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Eigensolutions of the Helmholtz equation for a multiply connected domain with circular boundaries using the multipole Trefftz method

J.T. Chen [a,b,*], S.K. Kao [a], W.M. Lee [c], Y.T. Lee [a]

[a] Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan
[b] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung, Taiwan
[c] Department of Mechanical Engineering, China University of Science and Technology, Taipei, Taiwan

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ABSTRACT

In this paper, 2D eigenproblems with the multiply connected domain are studied by using the multipole Trefftz method. We extend the conventional Trefftz method to the multipole Trefftz method by introducing the multipole expansion. The addition theorem is employed to expand the Trefftz bases to the same polar coordinates centered at one circle, where boundary conditions are specified. Owing to the introduction of the addition theorem, collocation techniques are not required to construct the linear algebraic system. Eigenvalues and eigenvectors can be found at the same time by employing the singular value decomposition (SVD). To deal with the eigenproblems, the present method is free of pollution of spurious eigenvalues. Both the eigenvalues and eigenmodes compare well with those obtained by analytical methods and the BEM as shown in illustrative examples.

1. Introduction

Acoustic problems become more and more important issues in the design phase for the new product. Many scholars have studied the sound radiation behavior and tried to find the connection between the sound radiation and vibration. They aimed to find an approach to decouple the sound radiation. Many well-developed numerical methods such as the finite element method (FEM), finite difference method (FDM) and boundary element method (BEM) can be adopted. Especially, the BEM has become popular in recent years due to its advantage of the reduction of dimensionality. However, spurious and fictitious frequencies occur and stem from the problem of non-uniqueness solution. If an incomplete set is adopted in the solution representation such as the real-part BEM [1] or the multiple reciprocity method (MRM) [2–7], spurious eigensolutions occur in solving eigenproblems with simply connected domain. Even though the complex-valued kernel is adopted in BEM, the spurious eigensolution also occurs for eigenproblems with the multiply connected domain [8] as well as the appearance of fictitious frequency for exterior acoustics [9]. Spurious eigensolutions and fictitious frequencies in the integral formulation belong to spectral pollution since it cannot be suppressed by refining the mesh. The origin of spurious modes arises from an improper approximation of null space of the integral operator [10]. This paper focuses on finding a meshless method free of spurious eigenvalues.

In the recent years, the meshless methods started to capture the interest of the researchers in the community of computational mechanics because these methods are mesh free and only boundary nodes are necessary [11–14]. Among meshless methods, the Trefftz method is a boundary-type solution procedure using only the T-complete functions satisfying the governing equation [15]. Since Trefftz presented the Trefftz method for solving boundary value problems in 1926 [16], various Trefftz methods such as direct formulations and indirect formulations [17] have been developed. The key issue in the use of the indirect Trefftz method is the definition of T-complete function set, which ensures the convergence of the subsequent expansions towards the analytical solutions. Many applications to the Laplace equation [18], the Helmholtz equation [19], the Navier equation [20,21] and the biharmonic equation [22] were done. Readers can consult with the Li et al.'s book [15]. However, all the applications seemed to be limited on the simply connected domain. The concept of the multipole method to solve exterior problems was firstly devised by Zàviška [23] and was used for the interaction of waves with arrays of circular cylinders by Linton and Evans [24]. Recently, Martin [25] reviewed several methods to solve multiple scattering problems in acoustics, electromagnetism, seismology and hydrodynamics. However, the interior eigenproblems were not mentioned therein. Extension to interior multiply connected eigenproblems by using the multipole Trefftz method is also our concern in this paper.

* Corresponding author at: Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan. Tel.: +886 2 24622192x6177; fax: +886 2 24632375.
E-mail address: jtchen@mail.ntou.edu.tw (J.T. Chen).

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This paper employs the addition theorem to expand the Bessel (J) and Hankel (H) functions [26] in the solution representation for matching the boundary conditions in an analytical way. The so-called multipole Trefftz technique is analytical and effective in solving problems with the multiply connected domain. Numerical experiments were performed to verify the present method. For the multiply connected eigenproblems, the mode shapes were plotted and compared with the other available results, e.g. exact solutions and BEM data [27,28].

2. Multipole Trefftz method for multiply connected eigenproblems with circular boundaries

2.1. Problem statement

The governing equation for the eigenproblem is the Helmholtz equation as follows:

$$(\nabla^2+k^2)u(\boldsymbol{x})=0, \quad \boldsymbol{x}\in D, \tag{1}$$

where ∇^2, k and D are the Laplacian operator, the wave number and the domain of interest, respectively. The multiply connected domain with circular boundaries is depicted in Fig. 1. The radius of the jth circle and the position vector of its center are R_j and $\boldsymbol{O}_j$, respectively.

2.2. Conventional Trefftz method for the simply connected domain

In the Trefftz method, the field solution $u(\boldsymbol{x})$ for a simply connected domain is superimposed by the T-complete functions, $\varphi_m(\boldsymbol{x})$, as follows:

$$u(\boldsymbol{x})=\sum_{m=-M}^{M}\alpha_m\varphi_m(\boldsymbol{x}), \tag{2}$$

where $\varphi_m(\boldsymbol{x})$ is the Trefftz base with respect to the origin O, $(2M+1)$ is the number of complete functions and α_m is the mth unknown coefficient which can be determined by matching the boundary conditions. Since this paper focuses on problems with circular boundaries, the polar coordinates are utilized and the field point $\boldsymbol{x}$ is expressed as $\boldsymbol{x}=(\rho,\phi)$. For the circular boundary with a radius R, the complete functions for 2D Helmholtz problems are shown below:

$$\varphi_m=\begin{cases}\varphi_m^I(\rho,\phi)=J_m(k\rho)e^{im\phi}, & \rho<R, \quad \text{interior case}, \quad m=0,\pm1,\pm2,\ldots,\pm M,\\ \varphi_m^E(\rho,\phi)=H_m^{(1)}(k\rho)e^{im\phi}, & \rho>R, \quad \text{exterior case}, \quad m=0,\pm1,\pm2,\ldots,\pm M,\end{cases} \tag{3}$$

where the superscripts of "I" and "E" denote the interior and exterior domains, respectively, and i is the imaginary number with $i^2=-1$.

2.3. Graf's addition theorem

According to the Graf's addition theorem for $J_m(k\rho_p)e^{im\phi_p}$ and $H_m^{(1)}(k\rho_p)e^{im\phi_p}$, we have

$$J_m(k\rho_p)e^{im\phi_p}=\sum_{n=-\infty}^{\infty}J_{m-n}(kb_{pq})e^{i(m-n)\theta_{pq}}J_n(k\rho_q)e^{in\phi_q}=\sum_{n=-\infty}^{\infty}J_{m-n}(k\rho_q)e^{i(m-n)\phi_q}J_n(kb_{pq})e^{in\theta_{pq}}, \tag{4}$$

$$H_m^{(1)}(k\rho_p)e^{im\phi_p}=\begin{cases}\sum_{n=-\infty}^{\infty}J_{m-n}(kb_{pq})e^{i(m-n)\theta_{pq}}H_n^{(1)}(k\rho_q)e^{in\phi_q}, & b_{pq}<\rho_q,\\ \sum_{n=-\infty}^{\infty}H_{m-n}^{(1)}(kb_{pq})e^{i(m-n)\theta_{pq}}J_n(k\rho_q)e^{in\phi_q}, & b_{pq}>\rho_q,\end{cases} \tag{5}$$

where (b_{pq},θ_{pq}) is the position vector (polar coordinates) of the qth center with respect to the pth center as shown in Fig. 2.

2.4. Singular value decomposition

Suppose $[\Phi]$ is an $m\times n$ matrix whose entries come from the field Ω, which is the field of complex numbers. Then there exists a factorization of the form

$$[\Phi]=[U][\Sigma][V]^H \tag{6}$$

where $[\sum]$ is the $m\times n$ diagonal matrix with nonnegative real numbers on the diagonal, the superscript "H" is the Hermitian operator, $[U]$ and $[V]$ are the $m\times m$ and $n\times n$ unitary matrices, respectively, and their column vectors which satisfy

$$\{u_i\}^H\cdot\{u_j\}=\delta_{ij} \tag{7}$$

$$\{v_i\}^H\cdot\{v_j\}=\delta_{ij} \tag{8}$$

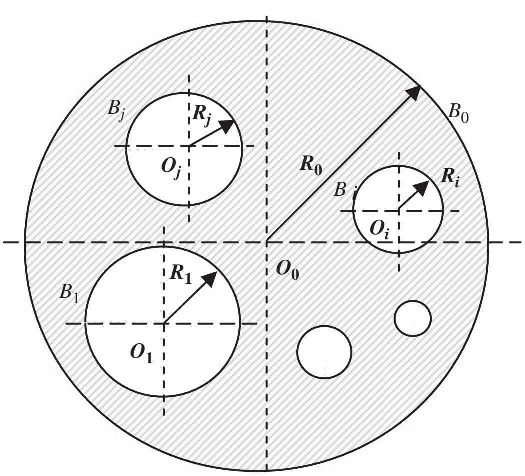


Fig. 1. A multiply connected domain with circular boundaries.

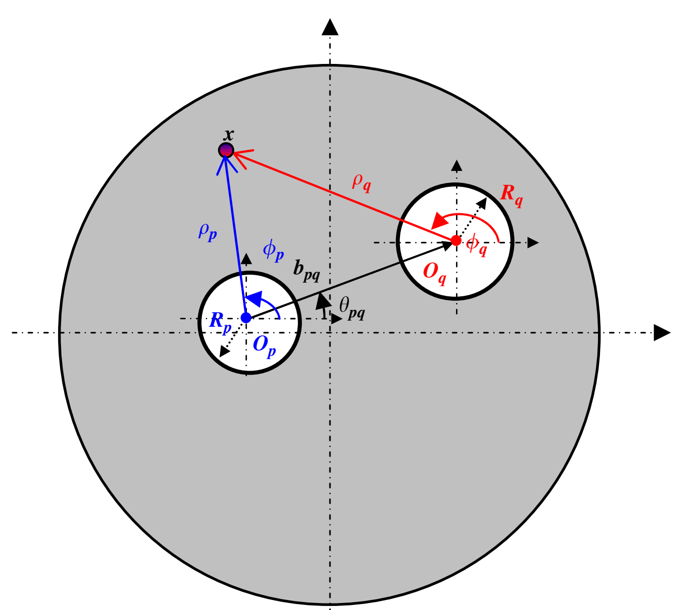


Fig. 2. Notations of the Graf's addition theorem.

in which $[U]^H[U]=[\mathbf{I}]_{m\times m}$ and $[V]^H[V]=[\mathbf{I}]_{n\times n}$. For an eigenproblem, we can obtain a nontrivial solution for the homogeneous system from a column vector $\{v_i\}$ of $[V]$ when the singular value (σ_i) is zero. Such a factorization is called a singular value decomposition of $[\Phi]$. We employ the SVD technique to simultaneously obtain the eigenvalues and eigenvectors.

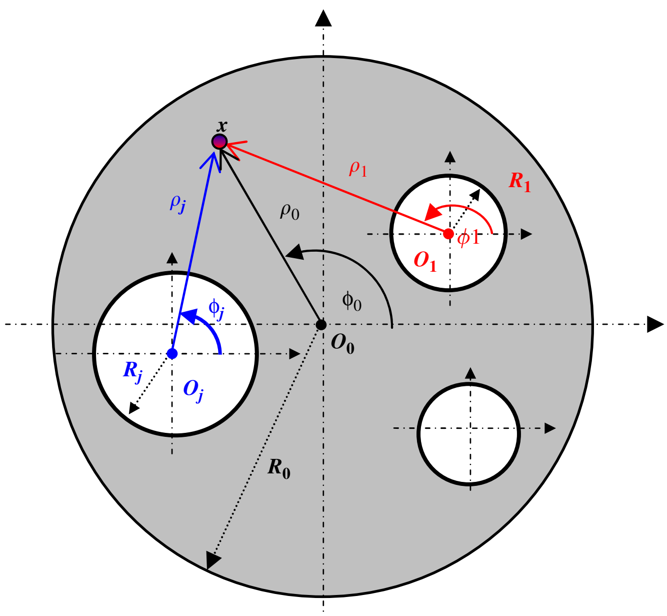


Fig. 3. Notations for the solution using the multipole Trefftz method.

2.5. Multipole Trefftz method

Since the multiply connected domain is considered, both the interior and exterior complete functions are required. The field solution can be represented by

$$u(\boldsymbol{x};\rho_0,\phi_0,\rho_1,\phi_1,\ldots,\rho_N,\phi_N) = \sum_{m=-\infty}^{\infty}\alpha_m^0 J_m(k\rho_0)e^{im\phi_0}+\sum_{j=1}^{N}\sum_{m=-\infty}^{\infty}\alpha_m^j H_m^{(1)}(k\rho_j)e^{im\phi_j}, \tag{9}$$

where α_m^j is the unknown coefficient of the mth complete function for $\boldsymbol{O}_j$ and the position vector of the field point $\boldsymbol{x}$ with respect to $\boldsymbol{O}_j$ is noted (ρ_j,ϕ_j), j=0,1,2, …, N, as shown in Fig. 3. In order to enforce the boundary condition on B_0 (ρ_0=R_0), we must express each term as a function of (R_0,ϕ_0) for the solution representation. By translating $H_m^{(1)}(k\rho_n)e^{im\phi_n}$ in terms of functions of (ρ_0,ϕ_0) using the addition theorem of Eq. (5), we have

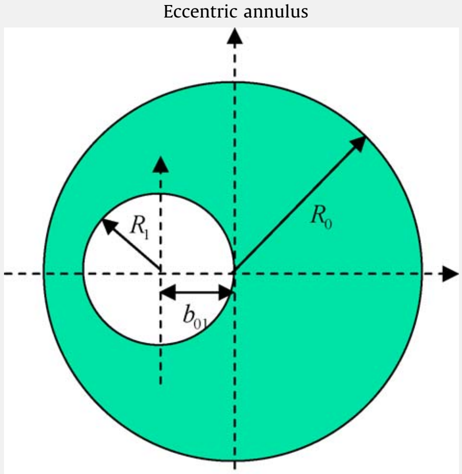
$$u(\boldsymbol{x};R_0,\phi_0)=\sum_{m=-\infty}^{\infty}\alpha_m^0 J_m(kR_0)e^{im\phi_0} +\sum_{j=1}^{N}\sum_{m=-\infty}^{\infty}\alpha_m^j\sum_{n=-\infty}^{\infty}J_{m-n}(kb_{j0})e^{i(m-n)\theta_{j0}}H_n^{(1)}(kR_0)e^{in\phi_0}, \quad x\in B_0, \tag{10}$$

where j, m and n in the three summation symbols denote indexes of the number of the circular holes, number of the Trefftz bases and number of terms in the addition theorem, respectively. For the Dirichlet problem, the boundary condition on B_0 is u_0=0. By comparing the coefficient of $e^{im\phi_0}$, we have

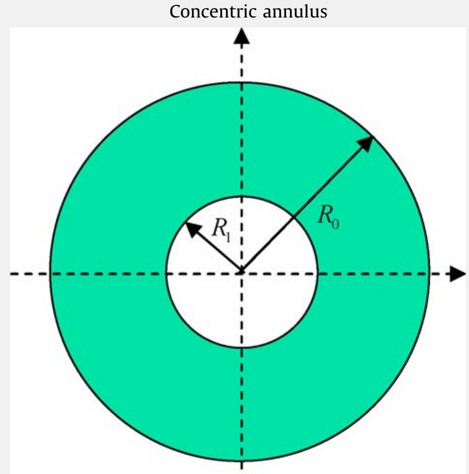
$$\alpha_m^0 J_m(kR_0)+H_m^{(1)}(kR_0)\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\alpha_n^j J_{n-m}(kb_{j0})e^{i(n-m)\theta_{j0}}=0, \quad m=0,\pm 1,\pm 2,\ldots. \tag{11}$$

Table 1
The former five eigenvalues for a multiply connected problem with an eccentric annulus and a concentric annulus using different approaches.

Eccentric annulus



Concentric annulus



	Eccentric annulus: Multipole Trefftz method (M=5)	Eccentric annulus: BEM [27]	Concentric annulus: Multipole Trefftz method (M=5)	Concentric annulus: BEM [27]	Analytical solution
k_1	1.74	1.75	2.05	2.06	2.04884
k_2	2.13	2.14	2.22	2.23	2.22375
k_3	2.46	2.47	2.22	2.23	2.22375
k_4	2.77	2.78	2.66	2.67	2.65993
k_5	2.96	2.98	2.66	2.67	2.65993

If we consider to enforce the boundary condition on B_l ($\rho_l=R_l$), $J_m(k\rho_0)e^{im\phi_0}$ and $H_m^{(1)}(k\rho_j)e^{im\phi_j}$ in Eq. (9), j=0,1,2, …, N and $j \neq l$, are required to translate into (ρ_l,ϕ_l) system using the addition theorem. The field solution of Eq. (9) yields

$$\begin{aligned} u(\boldsymbol{x};R_l,\phi_l) = {} & \sum_{m=-\infty}^{\infty} \alpha_m^0 \sum_{n=-\infty}^{\infty} J_n(kR_l)e^{in\phi_l} J_{m-n}(kb_{0l})e^{i(m-n)\theta_{0l}} \\ & + \sum_{m=-\infty}^{\infty} \alpha_m^l H_m^{(1)}(kR_l)e^{im\phi_l} \\ & + \sum_{\substack{j=1 \\ j\neq l}}^{N} \sum_{m=-\infty}^{\infty} \alpha_m^j \sum_{n=-\infty}^{\infty} f_{mn}(R_l,\phi_l,b_{jl},\theta_{jl}), \quad x \in B_l, \end{aligned} \tag{12}$$

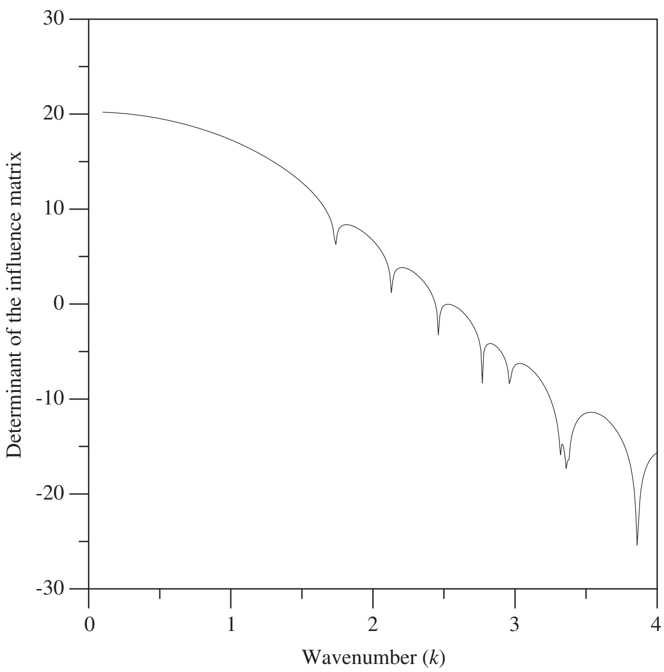


Fig. 4. Determinant versus the wave number by using the multipole Trefftz method for the eccentric case.

where

$$f_{mn}(R_l,\phi_l,b_{jl},\theta_{jl}) = \begin{cases} H_n^{(1)}(kR_l)e^{in\phi_l} J_{m-n}(kb_{jl})e^{i(m-n)\theta_{jl}}, & b_{jl} < R_l, \\ J_n(kR_l)e^{in\phi_l} H_{m-n}^{(1)}(kb_{jl})e^{i(m-n)\theta_{jl}}, & b_{jl} > R_l. \end{cases} \tag{13}$$

By satisfying the boundary condition u_l=0 and comparing with coefficients, we have

$$\begin{aligned} & J_m(kR_l) \sum_{n=-\infty}^{\infty} \alpha_n^0 J_{n-m}(kb_{0l})e^{i(n-m)\theta_{0l}} + \alpha_m^l H_m^{(1)}(kR_l) \\ & \quad + \sum_{\substack{j=1 \\ j\neq l}}^{N} \sum_{n=-\infty}^{\infty} \alpha_n^j f_{mn}(R_l,\phi_l,b_{jl},\theta_{jl}) e^{-im\phi_l} = 0, \\ & \quad m = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{14}$$

Eqs. (11) and (14) form a system of equations of simultaneous linear algebraic equations for the coefficients α_m^0 and α_m^j, m=0, ± 1, $\pm 2, \ldots \pm M$ and n=0, ± 1, $\pm 2, \ldots \pm M$, as shown below:

$$[\Phi]_{[(N+1)\times(2M+1)]\times[(N+1)\times(2M+1)]}\{\mathbf{c}\}_{[(N+1)\times(2M+1)]\times 1} = \{0\}, \tag{15}$$

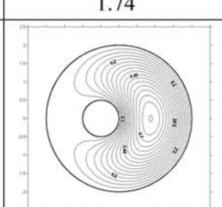
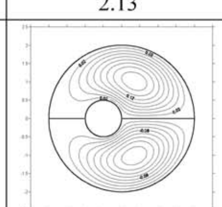
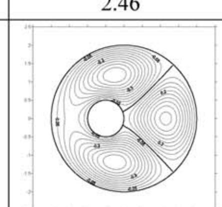
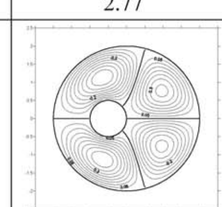
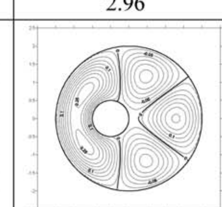
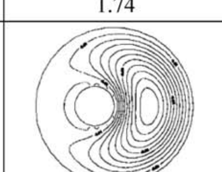
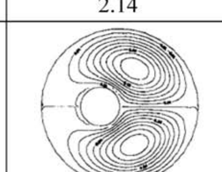
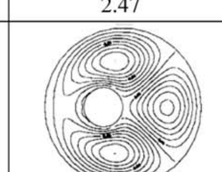
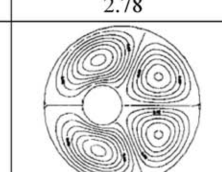
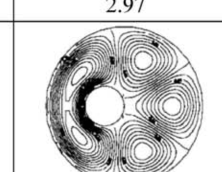
where

$$[\Phi] = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0N} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{N0} & \Phi_{N1} & \cdots & \Phi_{NN} \end{bmatrix}, \tag{16}$$

$$\{\mathbf{c}\} = \begin{Bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{Bmatrix}. \tag{17}$$

in which the dimension of $[\Phi]$ is $(N+1)\times(2M+1)$ by $(N+1)\times(2M+1)$, $\{\mathbf{c}\}$ denotes the column vector of unknown coefficients with a dimension of $(N+1)\times(2M+1)$ by 1. The submatrix, $[\Phi_{pq}]$, denotes the potential of the pth circular boundary with respect to O_q. The formation of $[\Phi_{pq}]$ can be

Table 2
The former five modes for a multiply connected problem with an eccentric hole.

Mode No.	1	2	3	4	5
eigenvalue	1.74	2.13	2.46	2.77	2.96
Multipole Trefftz method ($M=5$)					
eigenvalue	1.74	2.14	2.47	2.78	2.97
BEM [27]					

written as

$$[\Phi_{pq}] = \begin{cases} \begin{bmatrix} J_{-M}(kR_0)e^{-iM\phi_0} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_M(kR_0)e^{iM\phi_0} \end{bmatrix}, & p=q=0, \\ \begin{bmatrix} H^{(1)}_{-M}(kR_p)e^{-iM\phi_p} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H^{(1)}_M(kR_p)e^{iM\phi_p} \end{bmatrix}, & p=q\neq 0, \\ \begin{bmatrix} J_{-M}(kR_P)e^{-iM\phi_P}J_{-M+M}(kb_{0p})e^{i(-M+M)\theta_{0p}} & \cdots & J_{-M}(kR_P)e^{-iM\phi_P}J_{M+M}(kb_{0p})e^{i(M+M)\theta_{0p}} \\ \vdots & \ddots & \vdots \\ J_M(kR_P)e^{iM\phi_P}J_{-M-M}(kb_{0p})e^{i(-M-M)\theta_{0p}} & \cdots & J_M(kR_P)e^{iM\phi_P}J_{M-M}(kb_{0p})e^{i(M-M)\theta_{0p}} \end{bmatrix}, & p\neq 0 \text{ and } q=0 \\ \begin{bmatrix} H^{(1)}_{-M}(kR_p)e^{-iM\phi_p}J_{-M+M}(kb_{qp})e^{i(-M+M)\theta_{qp}} & \cdots & H^{(1)}_{-M}(kR_p)e^{-iM\phi_p}J_{M+M}(kb_{qp})e^{i(M+M)\theta_{qp}} \\ \vdots & \ddots & \vdots \\ H^{(1)}_M(kR_p)e^{iM\phi_p}J_{-M-M}(kb_{qp})e^{i(-M-M)\theta_{qp}} & \cdots & H^{(1)}_M(kR_p)e^{iM\phi_p}J_{M-M}(kb_{qp})e^{i(M-M)\theta_{qp}} \end{bmatrix}, & \text{otherwise.} \end{cases} \tag{18}$$

Moreover, the gradient of $u(\boldsymbol{x})$ is

$$\nabla u = \nabla u(\boldsymbol{x};\rho_0,\phi_0,\rho_1,\phi_1,\ldots,\rho_N,\phi_N)$$
$$= \nabla\left[\sum_{m=-\infty}^{\infty}\alpha^0_m J_m(k\rho_0)e^{im\phi_0} + \sum_{j=1}^{N}\sum_{m=-\infty}^{\infty}\alpha^j_m H^{(1)}_m(k\rho_j)e^{im\phi_j}\right]. \tag{19}$$

For the Neumann problem, we have the normal derivative

$$\nabla u\cdot n_x = \nabla\left[\sum_{m=-\infty}^{\infty}\alpha^0_m J_m(k\rho_0)e^{im\phi_0} + \sum_{j=1}^{N}\sum_{m=-\infty}^{\infty}\alpha^j_m H^{(1)}_m(k\rho_j)e^{im\phi_j}\right]\cdot n(\boldsymbol{x}),$$
$$m=0,\ \pm 1,\ \pm 2,\ldots. \tag{20}$$

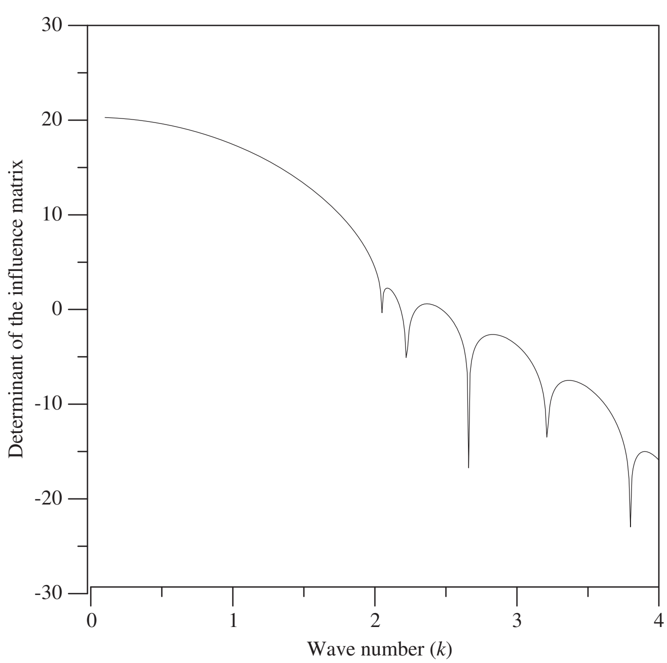


Fig. 5. Determinant versus the wave number by using the multipole Trefftz method for the concentric case.

For satisfying the boundary conditions on B_0 (t_0=0) and B_l (t_l=0) and comparing with coefficients, we have

$$\alpha^0_m J'_m(kR_0) + H^{(1)'}_m(kR_0)\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}\alpha^j_n J_{n-m}(kb_{j0})e^{i(n-m)\theta_{j0}} = 0,$$
$$m=0,\ \pm 1,\ \pm 2,\ldots,\ \pm M,\quad \boldsymbol{x}\in B_0 \tag{21}$$

and

$$J'_m(kR_l)\sum_{n=-\infty}^{\infty}\alpha^0_n J_{n-m}(kb_{0l})e^{i(n-m)\theta_{0l}} + \alpha^l_m H^{(1)'}_m(kR_l)$$
$$+\sum_{\substack{j=1\\ j\neq l}}^{N}\sum_{n=-\infty}^{\infty}\alpha^j_n \frac{\partial}{\partial\rho_l} f'_{mn}(\rho_l,\phi_l,b_{jl},\theta_{jl})e^{-im\phi_l}\big|_{\rho_l=R_l} = 0,$$
$$m=0,\ \pm 1,\ \pm 2,\ldots \pm M,\quad \boldsymbol{x}\in B_l,\quad l=1,2,\ldots,N, \tag{22}$$

where

$$f'_{mn}(R_l,\phi_l,b_{jl},\theta_{jl}) = \begin{cases} H^{(1)'}_n(kR_l)e^{in\phi_l}J_{m-n}(kb_{jl})e^{i(m-n)\theta_{jl}}, & b_{jl}<R_l, \\ J'_n(kR_l)e^{in\phi_l}H^{(1)}_{m-n}(kb_{jl})e^{i(m-n)\theta_{jl}}, & b_{jl}>R_l. \end{cases} \tag{23}$$

Eqs. (21) and (22) form a system of simultaneous linear algebraic equations for the coefficients α^0_m and α^j_m, m=0, ± 1, $\pm 2,\ldots,\pm M$. In the implementation, the value of M is chosen five to obtain acceptable results in following examples. By applying the SVD technique to decompose the matrix $[\Phi]$, the determinant versus k is used to detect eigenvalues and nontrivial vector of $\{\boldsymbol{c}\}$. To save the CPU time for the direct-searching approach, an adaptive increment of k is used. In the adaptive scheme for the direct-seaching approach, a larger value of Δk is adopted to find the possible drop in the first trial. Then, a smaller value of Δk is considered in the area near the drop location. The eigenmode is obtained by searching the right unitary vector for $\{\boldsymbol{c}\}$ corresponding to the zero singular value. The number of the zero singular values implies the number of multiplicity roots.

3. Numerical examples

We consider two cases of Helmholtz eigenproblems with a multiply connected domain subjected to the Dirichlet boundary conditions.

Case 1. A circular membrane with an eccentric hole (special case: annulus).

Table 3
The former five modes for a multiply connected problem with a concentric hole.

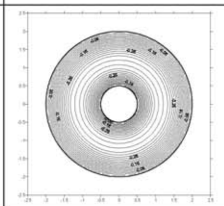
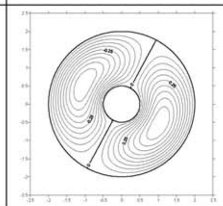
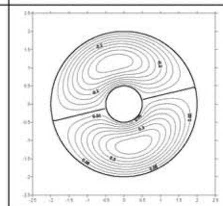
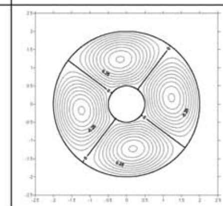
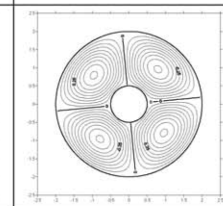
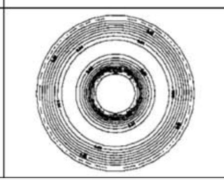
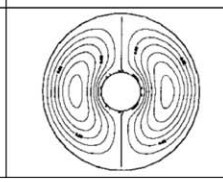
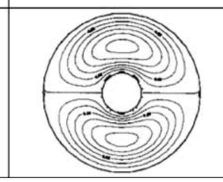
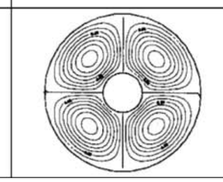
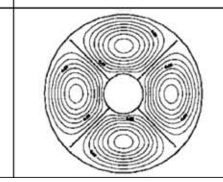
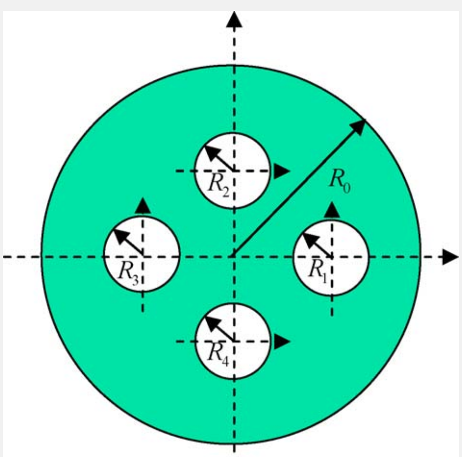
Mode No.	1	2	3	4	5
eigenvalue	2.05	2.22	2.22	2.66	2.66
Multipole Trefftz method ($M=5$)					
eigenvalue	2.06	2.23	2.23	2.67	2.67
BEM [27]					

Table 4
The former five eigenvalues for a multiply connected problem with four equal holes using different approaches.



	Multipole Trefftz method (M=5)	BEM [28]
k_1	4.499	4.47
k_2	5.369	5.37
k_3	5.369	5.37
k_4	5.549	5.54
k_5	5.949	5.95

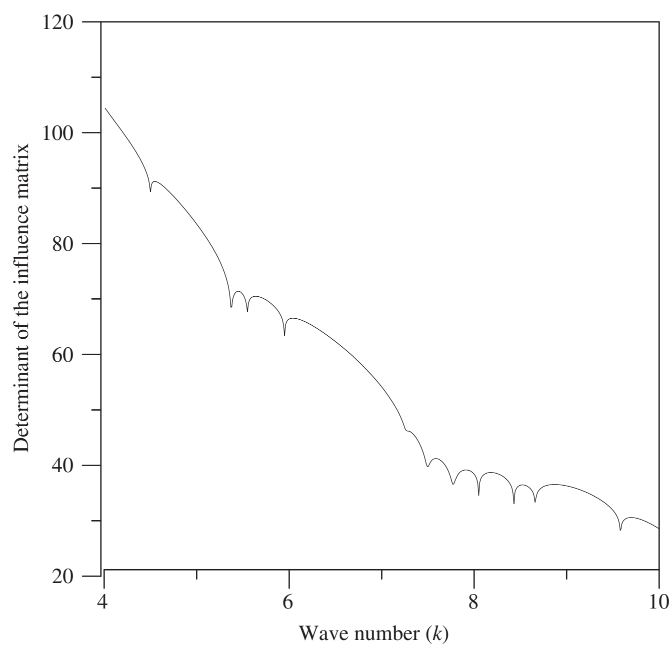


Fig. 6. Determinant versus the wave number by using the multipole Trefftz method for the multiply connected case with four equal holes.

The eccentric domain is shown in Table 1. The radii of the outer and inner circular boundaries are R_0=2 m and R_1=0.5 m, respectively. The eccentricity e=b_{01}=b_{10} is 0.5 m. Both the boundary conditions are u_j=0, j=0,1. Extraction of eigenvalues free of pollution of spurious eigenvalues by using the present method is shown in Fig. 4. The eigenvalues and modes are obtained as shown in Tables 1 and 2. By selecting M=5, the results of this approach agree well with those of BEM [27].

A special case of eccentric ring is an annular domain which is also considered in Table 1 and the radii of the outer and inner circles are the same as those of the eccentric case. Since the two circles are concentric, the distance between the two poles is zero (b_{01}=b_{10}=0). The linear algebraic system reduces to that derived by the conventional Trefftz method. Moreover, the analytical solution could be derived by using this approach. Eqs. (11) and (14) can be rewritten as

$$\alpha_m^0 J_m(kR_0)+\alpha_m^1 H_m^{(1)}(kR_0)=0, \quad m=0, \pm 1, \pm 2, \ldots \pm \infty, \tag{24}$$

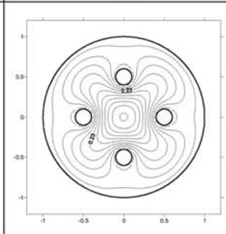
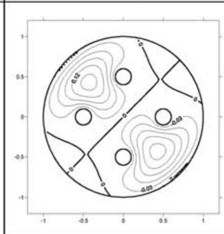
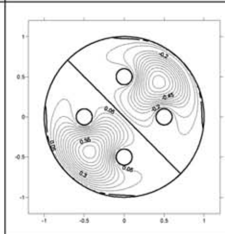
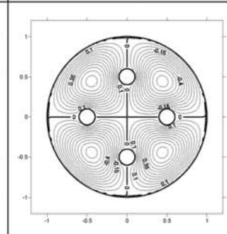
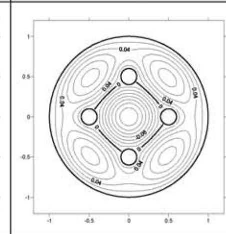
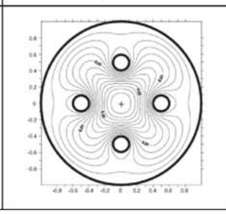
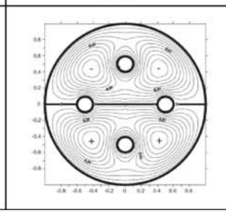
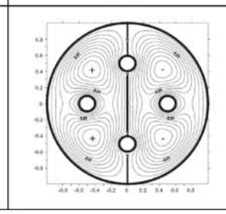
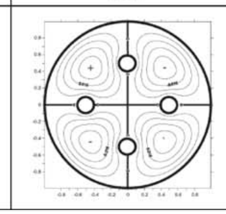
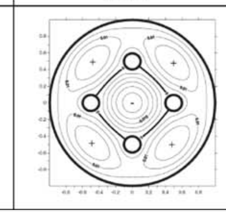
$$\alpha_m^0 J_m(kR_1)+\alpha_m^1 H_m^{(1)}(kR_1)=0, \quad m=0, \pm 1, \pm 2, \ldots \pm \infty. \tag{25}$$

According to Eqs. (24) and (25), the analytical eigenequation is derived as below:

$$J_m(kR_0)H_m^{(1)}(kR_1)-J_m(kR_1)H_m^{(1)}(kR_0)=0, \quad m=0, \pm 1, \pm 2, \ldots \pm \infty. \tag{26}$$

The analytical eigenvalues are also shown in Table 1. By using the SVD technique, the determinant of the influence matrix versus the wave number is shown in Fig. 5. The true eigenvalues and modes are shown in Tables 1 and 3, respectively. Although the mode shape corresponding to the eigenvalues k_2 and k_3 seem different from the results of the BEM, mode shapes of the

Table 5
The former five modes for a multiply connected problem with four equal holes.

Mode No.	1	2	3	4	5
eigenvalue	4.499	5.369	5.369	5.549	5.949
Multipole Trefftz method ($M=5$)					
eigenvalue	4.47	5.37	5.37	5.54	5.95
BEM [28]					

present method can be linearly superimposed by using the two independent modes of BEM, and vice versa.

Case 2. A circular membrane with four circular holes.

The outer boundary with a radius R_0=1m and four holes of equal size with radii R_j=0.1m, j=1,2,3,4 are considered and the former five eigenvalues are shown in Table 4. The positions of the four centers of the circular holes are (0.5,0), (0,0.5), ($-$0.5,0) and (0,$-$0.5). Chen et al. [28] also used the BEM for finding the eigenvalues of Dirichlet problems. The eigenvalules extracted out by the SVD are shown in Fig. 6. Eigenvalues and eigenmodes using the BEM and the present method are shown in Tables 4 and 5, respectively. Although the shapes of modes 2 and 3 seem different from the results of the BEM, the modes of the present method can be linearly superimposed by using the two independent modes of BEM, and vice versa. Good agreement is made.

4. Concluding remarks

In this paper, the Graf's addition theorem was used to reform the awkward situation of the classical Trefftz method for multiply connected problems. This approach was coined the multipole Trefftz method. The multipole Trefftz method has successively provided an analytical model for solving eigenvalues and eigenmodes of a circular membrane containing multiple circular holes. The numerical experiments of the multiply connected problems were performed to demonstrate the validity of the present approach. Good agreements between the results of the multipole Trefftz method and the BEM were made. In addition, the ability of detecting the root of multiplicity can be achieved in the multipole Trefftz method by using the SVD technique free of pollution of spurious eigenvalues. Numerical results show high accuracy and fast rate of convergence thanks to the analytical approach.

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