needs to solve the two systems of linear algebraic equations. The mathematical analysis of the SBM is now under study and will be reported in a subsequent paper.

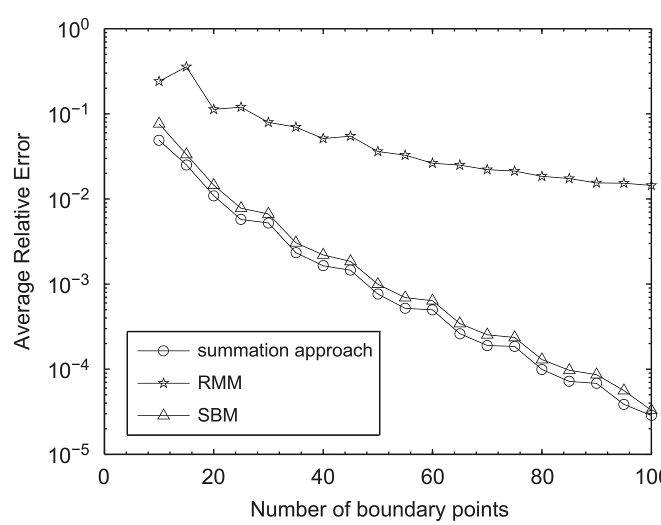

Fig. 1. Average relative error curves for Case 1.

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