ARTICLE IN PRESS

Engineering Analysis with Boundary Elements ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Development and implementation of some BEM variants—A critical review

Kok Hwa Yu [a,*], A. Halim Kadarman [a], Harijono Djojodihardjo [b]

[a] School of Aerospace Engineering, Engineering Campus, Universiti Sains Malaysia, 14300 Penang, Malaysia
[b] Aerospace Engineering, Faculty of Engineering, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia

ARTICLE INFO

Article history:
Received 20 August 2009
Accepted 4 May 2010

Keywords:
Boundary element method (BEM)
Collocation BEM
Galerkin BEM
Dual reciprocity BEM
Complex variable BEM
Analog equation method
BEM development
Review

ABSTRACT

Due to rapid development of boundary element method (BEM), this article explores the evolution of BEM over the past half century. We here summarize the overall development and implementation of several well-known BEM variants that includes collocation BEM, galerkin BEM, dual reciprocity BEM, complex variable BEM and analog equation method. Their theoretical and mathematical backgrounds are carefully described and a generalized Laplace's equation (and Poisson's equation) is utilized in demonstrating the different approaches involved. An up-to-date review on characteristics and implementation for each of the five variants is presented and also highlighted their significant contributions in boundary element research. In addition, this article tries to cover whole aspect of interests including efficiency, applicability and accuracy in order to give better understanding of BEM evolution. Comparisons and techniques of improvement for these variants are also discussed.

1. Introduction

Boundary element method (BEM) is one of the numerical techniques designed for solving boundary value problems in partial differential equations (PDEs). Applied in various engineering and science disciplines, BEM can be considered as a major numerical method alongside with better known finite element method (FEM) and finite difference method (FDM) jointly providing effective computational solution for a wide class of engineering and scientific problems. Like many other numerical techniques, the interest towards BEM increased gradually over recent decades catalyzed by rapid advances in computer technology. In fact, the statistical data based on the *Web of Science* search shows that the amount of annual published literature described by BEM saw an exponential growth until late 1990s and exceeded 700 literatures annually in subsequent years [39]. In this relatively short period of time, the evolution of BEM is tremendous. Several comprehensive reviews on BEM recently prepared by scholars have documented various theoretical basis and early development of BEM [39,50,106,256]. Many of the mathematical approaches presented in BEM are associated with work of famous mathematicians and scientists. The contributions of mathematicians like Laplace, Green, Fredholm, Fourier, Kellogg and Betti could be traced in the theoretical and mathematical foundation of boundary integral equation (BIE) in the early 20th century (See [39] for more detailed description on their contributions). However, pioneering work by Jaswon [113] and Symm [243] in 1963 has been marked as the formal beginning of the boundary element era. They demonstrated direct formulation using Green's third identity for two-dimensional Laplace's equation, hence, credited them as the first to formulate the potential problem in terms of direct BIE that set them apart from existing indirect BIE. Although, in the beginning, their work did not receive much attention as they deserved, their concept on direct approach has inspired Rizzo [219] to adopt similar approach using displacements and tractions in integral equation. Soon afterwards, his work has attracted and inspired researchers to investigate the potential of the new boundary integral approach. Considered as a major breakthrough in BEM, his work is then adopted by many researchers and saw swift progress in the development of boundary integral equation approach.

Following these early works, extensive researches and development works were carried out in 1970s to construct the basis of modern BEM on various aspects including basic principles, applications and numerical techniques. One must bear in mind that before BEM existed, Green's formulation used only to reduce the differential equation in the domain-to-boundary integral equation. Only until late 1970s, the discretization of these boundary integrals using numerical techniques creates so-called BEMs. The terminology of boundary element method, previously referred as boundary integral equation method or boundary integral method first appeared in the works of Brebbia and Dominguez [28] and Banerjee and Butterfield [18] in 1977. However, the article prepared by Banerjee and Butterfield was still based on indirect method boundary equations while Brebbia and Dominguez presented the potential problems through a weighted residual approach using direct version. A year later, Brebbia [25] published the first textbook of BEM focusing on basic

* Corresponding author. Tel.: +60 165419737.
E-mail address: yukokhwa@yahoo.com (K.H. Yu).

0955-7997/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2010.05.001

principles and its application to potential and elasticity problems, created a major advance in BEM. At the same time, enormous research efforts have been directed in expanding the development of BEM. One of the significant achievements was the implementation of substructure technique in BEM presented by Lachat [143,144] which overcome one of the major drawbacks in BEM: nonsymmetrical matrix. On the other hand, apart from conventional BEM in frequency domain, the time domain BEM was later introduced by Cole et al. [46] for anti-plane strain problem in two-dimensional elastodynamics and then improved by Mansur and Brebbia [153,154] to accommodate scalar wave problem. Following the extension of this formulation, the BEM can be employed to investigate the transient behavior and also nonlinear problems. The detailed theoretical basis for these problems (nonlinear and time-dependent problems) can be obtained in textbooks written by Brebbia and his colleagues in [30,31]. Soon, the BEM was further extended to cover a wide scope of solid mechanics [6,249] and fluid mechanics [32] problems. Thanks to early development prepared by numerous researchers including Brebbia and his fellow colleagues who directly involved in constructing the blueprint of modern BEM which then well received by the scientific community that nowadays the BEM has been adopted by many researchers covering countless specialized areas such as acoustics [8,45,253,262], contact mechanics [152], dynamics analysis [61], solids and structures [9,231], soil-structure interactions [84], nonlinear fluid dynamics [20], heat transfer [262,263] and quantum mechanics [215]. With the solid foundation and rich heritage, BEM emerged as a powerful method and thus become a strong alternative to the FEM and FDM.

2. Boundary integral equation

Unlike FEM and FDM, as the numerical implementation of boundary integral equations, BEM requires surface-only discretization. The BIE re-formulations of boundary value problems for partial differential equations are valid everywhere—interior and exterior of the domain and also on the boundary, giving it a big advantage since most of the engineering and scientific applications can be described by PDEs. Note, however, that not all PDEs can be transformed into integral equations, and therefore, it is crucial for researchers to be able to understand the classification of PDEs and the concept behind each class. In most mathematics books, these PDEs can be classified as being elliptic, parabolic, or hyperbolic type according to the form of the equation. In second-order PDE (the order of a PDE indicates the order of the highest order derivative found in the PDE), the differential equation is considered as: (1) elliptic equation when the coefficients of both no mixed second-order derivatives are nonvanishing and of the same sign. (2) parabolic equation when only one second-order non-mixed derivative term is present and of opposite sign. (3) hyperbolic equation when the coefficients of the non-mixed two second-order derivatives are nonvanishing and opposite in sign. Moreover, elliptic PDEs have boundary conditions specified around a closed boundary, whilst hyperbolic and parabolic PDEs have at least one open boundary. Thus, elliptic equations are often used to describe systems in the equilibrium or steady state whereas the parabolic equations being utilized for demonstrating physical systems with a time variable, diffusion like phenomena, and for the hyperbolic equations, they are frequently used to describe oscillatory systems especially wave-like phenomena. In addition to the classification mentioned above, PDEs can be presented as single PDE or systems of PDEs with multiple variables. Although the integral equation re-formulation can only be derived for certain classes of PDE, it is much easier to apply and more computationally efficient, if applicable. Therefore, it is critically important to know the characteristics of each differential equation in order to employ the right numerical technique. To date, the integral equations have been successfully applied to describe various physical disciplines such as elastostatics, electrostatics, electrodynamics, elasticity, plasticity, heat transfer, acoustics, fluid dynamics and so on. Table 1 illustrates some applicable areas for linear and nonlinear problems distinguished into classes of PDEs.

Table 1
Applicable areas for different classes of PDEs.

Linear	Nonlinear
Single second-order PDEs	
(A) Elliptic	(A) Hyperbolic
Laplace's equation	*Burgers' equation*
• Electrostatics	• Fluid mechanics
• Incompressible potential flow	
• Heat Conduction (Steady state, no heat generation)	
Poisson's equation	
• Electrostatics	
• Electromagnetics	
• Heat Conduction (Uniform thermal conductivity, steady state, heat generation within the solid)	
Helmholtz equation	
• Acoustics (Interior and exterior problem)	
• Electromagnetics	
• Optics	
(B) Parabolic	
Diffusion/Heat equation	
• Heat Conduction (Uniform thermal conductivity, no heat sources)	
Schrödinger's equation	
• Quantum mechanics	
(C) Hyperbolic	
Wave equation	
• Acoustics	
• Electromagnetics	
• Fluid dynamics	
Single fourth-order PDEs	
Biharmonic equation	
• Thin plate problem	
• Fluid-flow problem	
System of PDEs	
Navier's equation	*Navier–Stokes equation*
• Elasticity problem	• Fluid dynamics
Maxwell's equation	*Euler's equation*
• Electromagnetics	• Fluid dynamics

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

2.1. Laplace's equation

Despite the variety, Laplace's equation is widely preferred by scholars when introducing the BEM to the reader and we noticed, in general, the elliptic equations have a well-developed theory and some equivalent forms of BIE. Hailed as the most simple and prominent example, Laplace's equation is typically utilized in formulating the potential problem in which the solution is harmonic. Thus, in studying the development of BEM, Laplace's equation (and Poisson's equation) is generally utilized in demonstrating the different approach adopted by each BEM variants. For convenience, the standard Laplace's equation with the corresponding notation described by Brebbia in [25,27,30] is then employed in this section to assist the understanding of concept behind the transformation of PDEs into integral equations form. Following this, let us begin this section with basic equation in which two-dimensional Laplace's equation is considered. Laplace's equation can be expressed as follows:

$$\nabla^2 u = 0, \quad u \text{ in } \Omega \tag{1}$$

where the domain Ω is shown in figure below. Fig. 1

Here two BEM approaches can be used in solving Laplace's equation: 'direct' and 'indirect'. For direct approach, the integral equation is derived using Green's second theorem,

$$\int_\Omega (u^* \nabla^2 u - u \nabla^2 u^*) d\Omega = \int_\Gamma \left(u^* \frac{\partial u}{\partial n} - u \frac{\partial u^*}{\partial n} \right) d\Gamma \tag{2}$$

and leads to

$$\int_\Omega u^* \nabla^2 u d\Omega = \int_{\Gamma_2} q u^* d\Gamma - \int_{\Gamma_1} u q^* d\Gamma \tag{3}$$

with both u and u_* satisfy Laplace's equation where the potential function, u, satisfies $\nabla^2 u = 0$ everywhere in the solution domain while function u_*, known as fundamental solution satisfies $\nabla^2 u_* = 0$ everywhere except at source points. Similarly, the derivative for fundamental solution can be obtained using $q_* = \partial u_* / \partial n$. Here Green's second identity is utilized in reducing the dimensionality of the problem. Integrating the left hand side of Eq. (3), one obtains

$$\int_\Omega (\nabla^2 u^*) u d\Omega = \int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma \tag{4}$$

As function u_* is the fundamental solution to Laplace's equation, the solution can be written

$$\nabla^2 u^* = -2\pi \Delta^i \tag{5}$$

where the Dirac delta function, Δ^i tends to infinity at point $x = x^i$ and equal to zero anywhere else and the integral of Δ^i however is

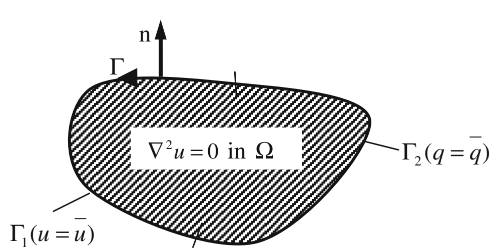

Fig. 1. Geometrical notation for Laplace's equation with two different types of boundary conditions

(i) $u = \bar{u}$ on Γ_1 (Dirichlet boundary condition).
(ii) $q = \frac{\partial u}{\partial n} = \bar{q}$ on Γ_2 (Neumann boundary condition).

where boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and n is the normal vector to the boundary with potential function u is specified (Dirichlet) or its derivative q is specified (Neumann), both indicated with bar sign.

equal to one. Thus, the fundamental solution can be taken as $u^* = -\ln r/2\pi$ for two-dimensional cases where r is the distance from source point x to field point y. This function is also sometimes known as Green's function which can be expressed as $G(x,y) = -\ln r/2\pi$. Therefore, Eq. (4) can be reformulate and rewritten as

$$c(x)u(x) = \int_\Gamma q(y) u^*(x,y) d\Gamma - \int_\Gamma u(y) q^*(x,y) d\Gamma \tag{6}$$

where

$$c(x) = \begin{cases} 0 & \text{if } x \text{ is in exterior of domain } \Omega \\ 1/2 & \text{if } x \text{ is on smooth boundary } \Gamma \\ 1 & \text{if } x \text{ is in domain } \Omega \end{cases}$$

Eq. (6) is known as the boundary integral equation for Laplace's equation where the coefficient $c(x)$ depends on the position of point x. In mathematical point of view based on the description written by Arnold and Wendland [15], the Eq. (4) can be categorized into three typical classes of mathematical problems. For coefficient $c = 0$, it best describes the exterior problems (e.g. incompressible flow, electromagnetic fields, plane elasticity, steady-state thermoelasticity and exterior acoustics problem) in which the singular integral equations with a Cauchy's kernel appeared. For $c = 1/2$, the BIE is applicable to surface problems including normal derivative of double layer potentials in acoustics, ideal flows and plane elasticity. Meanwhile, its application on interior problem consisting of an integro-differential operators of second order, is obtained once using coefficient $c = 1$. Other than that, special cases involving first kind integral equation with logarithmic kernel (Symm's integral equation) can be established while assuming $c = -1/2$ and is frequently applied in areas like conformal mapping, torsion problems, plane elasticity, Stokes flows and electrostatics.

On the other hand, the indirect approach (a.k.a. classical or source method) is written in terms of source density functions defined by the boundary. The source density however has no specific physical significance. In the indirect approach, reformulation of integral equation is obtained by writing layer potential u by including unknown density σ,

$$u = \int_\Gamma \sigma u^* d\Gamma \tag{7}$$

The above integral equation is the solution for the Neumann problem whereas the solution of the Dirichlet problem can be obtained when the unknown function u is formulated by adding double-layer potential with unknown density μ,

$$u = \int_\Gamma \mu \frac{\partial u^*}{\partial n} d\Gamma \tag{8}$$

This indirect approach has been well known and leads to the formulation of lifting surface methods pioneered by Hess and Smith [91]. Moreover, the work of Djojodihardjo and Widnall [60] is one of the example of early manifestations of indirect BEM approach. Although the direct and the indirect methods have different techniques in solving PDEs, both can be derived from their integral expression. In fact, Brebbia and Butterfield [26] proved that the direct and indirect BEMs are equivalent. For interested reader, we refer to the standard textbooks [27,30] for further elaboration on both approaches.

3. Development of BEM variants

Using similar boundary integral equation mentioned, numerous BEM variants were created thanks to enormous efforts by researchers for the past three decades. In this review paper, the overall development of five well-known BEM variants composing collocation boundary element method (CBEM), galerkin boundary element method (GBEM), dual Reciprocity boundary element method (DRBEM), complex Variable boundary element method (CVBEM) and analog equation method (AEM) were discussed. Among a choice of nearly 40 BEM variants, they were selected based on their significant contributions in boundary element research and at the same time, possesses solid foundations with establishment duration over 15 years. Practically, these BEM variants can be divided into two major groups—boundary integrals and domain integrals [217]. The boundary-type integrals can be discretized directly using boundary elements and they often contains kernels which are singular or hypersingular (singularity of an order greater than one). As for domain integrals, they will need to be transformed into a boundary one via analytical or numerical approaches before discretizing them into boundary elements. In the analytical approach, the conversion of domain-to-boundary is mainly based on Green's identities. Meanwhile, the technique for transferring the domain integrals into equivalent surface integrals using a set of radial basic approximation functions presents a numerical treatment of domain integrals. In dealing with boundary or domain integral equations, this article tries to outline the evolution of the theoretical and mathematical basis of each BEM variants along with the way they were being implemented in a systematic way. As the boundary is discretized, the BIE is then numerically approximated, leading to the concern of its stability and convergence properties. The accuracy of each BEM variants which mainly based on their choice of numerical technique is widely discussed in the subsequent subtopics. Fig. 2 outlines the structure of several BEM variants focusing in giving a clearer picture of BEM evolution.

3.1. Collocation boundary element method

The collocation boundary element method (CBEM) is one of the earliest variant in BEM and often known as traditional collocation boundary element method (TCBEM) or classical BEM. The implementation of collocation method in BEM can be traced as far as in the sixties, where this method was pioneered by Jaswon [113] and Symm [243] when they discretized the boundary of a domain by a set of points with straight lines in potential theory problems and then by Rizzo [219] as corresponding numerical solution in elastostatics. At the beginning of BEM development, this basic concept of implementing collocation method in BEM has been promising due to its generality and simplicity. For this particular variant, the disretization of boundary integral equations is based on the nodal collocation method in which the approximated boundary integral equation is evaluated on the interpolation nodes. A number of discrete points (nodes of elements) on the problem boundary were chosen and it was assumed that the integral equation is satisfied at these points. In order to do so, the governing equation for direct BEM obtained from Eq. (6) can be rearranged as

$$cu+\int_{\Gamma} uq^* d\Gamma = \int_{\Gamma} qu^* d\Gamma \tag{9}$$

Then, the boundary is discretized into a series of N elements. Taking the elements mid points as collocation nodes, Eq. (9) becomes

$$cu+\sum_{j=1}^{N}\int_{\Gamma_j} uq^* d\Gamma = \sum_{j=1}^{N}\int_{\Gamma_j} u^* q d\Gamma \tag{10}$$

for $j=1,\ldots,N$. Assuming the u and q values are constant within each element, we can rewrite it as

$$cu+\left(\sum_{j=1}^{N}\int_{\Gamma_j} q^* d\Gamma\right)u_j = \left(\sum_{j=1}^{N}\int_{\Gamma_j} u^* d\Gamma\right)q_j \tag{11}$$

which can be summarized in the form:

$$[A]\{u\}=[B]\{q\} \tag{12}$$

The numerical method mentioned above is the classical collocation procedure which also known as the nodal collocation that leads to the linear system connecting functions u and q on boundary collocation nodes. The complete and detail mathema-

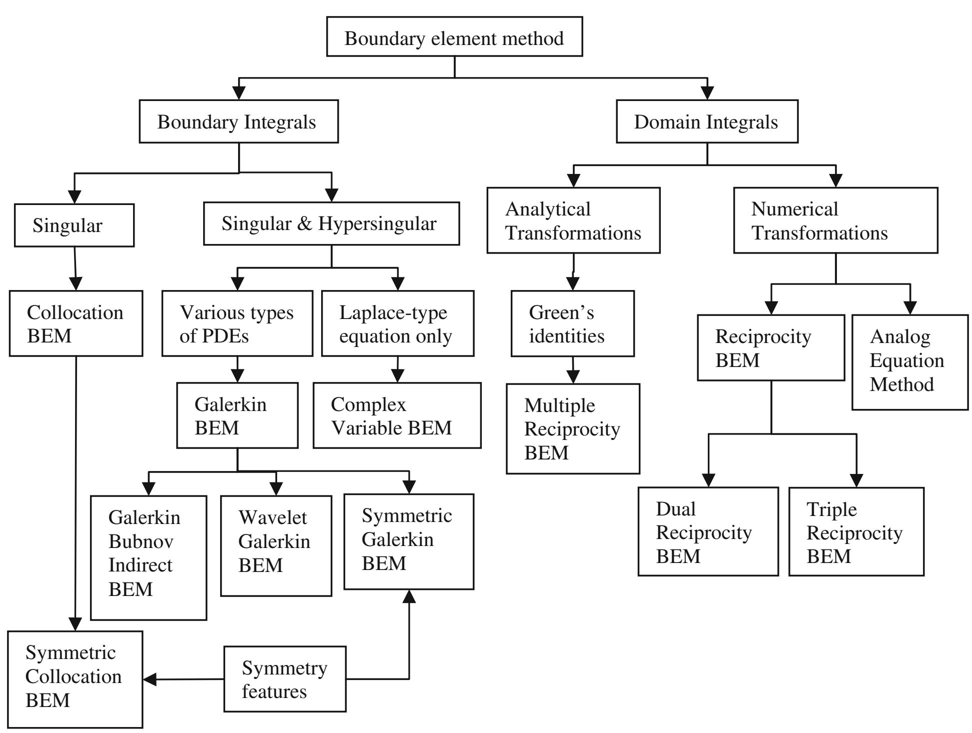

Fig. 2. Classification of BEM variants.

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

tical description on the nodal collocation in direct BEM can be found elsewhere, e.g. in [27,30,174]. Due to its simplicity, many researchers found it easy to implement and as a result, most of the boundary value problems at the beginning of BEM development are based on collocation method. Therefore, the collocation BEM has been adopted in diverse engineering-related discipline, such as elastostatics [49,142,219], transient elastodynamics [48,51,155], thermoelasticity [220], plate bending [90], thin plate analysis [187], fracture mechanics [156,257], Stokes flow [68,140,214], plane wave radiation and scattering problem [198] and acoustics [10,45,199]. In addition, the collocation-based direct BEM was extended for nonlinear boundary value problem [225] (i.e. elastoplastic problem [19,44,248]). Practically, the numerical solution for BIE usually employs either constant, linear, quadratic or higher element discretization of the surface as boundary elements, depending on their expected accuracy. One of the most preferable techniques in this variant is the piecewise polynomial collocation method which appears to be more adaptable to specific problems and achieves better outcome in the standard cases especially in the linear case [227]. In spite of all those mentioned above, which may appear to be an ideal numerical solution in solving the BIE, the nodal collocation was not employed in indirect BEM in the past. This is due to the difficulties associated with the evaluation of the hypersingular integrals of the indirect BEM. Furthermore, some of the researchers raise concern on the accuracy of the collocation approach. In fact, significant number of published articles mainly discuss about the convergence analysis of collocation approach; and among them, extensive analysis was performed by Arnold et al. [14,16,17], Ruotsalainen [224] and Juhl [115], and covered various aspects and provides strong evidences on this issue in their report. In conclusion, they pointed out that the collocation BEM often produces results with poor convergence and it is unlikely to employ this technique on bodies with edges as it display a strong singularity at corner points. Due to this, the collocation method appears to be less efficient. In response to that, Juhl [115] suggested two different approaches with the aim to boost the rate of convergence for this particular case (bodies with edges). One applies specialized shape functions replacing the mid-element node and another adopted the so-called generalized quarter point technique by shifting the mid-element node approximately one-fourth of an element length. Moreover, the lack of symmetry problem later noticed in the CBEM often cost the lost of computer efficiency, especially in the boundary element method/finite element method (BEM/FEM). Therefore, a new variant in the CBEM family known as the symmetric collocation BEM (SCBEM) was presented by Yu [274] in solving the elastodynamic problems. This approach retains the advantages of the collocation approach while employing symmetry coefficient matrix aiming to save memory storage. In addition, a modified CBEM was recently presented by Rüberg and Schanz [223] for static and dynamic problems which is applicable to vector problems (i.e. elasticity) and dynamic problems (i.e. acoustics and elastodynamics). Compared to standard collocation procedure, the modified CBEM is based on a mixed approximation formulation on primary variable (e.g. potentials or displacements) and on dual variable (e.g. surface fluxes or tractions).

3.2. Galerkin boundary element method

Another approach implemented in BEM discretization procedure using one of the weighted residual techniques called Galerkin method, formed a new variant—Galerkin boundary element method (GBEM). Apparently, the Galerkin approach for boundary integral equation was originated from the work of Bui [34] in fracture analysis where the problem of a plane crack with arbitrary shape is investigated. However, this approach, at that time, still unnoticed by the scientific community as researchers were at the stage where they started to employ the collocation approach for boundary value problems. The work of Wendland [258] on asymptotic convergence of Galerkin approach, however, changes the fortune of this particular approach. Compared to collocation approach, Galerkin approach shows its potential in solving boundary value problems in terms of singular and hypersingular integral equations and is superior than collocation approach for obtaining more accurate results. Comparison between Collocation and Galerkin approach done by several scholars in various aspects such as computing time [15], stiffness matrices [55], time-domain BEM formulation applied to the wave propagation problem [232] and its stability [275] further explain the characteristic and performance on both methods. Unlike the collocation procedure, the Galerkin approach does not involve specific point satisfying the integral equation but requires double integrals on $\Gamma\times\Gamma$. In order to formulate the standard GBEM, the BIE for Laplace's equation from Eq. (6) can written as

$$P(x)=c(x)u(x)+\int_{\Gamma}u(y)q^{*}(x,y)d\Gamma-\int_{\Gamma}q(y)u^{*}(x,y)d\Gamma \tag{13}$$

and the corresponding equation for the normal derivative of u with respect to point x could be obtained as

$$F(x)=c(x)\frac{\partial u}{\partial N}(x)+\int_{\Gamma}u(y)\frac{\partial q^{*}}{\partial N}(x,y)d\Gamma-\int_{\Gamma}q(y)\frac{\partial u^{*}}{\partial N}(x,y)d\Gamma \tag{14}$$

in which both $P(x)$ and $F(x)$ are the residual of the Eq. (13) and (14). According to the Galerkin scheme, the integration of weighted residual statement over the surface is equal to zero where the statement is formed by multiplying the residuals with the chosen weight functions. Thus, it can be formulated in the form

$$\int_{\Gamma}\psi_k(x)P(x)dx=0 \tag{15}$$

$$\int_{\Gamma}\psi_k(x)F(x)dx=0 \tag{16}$$

where Eq. (15) can be employed for Dirichlet-type surface and Eq. (16) for Neumann-type surface with $\psi_k(x)$ as the preferred weight functions. These weighting functions can be expressed by the shape functions obtained by interpolating the boundary functions u and its normal derivative. Due to this, the Galerkin approach is generally more accurate than Collocation approach as the error of satisfying the Eq. (6) are minimized. Further description on the Galerkin procedure in direct BEM can be obtained in [188] for plane elastostatic problem. Emerged as an alternative approach to the well established CBEM, the Galerkin approach in BIE then extended for other standard cases such as in elastoplastic problem [151], electrostatics problem [21], Stokes flow [52], acoustic scattering problem [38,83,114], electromagnetic scattering problem [33], acousto-aeroelastic problem [58,59], etc. For indirect approach, the implementation of the Galerkin formulation can be seen in articles by Lean et al. [146] for electrical engineering problems and nonlinear heat conduction problem by Routsalainen and Saranen [226]. Although regarded as more complicated and significantly slower techniques compared to collocation, the Galerkin method which concentrated on the evaluation of double integrals has its advantage by offering treatment to the hypersingular integrals with standard continuous elements—constant, linear, quadratic or higher order elements. Hence, the capability in solving the hypersingular equations is proven to be advantageous for the Galerkin approach in dealing with crack analysis where singular equation alone provides insufficient information [47,239]. Furthermore, the description written intended for the numerical integration on both singular and hypersingular BIE using the Galerkin approach for 2D problems is available in [3,4] and 3D problems in [2]. Similar work

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

regarding evaluation of singular and hypersingular based on Galerkin technique is also discussed by Gray and his coworkers [75,77,79,81]. Opposed to the Collocation approach, the Galerkin approach holds advantage where it is easier for Galerkin approach to formulate a natural symmetry BIE as the x and y are treated evenly. This is noticed since the early birth of GBEM which is highlighted by Sirtori [235] in the linear elastic analysis of the homogenous continuum and Hartmann et al [89] for elasticity in beams. The described symmetry matrices in elasticity then adopted and further developed by Maier and his coworkers in elastoplastic analysis [151], elasticity [236], plasticity [150] and gradient plasticity [149]. The extended Galerkin formulation is called Symmetric Galerkin Boundary Element Method (SGBEM) associated with its symmetric matrices features. In this BEM variant, the fundamental solution also known as Green's function, and its derivatives satisfy the following symmetry properties:

$$u^*(x,y)=u^*(y,x), \quad \partial u^*(x,y)/\partial n=-\partial u^*(y,x)/\partial n,$$
$$\partial u^*(x,y)/\partial N=-\partial u^*(y,x)/\partial N, \quad \partial u^*(x,y)/\partial N=-\partial u^*(y,x)/\partial n,$$
$$\partial^2 u^*(x,y)/\partial N\partial n=\partial^2 u^*(y,x)/\partial N\partial n \tag{17}$$

With these additional symmetry features, it will significantly lessen the computational work and thus, SGBEM has been chosen and implemented in a wide range of scientific applications: fracture analysis [72,76,80,201,240], steady and incompressible flow [35], plate bending analysis [186,197], analysis of Kirchoff elastic plates[71], heat conduction [241], viscoelastic problems [195], elastodynamics [276] and plasticity [148]. Moreover, interested readers may refer to extensive literature review on SGBEM documented by Bonnet et al. [24] which gives a greater understanding on this particular method with broad citation. In their review paper, the discussions on SGBEM are mainly focused on areas divided into several subtopics including linear elastic problems, elastic-plastic quasi-static analysis, fracture mechanics, time-dependent problems, variational approaches, singular integrals, approximation issues, sensitivity analysis, coupling of boundary and finite elements and computer implementations. Subsequently, a textbook totally devoted to SGBEM was recently written by Sutradhar et al. [242], and provides an updated information regarding SGBEM including the current development with fairly detailed survey of visualization techniques with corresponding software. In addition, a symmetric Galerkin MATLAB educational program called BEAN is also presented in the book and available for download at http://www.ghpaulino.com/SGBEM_book. In an attempt to expand the SGBEM applicability, the SGBEM was then extended by Pérez-Gavilán and Aliabadi [196] in which the modified method can be demonstrated for multi-connected bodies. However, in their following reports [193,194], they also pointed out that the applications of symmetric Galerkin boundary element method to multiple connected regions, where at least one closed boundary is subjected to Neumann boundary condition only, will result in non-uniqueness. To overcome the non-uniqueness problem, they proposed a new approach which consists of partitioning the domain in two simply connected subregions to deal with the non-uniqueness in symmetric Galerkin scheme. In another case, Gray and Griffith [78] proposed a modified Galerkin approach in order to improve the computational efficiency of Galerkin approach which can be applied on singular and hypersingular equations. To demonstrate the improved version of the Galerkin approach, the Dirichlet problem will be considered here. From the Eq. (15), the coefficient matrix can be taken from the integrals

$$\int_\Gamma \psi_k(x)\int_\Gamma q(y)u^*(x,y)dy\,dx \tag{18}$$

$$\int_\Gamma \psi_k(x)\int_\Gamma u(y)q^*(x,y)dy\,dx \tag{19}$$

In the proposed approach, the modification involves implementing the Galerkin on outer x integration and inner$y$integration,

$$\int_\Gamma \psi_m(x)\int_\Gamma \psi_l(y)u^*(x,y)dy\,dx \tag{20}$$

$$\int_\Gamma \psi_m(x)\int_\Gamma \psi_l(y)q^*(x,y)dy\,dx \tag{21}$$

in which the boundary functions $u(y)$ and $q(y)$ have been replaced by their approximations,

$$u(y)=\sum_l u(y_l)\psi_l(y) \tag{22}$$

$$q(y)=\sum_l q(y_l)\psi_l(y) \tag{23}$$

This simple modification was reported to decrease computation time by 20–30 percent for computing the non-singular integrals and has significantly improved the computational performance, especially for moderate to large scale problems. Another effort to improve the computational efficiency of the GBEM proposed by Wang et al. [254] demonstrated in solving two-dimensional elastostatics problems involving numerous inhomogeneities. Block computation techniques were employed by computing the combined influences of groups of elements using asymptotic expansions, multiple shifts and Taylor series expansions.

In addition to all mentioned above, there are two other subclasses of Galerkin approach known as the Galerkin–Bubnov Indirect Boundary Element Method and Wavelet Galerkin Boundary element method (WGBEM). Often, the densely populated system matrices resulting from traditional discretizations of integral equations cost a major concern. As a solution, a new alternative technique employing wavelets as the basis functions and weighting functions in the Galerkin BIE discretization was proposed. The early initiative implementing wavelet Galerkin scheme in boundary integral equation can be traced back to the literature written by Lage and Schwab [145] in 1999 on wavelet-based Galerkin discretization of second kind integral equations for piecewise smooth surfaces. At the same time, a fully discrete wavelet Galerkin scheme for the 2D Laplacian was presented by Harbrecht and Scheider [88] and later extended to three dimensional [85]. From their work, a new approach which known as wavelet Galerkin boundary element method (WGBEM) was created and then utilized by researchers in engineering fields with successful implementation in electromagnetic shaping [66,67], acoustics scattering wave [86], elasticity [64] and electrostatic analysis [268]. To understand more on WGBEM, the Fredholm's integral equation of the first kind from Eq. (6) for Dirichlet problem is considered,

$$Au(x)=\int_\Gamma q(y)u^*(x,y)d\Gamma=f(x) \tag{24}$$

which appears as a single-layer equation and also can be represent as

$$Au(x)=c(x)u(x)+\int_\Gamma u(y)q^*(x,y)d\Gamma \tag{25}$$

Introducing the weighting function in the Galerkin discretization by referring to Eq. (15), the matrix system entries A_ψ can be formulated as

$$A_\psi=\int_\Gamma \psi_k\int_\Gamma q(y)u^*(x,y)d\Gamma \tag{26}$$

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

Thus, the integral equations with the single-scale basis can be expressed in the linear system form as

$$A_\psi u_\psi = f_\psi \tag{27}$$

However, in the WGBEM, wavelet bases are employed instead of using the single-scale basis which is commonly applied in the traditional Galerkin approach. There are dozens wavelets that can be chosen and primarily categorized as either orthogonal [63,64] or biorthogonal [87,88]. The selection of wavelets depends on their characteristics, mainly on orthogonality, vanishing moments and smoothness. Consequently, the implementation of wavelet bases in Galerkin discretization leads to sparse matrices. These nonrelevent sparse matrix entries which can be negligible and treated as zero will be discarded in the latter part by employing the wavelet matrix compression introduced by Beylkin et al. [23]. The matrix compression which can be done using A-priori compression or A-posterior compression is one of the important factors in order to achieve an efficient computational method [53,269]. Investigation on multiwavelet basis function was also presented by in the work of several scholars [5,145,270]. In general, WGBEM offers a new class of fast solution for the integral equations other than multipole and panel clustering.

Another variant known as Galerkin–Bubnov Indirect boundary element method (GB-IBEM) was first implemented in solving the antenna radiation over lossy half-space developed by Poljak [203]. However, the implementation of the Galerkin–Bubnov scheme in boundary integral equation can be seen in the work of Rolics [221,222]. For the Galerkin–Bubnov scheme, the weighting function is obtained in the same form with the weight function for Galerkin approach (but with arbitrary coefficients). In several literatures [36,205,206,208,209,211,212], GB-IBEM is preferred for treating Pocklington's integro-differential equation arising from electric field integral equation which can be written in the operator form as

$$KI = E \tag{28}$$

Given E as the excitation with K represents linear operator, unknown function I can be computed. However, the unknown function can also be expressed by the sum of a finite number (n) of linearly independent basis functions f_i with unknown complex coefficients I_i:

$$I_n = \sum_{i=1}^{n} I_i f_i \tag{29}$$

Therefore, employing the weighted residual approach with the Galerkin–Bubnov procedure, the operator Eq. (28) transforms into a system of algebraic equation

$$\sum_{i=1}^{n} I_i \int_\Omega K(f_i) f_m dz = \int_\Omega E f_m dz, \quad m = 1,2,\ldots,n \tag{30}$$

This approach which contains unknown sources can be pointed out as an indirect boundary element method. Full details for this approach can be obtained from textbooks [204,207]. Frequently used in dealing with the electric field integral equation, the Galerkin–Bubnov scheme demonstrated the electromagnetics analysis for antenna with partially penetrating the ground [36,205], above ground [211] and fully buried in the ground [206,208,209,212]. It was also extended for investigating the effect of electromagnetic field on human body [210].

3.3. Dual reciprocity BEM

In 1982, Brebbia and Nardini presented a new alternative approach in solving the elastodynamic problem at the international conference on soil dynamics and earthquake engineering and the same article [29] was published in 1983. The proposed approach also addressed in [158] where the statical fundamental solution is employed in solving the dynamic problems which demonstrated conversion of free vibration problem to frequency independent algebraic eigenvalue problem on homogeneous elastic bodies. Thus, making the boundary integrals only computed once and therefore producing an efficient and accurate alternative method. This new BEM variant was later named dual reciprocity boundary element method (DRBEM). Unlike any other variants, this method has the ability to handle the domain integrals even nonhomogeneous is involved. To explain the DRBEM, basic formulation of Poisson-type equation in the form

$$\nabla^2 u = b \tag{31}$$

subjected to similar boundary condition mentioned earlier for Laplace's equation. Function b in the integral equation can be represented by

(i) $b = -(\partial u/\partial x)$ in the convective problem ($\nabla^2 u = -(\partial u/\partial x)$)
(ii) $b = (1/k)(\partial u/\partial t)$ in the diffusion problem ($\nabla^2 u = (1/k)(\partial u/\partial t)$)
(iii) $b = -\mu^2 u$ in the Helmholtz equation ($\nabla^2 u + \mu^2 u = 0$)
(iv) $b = (\partial^2 u/\partial t^2) + \beta(\partial u/\partial t) + \varphi \sin u$ in the sine-Gordon equation ($(\partial^2 u/\partial t^2) + \beta(\partial u/\partial t) = \nabla^2 u - \varphi \sin u$)

Applying the standard boundary integral approach, an integral equation analogous to Eq. (6) can be obtained.

$$c_i u_i + \int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma = \int_\Omega b u^* d\Omega \tag{32}$$

Note that the right hand side of the equation contains domain integral. Here the domain integral can be transformed to boundary integral using the dual reciprocity method featuring the main component of DRBEM. In order to attain that, function b can be approximated as follows:

$$b \approx \sum_{j=1}^{N} \alpha_j f_j \tag{33}$$

where N is the number of collocation points, α_j are a series of unknown coefficients which can be determined by collocation method, while f_j are a set of approximating functions usually applied as the radial basis approximation functions

$$f_j = \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \ldots \tag{34}$$

with the particular solutions written as

$$\nabla^2 \hat{u}_j = f_j \tag{35}$$

Multiplication of Eq. (31) with fundamental solution of Laplace's equation u_* and integrating over the domain Ω produces the equation

$$\int_\Omega u^* \nabla^2 u d\Omega = \int_\Omega u^* b d\Omega \tag{36}$$

Substituting Eq. (33) and (35) into Eq. (36) yields,

$$\int_\Omega (\nabla^2 u) u^* d\Omega = \sum_{j=1}^{N+L} \alpha_j \int_\Omega (\nabla^2 \hat{u}_j) u^* d\Omega \tag{37}$$

Applying integration by parts twice for the domain integral term, Eq. (32) can be rewritten as

$$c_i u_i + \int_\Gamma u q^* d\Gamma - \int_\Gamma u^* q d\Gamma = \sum_{j=1}^{N} \alpha_j \left(c_i \hat{u}_{ij} + \int_\Gamma \hat{u}_j q_* d\Gamma - \int_\Gamma u^* \hat{q}_j d\Gamma \right) \tag{38}$$

As expected, domain integral in Eq. (32) is eliminated and replaced with boundary integral shown in Eq. (38). The subsequent discretization and its solution will follow standard BEM procedure. Full description of this particular method can be obtained from the work of Patridge et al. [191]. Since proposed in 1982, the DRBEM experienced continuous progress which tends to improve the current formulation. For example, although the linear radial basis function $f_j=1+r$ is commonly adopted in the beginning of DRBEM development, Zhang and Zhu [277] investigated and compared the accuracy by using different selection of terms from Eq. (34) where r term is replaced by r^3 which has proven to be more superior with minimum error. In fact, there are other radial basis functions applicable in DRBEM and they are

(i) Duchon radial cubics, $f_j=r^3$
(ii) Radial quadratic plus cubic, $f_j=1+r^2+r^3$
(iii) Thin plate splines, $f_j=r^2 \ln r$
(iv) Multiquadric, $f_j=(r^2+c^2)^{1/2}$
(v) Gaussian, $f_j=\exp(-r^2/c^2)$

where $c$is a specified constant. Apart from radial basis functions, many kind of approximation functions have been proposed including global basis functions (i.e., polynomial series [40,189], trigonometric series [65,189,247] and hyperbolic series) which can be used to replace locally based radial basis function. These global basis functions in generalized form with $m, n=0,1,2,\ldots$ were listed in Table 2. In addition, Partridge and Sensale [192] suggested a hybrid approximation functions by augmenting radial basis functions with appropriate global expansions. With so many kind of approximation functions available, a study on the criteria of selection for a given problem was summarized by Partridge [190]. He concluded that the main considerations in selecting the f function rely upon the characteristic of the domain integral where different cases (i.e. known functions, unknown functions without derivaties (steady case), unknown functions with derivatives (steady case), unknown functions with derivaties (transient case), nonlinear functions and internal nodes) would adopt different approach in order to get the best performance. Furthermore, the convergence properties of the radial basis functions and global basis functions demonstrated in [40,271] shows superior convergence properties when employing the global basis functions.

In other aspect, the implementation of DRBEM was extended to the case of potential problems with arbitrarily distributed sources [167] and also in the nonlinear problem, for example in the diffusion problem [264] involving nonlinear features such as nonlinear material, nonlinear boundary conditions, nonlinear sources inside the domain and moving interface problems. Apart from Poisson-type equation, DRBEM is also employed for integral equation involving biharmonic operator ∇^4, especially structural analysis of plates like bending problem [118], buckling problem [65], large deflection problem [255] and vibration problem [54]. Moreover, the application of DRBEM to date has successfully been implemented in areas including heat conduction problems [12,22,234,244–246,265–267], heat convection problem [41], convective-diffusive heat problem [228,229], fluid flow problems governed by the Navier–Stokes equation [42,70] and crack problem [7,252]. As DRBEM proved to have practical applicability,

Table 2
. Example of the global basis functions in generalized form.

Global basis function	Generalized Equation
Polynomial series	$f_j=x^m y^n$
Fourier sine series	$f_j=\sin mx \sin ny$
Fourier cosine series	$f_j=\cos mx \cos ny$
Hyperbolic sine series	$f_j=\sin\text{h}\ mx \sin\text{h}\ ny$
Hyperbolic cosine series	$f_j=\cos\text{h}\ mx \cos\text{h}\ ny$

it was later extended to three dimensional problems [1,73,250]. In addition, the DRBEM based on multi-domain approach was proposed to improve the performance of DRBEM and applied to nonlinear convective-diffusive equations [213], velocity-traction formulation of the Navier–Stokes equations [70] and non-Newtonian Stokes equations [69]. The review of multi-domain DRBEM in terms of performance and implementation in 3D problems can be obtained from [159].

Few years after the dual reciprocity method was established, Nowak proposed an addition to the dual reciprocity BEM family by presenting multiple reciprocity boundary element method (MRBEM) through introduction of a sequence of higher order fundamental solutions derived from the Laplace operator. Also known as multiple reciprocity method (MRM), this technique was first applied for transient heat transfer [169] and then employed to the Poisson [172] and Helmholtz equation [171]. As an extension of DRBEM, MRBEM preserved the reciprocity theorem although the technique implementing it is different. In MRBEM, sequences of higher order fundamental solutions and source function laplacians employed can be formulated as

$$\nabla^2 u^*_{k+1} = u^*_k \tag{39}$$

$$\nabla^2 b_k = b_{k+1} \tag{40}$$

where $k=0,1,2,\ldots$ Now, consider the same Poisson's equation from Eq. (28) with an extra subscript "k" for the function b,

$$\nabla^2 u = b_k \tag{41}$$

Start with $k=0$,

$$\nabla^2 u = b_0 \tag{42}$$

and the integral equation obtained from Eq. (32) can be written

$$c_i u_i + \int_\Gamma u q^*_0 d\Gamma - \int_\Gamma q u^*_0 d\Gamma = \int_\Omega b_0 u^*_0 d\Omega \tag{43}$$

while bearing in mind that $q^*_0 = \partial u^*_0/\partial n$ and the fundamental solution also with similar subscript "0". Using the relationship in Eq. (39), domain integral in Eq. (43) can be transformed to boundary integral by applying the reciprocity theorem, which gives

$$\int_\Omega b_0 u^*_0 d\Omega = \int_\Omega b_0(\nabla^2 u^*_1) d\Omega = \int_\Gamma \left\{ b_0 \frac{\partial u^*_1}{\partial n} - u^*_1 \frac{\partial b_0}{\partial n} \right\} d\Gamma + \int_\Omega u^*_1 \nabla^2 b_0 d\Omega \tag{44}$$

Utilizing the relationship in Eq. (40), one can obtain new function b_1 such that

$$b_1 = \nabla^2 b_0 \tag{45}$$

Consequently the domain integral from the right hand side of Eq. (44) can be computed by

$$\int_\Omega b_1 u^*_1 d\Omega = \int_\Gamma \left\{ b_1 \frac{\partial u^*_2}{\partial n} - u^*_2 \frac{\partial b_1}{\partial n} \right\} d\Gamma + \int_\Omega u^*_2 \nabla^2 b_1 d\Omega \tag{46}$$

These repeated procedures can be carried out until reaching desirable number of cycles. Similar with DRBEM, MRBEM also successfully employed for several integral equation such as Helmholtz equation [37,116,117,272], diffusion equation [111,112,185] and biharmonic equation [164,237,238]. With both methods capable of transforming domain integrals, both often draw comparison between them [74,173]. One significant difference shown is the inclusion of the internal points is rather necessary in the dual reciprocity approach but not needed in MRBEM. Moreover, MRBEM produced better degree of accuracy compared to DRBEM but required more computational effort. In spite of DRBEM is more widely implemented in various problems, MRBEM rather focuses its potential in selective areas such as in vibration problem [238], heat conduction problem [168,170, 181,216], thermoelasticity analysis [165,166,233] and elastoplastic analysis [180]. In [180], Ochiai and Kobayashi stipulated that the conventional MRBEM is not convenient to operate in the elastoplastic problems arising from inability in determining the distribution of initial stress and initial strain analytically. Thus, they proposed an improved MRBEM called triple reciprocity boundary element method (TRBEM) in handling the elastoplasticity problems [175]. According to this approach, the domain integral in each step is divided into point and area integrals. Two step functional approximations can be formulated as

$$\nabla^2 W_1 = -W_2 \tag{47}$$

$$\nabla^2 W_2 = -W_3 \tag{48}$$

The term W_1 is known function whereas functions W_2 and W_3 are unknown. From Eqs. (47) and (48), one can obtain

$$\nabla^4 W_1 = W_3 \tag{49}$$

In the elastoplastic analysis [175], W_1 represents deformation of a fictitious thin plate while W_3 as the unknown point load. Based on TRBEM, the unknown point load can be computed inversely by utilizing the displacement given. Aside from elastoplastic analysis, the TRBEM is exercised in heat conduction problem [177–179,183], thermoelastic [182], thermal stress [176] and convection-diffusion problem [184]. Based on their work, TRBEM can produce remarkable degree of accuracy using only the fundamental solutions of lower order [176,179].

3.4. Complex variable boundary element method

Complex variable boundary element method (CVBEM) is another well-known BEM variant. The CVBEM is a numerical technique based on Cauchy's integral formulae with complex variable to formulate boundary integral equation and mainly dedicated only to Laplace's and Poisson's equations. This method was originated from the work of Hromadka and Guymon [99] in 1984, initially for the solution of two-dimensional potential problem involving Laplace's partial differential equation and was further elaborated in [105]. Basically, the CVBEM adopted similar idea developed in Analytic Function Method (AFM). This method which was presented by van der Veer [251] is based on summation of simple products of complex linear polynomials and complex logarithm functions to form an analytic approximation function and connection between both of them (CVBEM and AFM) was described in [96]. Unlike real variable BEM procedures, the functions used to approximate the exact solution in CVBEM are analytic and thus, exactly satisfy the integral equation throughout the domain of the problem. In addition, the integration of the boundary integrals along each boundary elements can be solved analytically without any numerical process. In general, the theoretical basis of CVBEM can be described by commencing the complex variable analytic function

$$\omega(z) = \phi(x,y) + i\psi(x,y) \tag{50}$$

where $z=x+iy$ and $i=\sqrt{-1}$. Both potential function $\phi(x,y)$ and stream function $\psi(x,y)$ satisfy the Laplace equation

$$\nabla^2\phi = 0 \text{ and } \nabla^2\psi = 0 \tag{51}$$

and they are connected by the Cauchy-Reimann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{52}$$

With the analytic function $\omega(z)$ defined, the Cauchy integral formula can be written as

$$\omega(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(z)}{z-z_0}dz \tag{53}$$

where z_0 is the interior point. As the solution cannot be performed analytically, approximate numerical solution is chosen instead. Similar to the collocation approach, the boundary then divided into straight line segments by boundary nodal points. However, nodal point equation needs to be employed as both functions ϕ and ψ are unknown. For a standard case, the nodal point equation is normally obtained through linear trial function given by

$$\omega(z) = \omega_i\left(\frac{z_{i+1}-z}{z_{i+1}-z_i}\right) + \omega_{i+1}\left(\frac{z-z_i}{z_{i+1}-z_i}\right) \tag{54}$$

Besides the linear trial functions, other approximation techniques applicable in CVBEM such as circular contour integration and high order polynomial approximations were mentioned in [99]. Soon afterward, it was extended to higher order trial functions and the implementation for parabolic, cubic and Hermite cubic polynomial functions is demonstrated in the article written by Hromadka II and Yen [105]. In their report, they concluded that those four investigated trial function (including linear trial function) produced same level of accuracy. However, the CVBEM is not error-free and more research efforts were carried out in order to improve the CVBEM performance. From there, the error analysis was further investigated and the studies indicated that the error analysis measured in the CVBEM is due to modelling error [95] and approximation error [97] and therefore leads to its convergence study on Dirichlet and mixed boundary value problems in [94]. Error reduction was later examined by refining approximate boundary method [261] in which the modeling error can be easily reduced by adding nodal points along the boundary accordingly. In other aspect, the applicability of CVBEM has been extended to accommodate doubly connected analysis [120] and multiply connected analysis [107] for potential problems. This additional feature enables solution of problems involving multiple domains especially structure containing multiple holes, cracks and inclusions using CVBEM. To date, the CVBEM which can be applied to mixed boundary value problems (Dirichlet, Neumann or Robin) has been successfully implemented in potential problems including ideal fluid flow (for instance: air flows [82,157,230], groundwater flows [218], hydrodynamics [200], etc.), heat transfer [119], torsion analysis [43,62], crack problems [11,57,202], etc. Although regarded as specialized method devoted for the Laplace and Poisson equations, an attempt to use CVBEM for biharmonic problem ($\nabla^4 u=0$) for elasticity of plane and plate was reported by Lei and Mao-Kuang [147]. In their report, computational results of CVBEM on several different structures (i.e., circular disk, cantilever beam, simply supported square plate) under external concentrated forces or uniform loads show good agreement with the exact solution. The CVBEM was later imposed into other elasticity problem including thermo-elasticity [13,141], poroelasticity [108] and viscoelasticity [109]. In spite of all those improvements mentioned above, before 21th century, the CVBEM only concentrated in the 2D problem and the lack of multi-dimensional application feature was realized and recently, the problem is solved and the upgraded CVBEM can be applied for three-dimensional analysis [103]. The new additional feature in CVBEM is then employed by Hromadka and Whitley [102] for three dimensional steady-state potential flow problems using two-dimensional complex polynomials. In other related work, Huber and Hickman [110] developed a scheme called surface 'dust' to randomly generate dust points on 3D object's surface. In their report, they attached the Mathematica code for their scheme which demonstrated the new approach on typical sphere and open cylinder. In spite of recent progress in three dimensional, the CVBEM transformation in multi-dimension is still in the early stage of development and required more intensive study. In separate case, the program Mathematica was again utilized to investigate the implementation of CVBEM using collocation points not located at the usual boundary nodal point locations [56]. However, further research is needed to improve the current approach. In overall, this advanced mathematical modeling approach had seen considerable theoretical development for the past two decades and all mentioned aspects above along with complete derivation compiled in several published articles [92,259,260] and series of textbooks [93,98,100,101,104].

3.5. Analog equation method

Analog Equation Method (AEM) presented by Katsikadelis [125] was originally designated for solving the linear and nonlinear potential problem. Combining the features and advantages of few numerical techniques like FEM, BEM and FDM, this method tries to establish itself as one among few leading BEM variants. Similar to DRBEM, this method is developed to deal with domain integrals without involving domain discretization. Given a nonlinear differential equation or a linear one which is difficult to establish its fundamental solution, the actual problem can be substituted with an equivalent one described by a linear differential operator with known fundamental solution using AEM. In order to solve this equivalent equation, the unknown fictitious source density function needs to be computed first using indirect BEM while preserving the boundary and initial conditions. Described in [125], the AEM procedure can be laid out by first considering the boundary value problem,

$$N(u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = g(x,y) \text{ in } \Omega \tag{55}$$

$$\alpha u + \beta u_n = \gamma \text{ on } \Gamma \tag{56}$$

where $N$is in general a nonlinear second order differential operator and α, β and γ are functions specified on Γ. Now, function u can be assumed as the sought solution of the real domain problem and once Laplacian operator applied to this function, the Poisson equation similar to Eq. (31) can be obtained

$$\nabla^2 u = b \tag{57}$$

In AEM, Eq. (55) is known as the analog equation and the uniqueness of AEM is laid on the approach in constructing the source density function b. Here, the solution for Eq. (57) at instant time t can be written as a sum of the homogenous solution $\overline{u}$and a particular solution u_p.

$$u = \overline{u} + u_P \tag{58}$$

However, to solve Eq. (57), fictitious source can be established where we assume

$$b = \sum_{j=1}^{M} \alpha_j f_j \tag{59}$$

and the particular solution as

$$u_p = \sum_{j=1}^{M} \alpha_j \hat{u}_j \tag{60}$$

in which $\hat{u}_j$ are the particular solutions for $\nabla^2 \hat{u}_j = f_j$ with f_j as a set of approximating functions while α_j known as the time-dependent coefficients. On the other hand, the integral representation of the solution $\overline{u}$ is given as

$$c\overline{u} = -\int_\Gamma (u^* \overline{q} - \overline{u} q^*) ds \tag{61}$$

and substituting Eq. (58) into Eq. (61) and assuming $c=1$ yields

$$u = -\int_\Gamma (u^* \overline{q} - \overline{u} q^*) ds + \sum_{j=1}^{M} \hat{u}_j \alpha_j \tag{62}$$

Differentiating the above equation, one would obtain

$$u_x = -\int_\Gamma (u_x^* \overline{q} - \overline{u} q_x^*) ds + \sum_{j=1}^{M} (\hat{u}_j)_x \alpha_j \tag{63}$$

$$u_y = -\int_\Gamma (u_y^* \overline{q} - \overline{u} q_y^*) ds + \sum_{j=1}^{M} (\hat{u}_j)_y \alpha_j \tag{64}$$

$$u_{xx} = -\int_\Gamma (u_{xx}^* \overline{q} - \overline{u} q_{xx}^*) ds + \sum_{j=1}^{M} (\hat{u}_j)_{xx} \alpha_j \tag{65}$$

$$u_{yy} = -\int_\Gamma (u_{yy}^* \overline{q} - \overline{u} q_{yy}^*) ds + \sum_{j=1}^{M} (\hat{u}_j)_{yy} \alpha_j \tag{66}$$

$$u_{xy} = -\int_\Gamma (u_{xy}^* \overline{q} - \overline{u} q_{xy}^*) ds + \sum_{j=1}^{M} (\hat{u}_j)_{xy} \alpha_j \tag{67}$$

Then, in order to complete the solution process, the Eqs. (63–67) above can be further discretized and full description of AEM procedure with several demonstrated examples (such as membranes on elastic foundation, heat flow in bodies with non-homogenous material properties, heat flow in bodies with nonlinear material properties and steady-state spontaneous ignition) can be obtained in [125]. This method had been demonstrated in solving boundary value problems for many research areas, i.e. thermal conductivity in non-homogenous or nonlinear bodies [125], nonlinear flexural vibrations of plates [135], integration of nonlinear equations of motion [121], linear and nonlinear plate bending problems [132,160], plane elastostatic problems [128,129], finite deformation analysis of elastic cables [123,127], plate buckling problems [161], inverse problems [162], soap bubble problem [131], nonlinear analysis of shells [273], finite equationless problems in nonlinear bodies using only boundary data [133], large deflection analysis of beams [137], nonlinear static and dynamic analysis of membranes [122,130,136,138,139], ponding problem on membranes [134,163] and meshless approach on 2D elastostatic problem [124]. Since it is considered as boundary-only method, this method only deals with discretization and integration on the boundary only. However, the collocation points inside the domain may be used to improve the solution. This method was later further developed by Katsikadelis [126] to meshless analog equation method (MAEM) as a new RBFs (radial basic functions) method for solving 2D and 3D PDEs. Similar with other RBFs method, MAEM requires no domain boundary (BEM) dicretization and integration and thus avoids establishment of fundamental solutions and evaluation of singular integrals.

4. Summary and conclusion

The reviews presented in this article are rather basic concepts of several BEM variants with their significant contributions in boundary element research. The detailed developments for each of the variants were outlined in a systematic way to enhance the understanding of their different approaches. In general, BEM and more specific, the BEM variants underwent a remarkable progress over the past half century. They are well developed in both theoretical and applied aspects. Compared to FEM and FDM, BEM possess a clear advantage in modeling and meshing stage which reduces the dimensionality of problems considered. However, like many other numerical methods, BEM also has advantages and disadvantages. Table 3 gives a detailed comparison of the BEM with FEM and FDM regarding their theoretical basis, as well as advantages and disadvantages.

Table 3
. Comparisons between BEM, FEM and FDM.

	Boundary Element Method (BEM)	Finite Element Method (FEM)	Finite Difference Method (FDM)
Theoretical basis	Based on approximation on equivalent governing integral relation using boundary segments.	Based on approximation on equivalent governing integral relation using finite segments.	Based on finite difference approximation on governing differential equation using a grid of uniformly spaced nodes.
Advantages	• Discretisation of the boundary only • Ideal for infinite problem • Less computational time	• Integration of simple function • Symmetrical and sparse matrices	• Easiest method to implement • No numerical integration
Disadvantages	• Mostly nonsymmetrical matrices • Requires complicated integral relation • Difficult treatment of inhomogeneous and nonlinear problems	• Requires domain meshes • Not suitable for infinite problems • Requires integral relation from variational principle or weighted residual formulation • Computation process is time consumings	• Requires domain meshes • Not suitable for infinite problem • Requires very fine grids • Computation process is time consuming

Table 4
. Comparisons between CBEM, GBEM, DRBEM, CVBEM and AEM.

	Advantages	Disadvantages
CBEM	• Easiest to implement • Fast solution • Applicable to wide range of standard cases	• Poor convergence and accuracy • Nonsymmetrical and dense matrices • Difficult to deal with hypersingular integral
GBEM	• High accuracy • Able to handle singular and hypersingular integrals • Easier to produce symmetric coefficient matrix • Can be implemented for various problems including	• Slow solution
DRBEM	• Able to deal with domain integrals • Requires only boundary discretization • Applicable to wide range of problems	• Fully populated and nonsymmetrical matrices • Computationally expensive • Mathematically complicated
CVBEM	• High accuracy • Suitable for problems with stress singularities and concentrations	• Limited to Laplace-type problem
AEM	• Able to deal with domain integrals • Requires only boundary discretization • Suitable for linear and nonlinear problem	• Mathematically complicated • Limited applicability • Hard to implement

In this article, we briefly discussed the implementation for each of the BEM variants considered in each of the variants and have demonstrated many different solving approaches when dealing with boundary integral equation. To illustrate the differences, Table 4 gives a comparison between these variants regarding their advantages and disadvantages. For beginner in BEM community, the collocation BEM is highly recommended as it is the simplest among them and the easiest to be implemented. It is preferable for engineer looking for fast solution and at the same time avoiding complicated mathematical procedure. However, for advance and more accurate analysis, SGBEM and DRBEM are considered as the best choice in their particular field: boundary-based and domain-based problems, where they can practically demonstrated in every known application area. On the other hand, specific method such as CVBEM extends BEM application scope to integral equation with complex variable configuration. Although, it is limited to Laplace-based integral equation, yet it has the potential for expansion in a much wider application area. In spite of considerable progress towards effective computational method, these BEM variants experiences continuous development with several additional improvement features proposed by various researchers. With all mentioned above summarized in this article, this review is believed to assist the task for researchers in selecting the appropriate BEM approach in solving particular problem. In general, the BEM still lacks behind FEM in the field of numerical method. Although, the BEM has improved in a big step forward but still the initial objective to replace FEM has not been realized. More needs to be done in order to achieve the goal.

Aside from these variants, new variants have emerged as strong competitor in the numerical analysis. Those recommended are boundary node method, boundary contour method, boundary particle method, boundary knot method, cell boundary element method, hybrid boundary element method, fast multipole boundary element method, spectral stochastic boundary element method, scaled boundary finite-element method and variational indirect boundary element method. The future of BEM may be depends on these variants.

Acknowledgement

The authors thank the reviewers for their valuable comments and suggestions to improve this manuscript. Thanks is also extended to Mohammed Zubair for his critical review. The first author was supported by USM Fellowship and the second author would like to acknowledge Universiti Sains Malaysia—Research University Grant no. RU814042 and USM Incentive Grant for sponsoring this project.

References

[1] Agnantiaris JP, Polyzos D, Beskos DE. Three-dimensional structural vibration analysis by the dual reciprocity BEM. Comput Mech 1998;21(4-5):372–81.
[2] Aimi A, Diligenti M. Hypersingular kernel integration in 3D Galerkin boundary element method. J Comput Appl Math 2002;138(1):51–72.
[3] Aimi A, Diligenti M, Monegato G. New numerical integration schemes for applications of Galerkin BEM to 2D problems. Int J Numer Meth Eng 1997;40(11):1977–99.
[4] Aimi A, Diligenti M, Monegato G. Numerical integration schemes for the BEM solution of hypersingular integral equations. Int J Numer Meth Eng 1999;45(12):1807–30.

[5] Aimi S, Nixon SP. Multiwavelet Galerkin boundary element solution of Laplace's equation. Eng Anal Bound Elem 2006;30(2):116–23.
[6] Alarcon E, Brebbia CA, Dominguez J. The boundary element method in elasticity. Int J Mech Sci 1978;20(9):625–39.
[7] Albuquerque EL, Sollero P, Fedelinski P. Dual reciprocity boundary element method in Laplace domain applied to anisotropic dynamic crack problems. Comp Struct 2003;81(17):1703–13.
[8] Ali A, Rajakumar C. The boundary element method: applications in sound and vibration. Taylor & Francis Routledge; 2004.
[9] Aliabadi MH, Wrobel LC. The boundary element method: applications in solids and structures. John Wiley & Sons; 2002.
[10] Amini S. Boundary integral solution of the exterior Helmholtz problem. Comp Mech 1993;13(1-2):2–11.
[11] Ang WT. A complex variable boundary element method for antiplane stress analysis around a crack in some nonhomogeneous bodies. J Chin Inst Eng 1999;22(6):753–61.
[12] Ang WT. Non-steady state heat conduction across an imperfect interface: a dual-reciprocity boundary element approach. Eng Anal Bound Elem 2006;30(9):781–9.
[13] Ang WT, Clements DL, Cooke T. A complex variable boundary element method for a class of boundary value problems in anisotropic thermo-elasticity. Int J Comp Math 1999;70(3):571–86.
[14] Arnold DN, Saranen J. On the asymptotic convergence of spline collocation methods for partial differential equations. SIAM J Numer Anal 1984;21(3): 459–72.
[15] Arnold DN, Wendland WL. Collocation versus Galerkin procedures for boundary integral methods. In: Brebbia CA, editor. Boundary Element Methods in Engineering. Berlin-Heidelberg-New York: Springer; 1982. p. 18–33.
[16] Arnold DN, Wendland WL. On the asymptotic convergence of collocation methods. Math Comput 1983;41:349–81.
[17] Arnold DN, Wendland WL. The convergence of spline collocation for strongly elliptic equations on curves. Numer Math 1985;47:317–41.
[18] Banerjee PK, Butterfield R. Boundary element method in geomechanics. In: Gudehus G, editor. Finite elements in geomechanics. New York: Wiley; 1977. p. 529–70. Chapter 16.
[19] Banerjee PK, Cathie DN. A direct formulation and numerical implementation of the boundary element method for two-dimensional problems of elasto-plasticity. Int J Mech Sci 1980;22(4):233–45.
[20] Banerjee PK, Morino L. Boundary element methods in nonlinear fluid dynamics. Taylor & Francis; 2005.
[21] Beatovic D, Levin PL, Sadovic S, Hutnak RA. Galerkin formulation of the boundary element method for two-dimensional and axi-symmetric problems in electrostatics. IEEE Trans Elec Insul 1992;27(1):135–43.
[22] Behbahani-nia A, Kowsary F. A dual reciprocity BE-based sequential function specification solution method for inverse heat conduction problems. Int J Heat Mass Transf 2004;47(6-7):1247–55.
[23] Beylkin G, Coifman R, Rokhlin V. Fast wavelet transforms and numerical algorithms I. Commun Pure Appl Math 1991;44(2):141–83.
[24] Bonnet M, Maier G, Polizzotto C. Symmetric galerkin boundary element method. Appl Mech Rev 1998;51:669–704.
[25] Brebbia CA. The boundary element method for engineers. London/New York: Pentech Press/Halstead Press; 1978.
[26] Brebbia CA, Butterfield R. Formal equivalence of direct and indirect boundary element methods. Appl Math Modell 1978;2(2):132–4.
[27] Brebbia CA, Dominguez J. Boundary Elements—An introductory course. Southampton: CMP; 1989.
[28] Brebbia CA, Dominguez J. Boundary element methods for potential problems. Appl Math Modell 1977;1:372–8.
[29] Brebbia CA, Nardini D. Dynamic analysis in solid mechanics by an alternative boundary element procedure. Int J Soil Dyn Earthquake Eng 1983;2(4):228.
[30] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques—theory and applications in engineering. Springer-Verlag; 1984.
[31] Brebbia CA, Walker S. Boundary element techniques in engineering. London: Newnes-Butterworths; 1980.
[32] Brebbia CA, Wrobel LC. Boundary element method for fluid flow. Adv Water Resour 1979;2:83–9.
[33] Buffa A, Hiptmair R. Galerkin boundary element methods for electromagnetic scattering. In: Ainsworth M, Davies P, Duncan Dugald, editors. Topics in computational wave propagation: direct and inverse problems. Springer; 2009. p. 83–124.
[34] Bui HD. An integral equation method for solving the problem of a plane crack of arbitrary shape. J Mech Phys Solids 1977;25:29–39.
[35] Capuani D, Bigoni D, Brun M. Integral representations at the boundary for stokes flow and related symmetric galerkin formulation. Arch Mech 2005;57(5):363–85.
[36] Čavka D, Poljak D GB-IBEM model of vertical antenna above, below and penetrating ground. In: Rozic N, editor. Proceedings of 2008 international conference on software, telecommunications and computer network, split, 2008.
[37] Chang JR, Weih W, Wu YC, Chang JJ. Applications of the complete multiple reciprocity method for solving the 1D Helmholtz equation of a semi-infinite domain. Adv Eng Software 2001;32(2):111–7.
[38] Chen Z, Hofstetter G, Mang HA. Galerkin-type BE-formulation for acoustic radiation and scattering of structures with arbitary shape. In: Marburg S, Nolte B, editors. Computational acoustics of noise propagation in fluids—finite and boundary element methods. Berlin Heidelberg: Springer; 2008. p. 435–58.
[39] Cheng AHD, Cheng DT. Heritage and early history of the boundary element method. Eng Anal Bound Elem 2005;29(3):268–302.
[40] Cheng AHD, Lafe O, Grilli S, Dual-reciprocity BEM. based on global interpolation functions. Eng Anal Bound Elem 1994;13(4):303–11.
[41] Choi CY. Dual reciprocity boundary element analysis for the laminar forced heat convection problem in concentric annulus. KSME Int J 1999;13(6): 496–503.
[42] Choi CY, Balaras E. A dual reciprocity boundary element formulation using the fractional step method for the incompressible Navier-Stokes equations. Eng Anal Bound Elem 2009;33(6):741–9.
[43] Chou SI, Shamas-Ahmadi M. Complex variable boundary element method for torsion of hollow shafts. Nucl Eng Des 1992;136(3):255–63.
[44] Cisilino AP, Aliabadi MH. A boundary element method for three-dimensional elastoplastic problems. Eng Comput 1998;15(8):1011–30.
[45] Ciskowski RD, Brebbia CA. Boundary element methods in acoustics. Southampton: CMP; 1991.
[46] Cole DM, Kosloff DD, Minster JB. A numerical boundary integral equation method for elastodynamics. I. Bull Seism Soc Am 1978;68:1331–57.
[47] Costabel M, Stephan EP. An improved boundary element Galerkin method for three-dimensional crack problems. Integr Equations Oper Theor 1987;10(4):467–504.
[48] Cruse TA. A direct formulation and numerical solution of the general transient elastodynamic problem II. J Math Anal Appl 1968;22(2):341–55.
[49] Cruse TA. Numerical solutions in three dimensional elastostatics. Int J Solids Struct 1969;5(12):1259–74.
[50] Cruse TA. Recent advances in boundary element analysis methods. Comp Meth Appl Mech Eng 1987;62:227–44.
[51] Cruse TA, Rizzo FJ. A direct formulation and numerical solution of the general transient elastodynamic problem I. J Math Anal Appl 1968;22(1): 244–59.
[52] D'Elía J, Battaglia L, Storti MA, Cardona A. Galerkin boundary integral equations applied to three dimensional Stokes flows. Mecánica Computacional, 2007;XXVII:2397–2410.
[53] Dahmen W, Harbrecht H, Schneider R. Compression technique for boundary integral equations—optimal complexity estimates. SIAM J Numer Anal 2006;43(6):2251–71.
[54] Davies TW, Moslehy FA. Modal analysis of plates using the dual reciprocity boundary element method. Eng Anal Bound Elem 1994;14(4):357–62.
[55] De Paula FA, Telles JCF. A. comparison between point collocation and Galerkin for stiffness matrices obtained by boundary elements. Eng Anal Bound Elem 1989;6(3):123–8.
[56] Dean TR, Hromadka II TV. A collocation CVBEM using program Mathematica. Eng Anal Bound Elem 2010;34(4):417–22.
[57] Denda M, Dong YF. Complex variable approach to the BEM for multiple crack problems. Comp Meth Appl Mech Eng 1997;141(3-4):247–64.
[58] Djojodihardjo RH. Unified BE-FE aerodynamic-acoustic-structure coupling scheme for acoustic effects on aeroelastic structures. paper ICAS 2008 7.7.7.5, In: Proceeding of the 26th international congress of the aeronautical sciences, 14–19 September 2008, Anchorage, Alaska.
[59] Djojodihardjo RH. True and efficient solution of unified BEM-FEM acoustic-structural coupling using chief regularization. paper IAC-08.C2.3.7, In: Proceeeding of the 59th international astronautical congress, 28 September–3 October 2008, Glasgow, Scotland.
[60] Djojodihardjo RH, Widnall SE. A numerical method for the calculation of nonlinear, unsteady lifting potential flow problems. AIAA J 1969;7(10): 2001–9.
[61] Dominguez J. Boundary elements in dynamics. CMP; 1993.
[62] Dumir PC, Kumar R. Complex variable boundary element method for torsion of anisotropic bars. Appl Math Modell 1993;17(2):80–8.
[63] Ebrahimnejad L, Attarnejad R. A novel approach to solve linear system arising from BEM using fast wavelet transforms. Math Meth Appl Sci 2010;33(5):643–54.
[64] Ebrahimnejad L, Attarnejad R. Fast solution of BEM systems for elasticity problems using wavelet transforms. Int J Comp Math 2010;87(1):77–93.
[65] Elzein A, Syngellakis S. Dual reciprocity in boundary element formulations of the plate buckling problem. Eng Anal Bound Elem 1992;9(2):175–84.
[66] Eppler K, Harbrecht H. Fast wavelet BEM for 3D electromagnetic shaping. Appl Numer Math 2005;54(3-4):537–54.
[67] Eppler K, Harbrecht H. Wavelet-based boundary element methods for exterior electromagnetic shaping. Eng Anal Bound Elem 2008;32(8):645–57.
[68] Fachinotti V, Cardona A, D'Elía J, Paquay S. BEM for the analysis of fluid flow around MEMS. Mecánica Computacional 2007;XXIV:1104–19.
[69] Florez WF, Power H. Multi-domain mass conservative dual reciprocity method for the solution of the non-Newtonian Stokes equations. Bound Elem Commun 1994;5:57–61.
[70] Florez W, Power H, Chejne F. Multi-domain dual reciprocity BEM approach for the Navier–Stokes system of equations. Commun Numer Meth Eng 2000;16(10):671–81.
[71] Frangi A, Bonnet M. A galerkin symmetric and direct BIE method for kirchoff elastic plates: formulation and implementation. Int J Numer Meth Eng 1998;41:337–69.
[72] Frangi A, Novati G, Springhetti R, Rovizzi M. 3D fracture analysis by the symmetric galerkin BEM. Comp Mech 2002;28:220–32.

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

[73] Frayce D, Khayat RE, Derdouri A. A dual reciprocity boundary element approach to three-dimensional transient heat conduction as applied to materials processing. Numer Heat Transf. Part A, Appl 1996;29(3):243–64.
[74] Gáspár C. Multigrid technique for biharmonic interpolation with application to dual and multiple reciprocity method. Numer Algorithms 1999; 21(1-4):165–83.
[75] Gray LJ. Evaluation of singular and hypersingular Galerkin integrals: direct limits and symbolic computation. In: Sladek V, Sladek J, editors. Advances in boundary elements. Southampton: CMP; 1998. p. 33–84.
[76] Gray LJ, Chinta B, Kane JH. Symmetric galerkin fracture analysis. Eng Anal Bound Elem 1995;15:103–9.
[77] Gray LJ, Glaeser JM, Kaplan T. Direct evaluation of hypersingular Galerkin surface integrals. J Sci Comput 2004;25(5):1534–56.
[78] Gray LJ, Griffith BE. A faster Galerkin boundary integral algorithm. Commun Numer Meth Eng 1998;14(12):1109–17.
[79] Gray LJ, Maroudas D, Enmark MN. Galerkin boundary integral method for evaluating surface derivates. Comput Mech 1998;22(2):187–93.
[80] Gray LJ, Paulino GH. Symmetric Galerkin boundary integral fracture analysis for plane orthotropic elasticity. Comput Mech 1997;20:26–33.
[81] Gray LJ, Salvadori A, Phan A-V. Mantič. Direct evaluation of hypersingular Galerkin surface integrals II. EJBE 2006;4(3):105–30.
[82] Grecu L, Petrila T. A complex variable boundary element method for the problem of the free-surface heavy inviscid flow over an obstacle. Gen Math 2008;16(2):3–17.
[83] Ha-Duong T, Ludwig B, Terrasse I. A Galerkin BEM for transient acoustic scattering by an absorbing obstacle. Int J Numer Meth Eng 2003;57(13):1845–82.
[84] Hall WS, Oliveto G. Boundary elements methods for soil-structure interaction. Springer; 2003.
[85] Harbrecht H Wavelet Galerkin schemes for the boundary element method in three dimensions. Ph.D. Thesis, Technische Universitat Chemnitz, Germany, 2001.
[86] Harbrecht H, Hohage T. Fast methods for three-dimensional inverse obstacle scattering. J Integr Equat Appl 2007;19(3):237–60.
[87] Harbrecht H, Schneider R. Adaptive wavelet Galerkin BEM. In: Bathe K, editor. Computational fluid and solid mechanics 2003. Elsevier; 2003. p. 1982–6.
[88] Harbrecht H, Schneider R. Wavelet Galerkin schemes for 2D-BEM. In: Elschner J, Gohberg I, Silbermann B, editors. Problems and methods in mathematical physics: The Siegfried Prössdorf Memorial Volume—Proceedings of the 11th TMP, Chemnitz, Germany, March 25–28, 1999 (Operator Theory: Advances and Applications). Birkhäuser Basel; 2001.
[89] Hartmann F, Katz C, Protopsaltis B. Boundary elements and symmetry. Ingenieur-Archiv 1985;55:440–9.
[90] Hartmann F, Zotemantel R. The direct boundary element method in plate bending. Int J Numer Meth Eng 1986;23(11):2049–69.
[91] Hess JL, Smith AMO. . Calculation of potential flow about arbitrary bodies. Progress in aeronatical sciences, vol. 8. New York: Pergamon Press; 1966.
[92] Homentcovschi D, Cocora D, Măgureanu R. Some developments of the CVBEM-aplication to mixed boundary value problem for the Laplace equation. Eng Anal 1987;4(1):15–20.
[93] Hromadka II TV. A multi-dimensional complex variable boundary element method. Southampton: WIT Press; 2002.
[94] Hromadka II TV. Convergence properties of the CVBEM: Development. Eng Anal 1986;3(3):181–7.
[95] Hromadka II TV. Determining relative error bounds for the CVBEM. Eng Anal 1985;2(2):75–80.
[96] Hromadka II TV. Linking the complex variable boundary-element method to the analytic function method. Numer Heat Transf, Part A: Appl 1984;7(2): 235–40.
[97] Hromadka II TV. Modeling error in evaluation of the CVBEM matrix system. Eng Anal Bound Elem 1996;17(1):85–90.
[98] Hromadka II TV. The complex variable boundary element method. Springer-Verlag; 1984.
[99] Hromadka II TV, Guymon GL. A complex variable boundary element method: development. Int J Num Meth Eng 1984;20:25–37.
[100] Hromadka II TV, Lai C. The complex variable boundary element method in engineering analysis. New York: Springer-Verlag; 1987.
[101] Hromadka II TV, Whitley RJ. Advances in the complex variable boundary element method. New York: Springer-Verlag; 1998.
[102] Hromadka II TV, Whitley RJ. Approximating three-dimensional steady-state potential flow problems using two-dimensional complex polynomials. Eng Anal Bound Elem 2005;29(2):190–4.
[103] Hromadka II TV, Whitley RJ, Yen CC. Extending the complex variable boundary element method of three dimensions. May 2003 BEM conference, Detroit, MI. Conference proceedings, vol. XV. Southampton, UK: WIT Press; 2004. p. 105–13.
[104] Hromadka II TV, Yen CC. Complex variable boundary elements. Computational Mechanics Press; 1986.
[105] Hromadka II TV, Yen CC. Extension of the CVBEM to higher-order trial functions. Appl Math Modell 1988;12(6):619–26.
[106] Hsiao GC. Boundary element methods—An overview. Appl Numer Math 2006;56(10-11):1356–69.
[107] Hsieh CK, Kassab AJ. Complex variable boundary element methods for the solution of potential problems in simply and multiply connected domains. Comp Meth Appl Mech Eng 1991;86(2):189–213.
[108] Huang Y, Mogilevskaya SG, Crouch SL. Complex variable boundary integral method for linear viscoelasticity: Part I—basic formulations. Eng Anal Bound Elem 2006;30(12):1049–56.
[109] Huang Y, Mogilevskaya SG, Crouch SL. Complex variable boundary integral method for linear viscoelasticity: Part II—application to problems involving circular boundaries. Eng Anal Bound Elem 2006;30(12):1057–68.
[110] Huber M, Hickman R. A scheme to randomly generate surface dust points for the 3D complex variable boundary element method (CVBEM). Eng Anal Bound Elem 2006;30(12):1025–9.
[111] Itagaki M, Brebbia CA. Multiple reciprocity boundary element formulation for one-group fission neutron source iteration problems. Eng Anal Bound Elem 1993;11(1):39–45.
[112] Itagaki M, Sahashi N. Matrix-Type multiple reciprocity boundary element method for solving three-dimensional two-group neutron diffusion equations. J Nucl Sci Tech 1997;34(7):655–65.
[113] Jaswon MA. Integral equation method in potential theory – I. Proc R Soc Ser 1963;A275:23–32.
[114] Jin C. A direct boundary integral equation method for the acoustic scattering problem. Eng Anal Bound Elem 1993;12(1):39–46.
[115] Juhl P. A note on the convergence of the direct collocation boundary element method. J Sound Vib 1998;212(4):703–19.
[116] Kamiya N, Andoh E. A note on multiple reciprocity integral formulation for Helmholtz equation. Commun Numer Meth Eng 1993;9(1):9–13.
[117] Kamiya N, Andoh E, Nogae K. Three-dimensional eigenvalue analysis of the Helmholtz equation by multiple reciprocity boundary element method. Adv Eng Software 1993;16(3):203–7.
[118] Kamiya N, Sawaki Y. The plate bending analysis by the dual reciprocity boundary elements. Eng Anal Bound Elem 1988;5(1):36–40.
[119] Kassab AJ, Chesla S. An iterative CVBEM solution of nonlinear heat transfer problems. Eng Anal Bound Elem 1993;11(1):67–75.
[120] Kassab AJ, Hsieh CK. Application of the complex variable boundary element method for solving potential problems in doubly connected domains. Int J Numer Meth Eng 1990;29(1):161–79.
[121] Katsikadelis JT. A new time step integration scheme for structural dynamics based on the analog equation method. In: Kounadis AN, editor. Collection of papers dedicated to Prof. P.S. Theocaris. Athens: National Technical University of Athens; 1994. p. 80–100.
[122] Katsikadelis JT. Dynamic analysis of nonlinear membranes by the analog equation method. A boundary-only solution. Comput Model Eng Sci 2002;29(2):170–7.
[123] Katsikadelis JT. Finite deformation of elastic cables under 3-D loading. Proceeding of the 4th German–Greek–Polish symposium on advances on mechanics. Warsaw-Pultusk; 2001.
[124] Katsikadelis JT. The 2D elastostatic problem in inhomogeneous anisotropic bodies by the meshless analog equation method (MAEM). Eng Anal Bound Elem 2008;32(12):997–1005.
[125] Katsikadelis JT. The analog equation method—a powerful BEM-based solution technique for solving linear and nonlinear engineering problems. In: Brebbia CA, editor. Boundary element method XVI. Southampton: CLM Publications; 1994. p. 167–82.
[126] Katsikadelis JT The meshless analog equation method. A new highly accurate truly mesh-free method for solving partial differential equations. In: Brebbia CA, Katsikadelis JT, editors. Proceedings of the international conference on boundary elements and other mesh reduction methods, 10–12 May 2006, Esperides Hotel, Skiathos, Greece, BEM/MRM 28, WIT Press, Southampton.
[127] Katsikadelis JT, Apostolopoulos NA Finite deformation analysis of cables by the analog equation method. In: Kounadis A, editor. Proceedings of the first European Conference on Steel Structures, Eurosteel '95, Balkema, Rotterdam, 1995. p. 355–360.
[128] Katsikadelis JT, Kandilas CB Plane stress analysis of thin plates with variable thickness by the analog equation method. In: Theocharis PS, Gdoutos EE, editors. Proceedings of the fourth national congress on applied mechanics, Vol. 1, Mechanics of Solids, Demokritus University, Xanthi, 1995. p. 562–573.
[129] Katsikadelis JT, Kandilas CB. Solving the elastostatic problem by the analog equation method. In: Papadrakakis M, Topping BHV, editors. Advances in computational mechanics. Edinburgh: Civil-Comp Press; 1994. p. 269–78.
[130] Katsikadelis JT, Nerantzaki MS. A boundary-only solution to dynamic analysis of non-homogenous elastic membranes. Computer Modell Eng Sci 2000;1(3):1–9.
[131] Katsikadelis JT, Nerantzaki MS. A boundary element solution to the soap bubble problem. Comput Mech 2001;27(2):154–9.
[132] Katsikadelis JT, Nerantzaki MS. Non-linear analysis of plates by the analog equation method. Comput Mech 1994;14:154–64.
[133] Katsikadelis JT, Nerantzaki MS Solving Equationless problems from boundary only data. In: Proceeding of the European conference on computational mechanics, Münich, Germany, 1999 p. 818–819.
[134] Katsikadelis JT, Nerantzaki MS. The ponding problem on membranes. An analog equation solution. Comput Mech 2001;28:122–8.
[135] Katsikadelis JT, Nerantzaki MS, Kandilas CBA. BEM approach to non-linear vibrations of plates. In: Moan, editor. Structural dynamics—EURODYN '93. Rotterdam: Balkema; 1993. p. 659–71.

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

[136] Katsikadelis JT, Nerantzaki MS, Tsiatas GC. The analog equation method for large deflection analysis of membranes. A boundary only solution. Comput Mech 2001;27(6):513–23.
[137] Katsikadelis JT, Tsiatas GC. Large deflection analysis of beams with variable stiffness. An analog equation solution. In: Proceedings of the sixth national congress on mechanics, Thessaloniki, Greece, I, 2001. p. 172–177.
[138] Katsikadelis JT, Tsiatas GC. Nonlinear dynamic analysis of heterogeneous orthotropic membranes. In: Beskos DE, Brebbia CA, Katsikadelis JT, Manolis G, editors. Boundary elements XXIII. Southampton: WITpress; 2001. p. 139–48.
[139] Katsikadelis JT, Tsiatas GC. The analog equation method for large deflection analysis of hererogeneous orthotropic membranes. A boundary-only solution. Eng Anal Bound Elem 2001;25(8):655–67.
[140] Kim S, Karrila SJ. Integral equations of second kind for stokes flow: direct solutions for physical variables and removal of inherent accuracy limitations. Chem Eng Commun 1989;82(1):123–61.
[141] Koshelev V, Ghassemi A. Complex variable BEM for thermo- and poroelasticity. Eng Anal Bound Elem 2004;28(7):825–32.
[142] Kuhn G, Möhrmann W. Boundary element methods in elastostatics: Theory and applications. Appl Math Modelling 1983;7(2):97–105.
[143] Lachat JC A further development of the boundary integral technique for elastostatics. Ph.D. Thesis, University of Southampton, 1975.
[144] Lachat JC, Watson JO. Effective numerical treatment of boundary integral equations: a formulation for three-dimensional elastostatics. Int J Num Meth Eng 1976;10:991–1005.
[145] Lage C, Schwab C. Wavelet Galerkin algorithms for boundary integral equations. SIAM J Sci Comput 1999;20(6):2195–222.
[146] Lean MH, Friedman M, Wexler A. Application of the boundary element method in electrical problems. In: Banerjee PK, Butterfield R, editors. Developments in boundary element methods, 1. London: Applied Science Publishers; 1979. p. 207–50.
[147] Lei G, Mao-Kung H. A complex variable boundary element method for solving plane and plate problems of elasticity. Eng Anal Bound Elem 1991;8(6):266–72.
[148] Liu YH, Zhang XF, Cen ZZ. Lower bound shakedown analysis by the symmetric galerkin boundary element method. Int J Plasticity 2005;21: 21–42.
[149] Maier G, Miccoli S, Novati G, Perego U. Symmetric Galerkin boundary element method in plasticity and gradient plasticity. Comput Mech 1995;17(1-2):115–29.
[150] Maier G, Miccoli S, Novati G, Sirtori SA. Galerkin symmetric boundary element methods in plasticity: formulation and implementation. In: Kane JH, Maier G, Tosaka N, Atluri SN, editors. Advances in boundary element techniques. Berlin Heidelberg: Springer-Verlag; 1993. p. 288–328.
[151] Maier G, Polizzotto CA. Galerkin approach to boundary element elastoplastic analysis. Comp Meth Appl, Mech Eng 1987;60(2):175–94.
[152] Man KW. Contact mechanics using boundary elements. CMP; 1994.
[153] Mansur WJ, Brebbia CA. Formulation of the boundary element method for transient problems governed by the scalar wave equation. Appl Math Modell 1982;6(4):307–11.
[154] Mansur WJ, Brebbia CA. Numerical implementation of the boundary element method for two-dimensional transient scalar wave propagation problems. Appl Math Modell 1982;6(4):299–306.
[155] Mansur WJ, Brebbia CA. Transient elastodynamics using a time-stepping technique. In: Brebbia CA, Futagami T, Tanaka M, editors. Boundary elements. Berlin: Springer; 1983. p. 677–98.
[156] Martin PA, Rizzo FJ. On boundary integral equations for crack problems. Proc R Soc London. Ser A. Math Phys 1989;421:341–55.
[157] Mokry M. Complex variable boundary element method for external potential flows. AIAA J 1991;29(12):2027–8.
[158] Nardini D, Brebbia CA. A new approach to free vibration analysis using boundary elements. Appl Math Modell 1983;7(3):157–62.
[159] Natalini B, Popov V. Tests of performance of the boundary element dual reciprocity method—multi-domain approach for 3D problems. Mecánica Computacional 2007;26:1174–93.
[160] Nerantzaki MS. Solving plate bending problems by the analog equation method. In: Brebbia CA, editor. Boundary Element Method. Southampton: CLM Publications; 1994. p. 283–91.
[161] Nerantzaki MS, Katsikadelis JT. Buckling of plates with variable thickness – an analog equation solution. Eng Anal Bound Elem 1996;18(2):149–54.
[162] Nerantzaki MS, Katsikadelis JT. Solving inverse problems by use of the AEM. In: Tanaka M, Dulikravich G, editors. Inverse problems in engineering mechanics. Tokyo: Elsevier; 1998. p. 335–40.
[163] Nerantzaki MS, Katsikadelis JT. The ponding problem on floating membranes. Eng Anal Bound Elem 2002;27:589–96.
[164] Neves AC, Brebbia CA. The multiple reciprocity boundary element method in elasticity: a new approach for transforming domain integrals to the boundary. Int J Numer Meth Eng 1991;31(4):709–27.
[165] Neves AC, Nowak AJ. Steady-state thermoelasticity by multiple reciprocity method. Int J Numer Meth Heat Fluid Flow 1992;2(5):429–40.
[166] Neves AC, Wrobel LC, Nowak AJ. Transient thermoelasticity by multiple reciprocity method. Int J Numer Meth Heat Fluid Flow 1993;3(2): 107–19.
[167] Niku SM, Brebbia CA. Dual reciprocity boundary element formulation for potential problems with arbitrarily distributed sources. Eng Anal 1988;5(1):46–8.
[168] Nowak AJ. Solving linear heat conduction problems by the multiple reciprocity method. In: Wrobel LC, Brebbia CA, editors. Boundary element methods in heat transfer. Southampton, Boston: CMP; 1992. p. 63–132.
[169] Nowak AJ Temperature fields in domains with heat sources using boundary-only formulations. In: Brebbia CA. In: Proceeding of tenth BEM conference on computational mechanics publications Southampton and Boston, Springer-Verlag, Berlin, vol. 2, pp. 233-247, 1988.
[170] Nowak AJ. The multiple reciprocity method of solving transient heat conduction problems. In: Brebbia CA, Connors JJ, editors. Advances in boundary elements, vol. 2. Southampton: CMP; 1989.
[171] Nowak AJ, Brebbia CA. Solving Helmholtz equation by multiple reciprocity method. In: Carlomangno GM, Brebbia CA, editors. Computer and Experiments in Fluid Flow. Southampton: CMP; 1989. p. 265–70.
[172] Nowak AJ, Brebbia CA. The multiple-reciprocity. A new approach for transforming BEM domain integrals to the boundary. Eng Anal Bound Elem 1989;6(3):164–7.
[173] Nowak AJ, Partridge PW. Comparison of the dual reciprocity and the multiple reciprocity methods. Eng Anal Bound Elem 1992;10:155–60.
[174] Noye J. Computational techniques for differential equations. Elsevier; 1984.
[175] Ochiai Y. Formulation for three-dimensional elastoplastic analysis without internal cells by triple-reciprocity BEM. EJBE 2002;1:67–76.
[176] Ochiai Y. Three-dimensional steady thermal stress analysis by triple-reciprocity boundary element method. Int J Numer Meth Eng 2005;63(12): 1741–56.
[177] Ochiai Y. Two-dimensional steady heat conduction in functionally gradient materials by tripel-reciprocity boundary element method. Eng Anal Bound Elem 2004;28(12):1445–53.
[178] Ochiai Y. Two-dimensional unsteady heat conduction analysis with heat generation by triple-reciprocity BEM. Int J Numer Meth Eng 2001;51(2): 143–57.
[179] Ochiai Y, Kitayama Y. Three-dimensional unsteady heat conduction analysis by triple-reciprocity boundary element method. Eng Anal Bound Elem 2009;33(6):789–95.
[180] Ochiai Y, Kobayashi T. Initial stress formulation for elastoplastic analysis by improved multiple-reciprocity boundary element method. Eng Anal Bound Elem 1999;23(2):167–73.
[181] Ochiai Y, Sekiya T. Steady heat conduction analysis by improved multiple-reciprocity boundary element method. Eng Anal Bound Elem 1996;18(2): 111–7.
[182] Ochiai Y, Sladek V, Sladek J. Axial symmetric stationary thermoelastic analysis by triple-reciprocity BEM. Commun Numer Meth Eng 2006;22(6):547–66.
[183] Ochiai Y, Sladek V, Sladek J. Transient heat conduction analysis by triple-reciprocity boundary element method. Eng Anal Bound Elem 2006;30(3): 194–204.
[184] Ochiai Y, Takeda S. Meshless convection-diffusion analysis by triple-reciprocity boundary element method. Eng Anal Bound Elem 2009;33(2): 168–75.
[185] Ozgener B, Ozgener HA. The application of the multiple reciprocity method to the boundary element formulation of the neutron diffusion equation. Ann Nucl Energy 1994;21(11):711–23.
[186] Panzeca T, Milana V, Salerno M. A symmetric galerkin BEM for plate bending analysis. Eur J Mech A/Solids 2009;28:62–74.
[187] París F, De León S. Boundary element method applied to the analysis of thin plates. Comp Struct 1987;25(2):225–33.
[188] Parreira P, Guiggiani M. On the implementation of the Galerkin approach in the boundary element method. Comp Struct 1989;33(1):269–79.
[189] Partridge PW. Dual reciprocity BEM: local versus global approximation functions for diffusion, convection and other problems. Eng Anal Bound Elem 1994;14(4):349–56.
[190] Partridge PW. Towards criteria for selecting approximation functions in the dual reciprocity method. Eng Anal Bound Elem 2000;24(7-8):519–29.
[191] Partridge PW, Brebbia CA, Wrobel LC. The dual reciprocity boundary element method. CMP; 1991.
[192] Partridge PW, Sensale B. Hybrid approximation functions in the dual reciprocity boundary element method. Commun Numer Meth Eng 1997;13(2):83–94.
[193] Pérez-Gavilán JJ, Aliabadi MH. A symmetric galerkin BEM for harmonic problems and multiconnected bodies. Meccanica 2001;36:449–62.
[194] Pérez-Gavilán JJ, Aliabadi MH. A symmetric galerkin BEM for multi-connected bodies: a new approach. Eng Anal Bound Elem 2001;25:633–8.
[195] Pérez-Gavilán JJ, Aliabadi MH. A symmetric galerkin boundary element method for dynamic frequency domain viscoelastic problems. Comp Struct 2001;79:2621–33.
[196] Pérez-Gavilán JJ, Aliabadi MH. Symmetric galerkin BEM for multi-connected bodies. Commun Numer Meth Eng 2001;17(11):761–70.
[197] Pérez-Gavilán JJ, Aliabadi MH. Symmetric galerkin BEM for shear deformable plates. Int J Numer Meth Eng 2003;57:1661–93.
[198] Perrey-Debain E, Trevelyan J, Bettess P. Plane wave interpolation in direct collocation boundary element method for radiation and wave scattering: numerical aspects and application. J Sound Vib 2003;261(5):839–58.
[199] Perrey-Debain E, Trevelyan J, Bettess P. Use of wave boundary elements for acoustic computations. J Comput Acoustics 2003;11(2):305–21.
[200] Petrila T. Consideration of a CVBEM approximation for plane hydrodynamics. Eng Anal Bound Elem 2006;30(12):1045–8.

Please cite this article as: Yu KH, et al. Development and implementation of some BEM variants—A critical review. Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.05.001

[201] Phan A-V, Napier JAL, Gray LJ, Kaplan T, Symmetric-Galerkin BEM. simulation of fracture with frictional contact. Int J Numer Meth Eng 2003;57:835–51.
[202] Ping SJ, Hui LX, Heng CY. A complex variable boundary element method for solving interface crack problem 1999;96(2):167–78.
[203] Poljak D. New numerical approach in the analysis of a thin wire radiating over a lossy half-space. Int J Numer Meth Eng 1995;38(22):3803–16.
[204] Poljak D, Brebbia CA. Boundary element methods for electrical engineers. Southampton: WIT Press; 2005.
[205] Poljak D, Brebbia CA. Indirect Galerkin-Bubnov boundary element method for solving integral equations in electromagnetics. Eng Anal Bound Elem 2004;28(7):771–7.
[206] Poljak D, Doric V. Wire antenna model for transient analysis of simple grounding systems, Part II: The Horizontal grounding electrode. Progress in Electromagnetics Research 2006;64:167–89.
[207] Poljak D, Doric V, Antonijevic S. Computer aided design of wire structures —frequency and time domain analysis. Southampton: WIT Press; 2007.
[208] Poljak D, Doric V, Drissi KEK, Kerroum K, Medić I. Comparison of wire antenna and modified transimission line approach to the assessment of frequency response of horizontal grounding electrodes. Eng Anal Bound Elem 2008;32(8):676–81.
[209] Poljak D, Gizdic I, Roje V. Plane wave coupling to finite length cables buried in a lossy ground. Eng Anal Bound Elem 2002;26(9):803–6.
[210] Poljak D, Roje V Currents induced in human body exposed to the power line electromagnetic field. In: Proceedings of the 20th annual international conference of the IEEE engineering in medicine and biology society, Vol 20, No 6, 1998.
[211] Poljak D, Roje V Induced current and voltages along a horizontal wire above a lossy ground. In: Proceedings of the 21th international conference on bounda elements, BEM 21, pp. 185–194, Oxford UK, Aug 25–27, 1999.
[212] Poljak D, Roje V. Transient response of a thin wire buried in a real ground. Antennas Propag Soc Int Symp 2001;2(2001):396–9.
[213] Popov V, Power H, Baldasano JM. BEM solution of design of trenches in a multi-layered landfill. J Environ Eng ASCE 1998;124(1):59–66.
[214] Power H, Wrobel L. Boundary integral methods in fluid mechanics. Southampton: CMP; 1995.
[215] Ramdas Ram-Mohan L. Finite element and boundary element applications in quantum mechanics. Oxford University Press; 2002.
[216] Ramesh PS, Lean MH. Stability of the multiple reciprocity method for transient heat conduction. Commun Numer Meth Eng 1993;9(7):629–34.
[217] Rashed YF, Brebbia CA. Transformation of domain effects to the boundary. Southampton and Boston: WIT Press; 2003.
[218] Rasmussen TC, Yu G. Determination of groundwater flownets, fluxes, velocities, and travel times using the complex variable boundary element method. Eng Anal Bound Elem 2006;30(12):1030–44.
[219] Rizzo FJ. An integral equation approach to boundary value problems of classical elastostatics. Q Appl Math 1967;25:83–95.
[220] Rizzo FJ, Shippy DJ. An advanced boundary integral equation method for three-dimensional thermoelasticity. Int J Numer Meth Eng 1977;11(11): 1753–68.
[221] Rolicz P. Bubnov–Galerkin method for solving exterior alternating field problems. J Phys D: Appl Phys 1982;15:2401–11.
[222] Rolicz P. Use of the Bubnov–Galerkin method to skin effect problems for the case of unknown boundary conditions. IEEE Trans Magn 1982;18(2):527–30.
[223] Rüberg T, Schanz M. An alternative collocation boundary element method for static and dynamic problems. Comput Mech 2009;44(2):247–61.
[224] Ruotsalainen K. On the convergence of the collocation method for nonlinear boundary integral equations. J Comput Appl Math 1994;50(1-3):471–83.
[225] Ruotsalainen K, Saranen J. On the collocation method for a nonlinear boundary integral equation. J Comput Appl Math 1989;28:339–48.
[226] Routsalainen K, Saranen J. The boundary element method for some nonlinear initial boundary value problems. In: Constanda C, editor. Integral methods in science and engineering. Longman Scientific & Technical; 1994.
[227] Russell RD, Shampine LF. A collocation method for boundary value problems. Numer Math 1972;19:1–28.
[228] Šarler B, Kuhn G. Dual reciprocity boundary element method for convective-diffusive solid-liquid phase change problems. Part 1. Formulation. Eng Anal Bound Elem 1998;21(1):53–63.
[229] Šarler B, Kuhn G. Dual reciprocity boundary element method for convective-diffusive solid-liquid phase change problems. Part 2. Numerical examples. Eng Anal Bound Elem 1998;21(1):65–79.
[230] Sato K. Complex variable boundary element method for potential flow with thin objects. Comp Meth Appl Mech Eng 2003;192(11-12):1421–33.
[231] Schanz M. Wave propagation in viscoelastic and poroelastic continua: a boundary element approach.. Springer-Verlag; 2001.
[232] Schanz M, Rüberg T, Kielhorn L. Time Domain BEM: Numerical aspects of Collocation and Galerkin formulations. In: Manolis GD, Polyzos D, editors. Recent advances in boundary element methods. Springer; 2009. p. 415–32.
[233] Shiah YC, Lin YJ. Multiple reciprocity boundary element analysis of two-dimensional anisotropic thermoelasticity involving an internal arbitrary non-uniform volume heat source. Int J Solids Struct 2003;40(24): 6593–612.
[234] Singh KM, Tanaka M. Dual reciprocity boundary element analysis of inverse heat conduction problems. Comp Meth Appl Mech Eng 2001;190:5283–95.
[235] Sirtori S. General stress analysis method by means of integral equations and boundary elements. Meccanica 1979;14:210–8.
[236] Sirtori S, Maier G, Novati G, Miccoli SA. Galerkin symmetric boundary-element method in elasticity: formulation and implementation. Int J Numer Meth Eng 1992;35(2):255–82.
[237] Sladek V, Sladek J. Multiple reciprocity method in BEM formulations for solution of plate bending problems. Eng Anal Bound Elem 1996;17(2): 161–73.
[238] Sladek V, Sladek J, Tanaka M. Multiple reciprocity method for harmonic vibration of thin elastic plates. Appl Math Modell 1993;17(9):468–76.
[239] Stephan EP, Wendland WL. An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. Appl Anal 1984;18(3):183–219.
[240] Sutradhar A, Paulino GH. Symmetric galerkin boundary element computation of T-stress and stress intensity factors for mixed-mode cracks by the interaction integral method. Eng Anal Bound Elem 2004;28(11): 1335–50.
[241] Sutradhar A, Paulino GH, Gray LJ. On hypersingular surface integrals in the symmetric galerkin boundary element method: application to heat conduction in exponentially graded materials. Int J Numer Meth Eng 2005;62:122–57.
[242] Sutradhar A, Paulino GH, Gray LJ. Symmetric galerkin boundary element method.. Springer-Verlag; 2008.
[243] Symm GT. Integral equation method in potential theory—II. Proc R Soc Ser 1963;A275:33–46.
[244] Tanaka M, Kurokawa K, Matsumoto T. A time-stepping DRBEM for transient heat conduction in anisotropic solids. Eng Anal Bound Elem 2008;32(12): 1046–53.
[245] Tanaka M, Matsumoto T, Takakuwa S. Dual reciprocity BEM for time-stepping approach to the transient heat conduction problem in nonlinear materials. Comp Meth Appl Mech Eng 2006;195(37-40):4953–61.
[246] Tanaka M, Matsumoto T, Suda Y. A dual reciprocity boundary element method applied to the steady-state heat conduction problem of functionally gradient materials. EJBE 2002;1:128–35.
[247] Tang W A generalized approach for transforming domain integrals into boundary integrals in boundary element method. Ph.D. Thesis, Computational Mechanics Institute, Southampton, 1987.
[248] Telles JCF, Brebbia CA. On the application of the boundary element method to plasticity. Appl Math Modell 1979;3:466–70.
[249] Telles JCF, Brebbia CA. The boundary element method in plasticity. Appl Math Modell 1981;5(4):275–81.
[250] Tosecký A. Koleková, Schmid G, Kalinchuk V. Three-dimensional transient half-space dynamics using the dual reciprocity boundary element method. Eng Anal Bound Elem 2008;32(7):597–618.
[251] Van Der Veer P. Calculation method for two-dimensional groundwater flow. Rijkswaterstaat Communications No. 28/1978. Netherlands: Government Publishing Office, The Hague; 1978.
[252] Vera-Tudela CAR, Telles JCF. A numerical Green's function and dual reciprocity BEM method to solve elastodynamic crack problems. Eng Anal Bound Elem 2005;29(3):204–9.
[253] Von Estorff O. Boundary elements in acoustics: advances and applications. WIT Press; 2000.
[254] Wang J, Crouch SL, Mogilevskaya SG. A fast and accurate algorithm for a Galerkin boundary integral method. Comput Mech 2005;37:96–109.
[255] Wang W, Ji X, Tanaka M. A dual reciprocity boundary element approach for the problems of large deflection of thin elastic plates. Comput Mech 2000;26(1):58–65.
[256] Watson JO. Boundary elements from 1960 to the present day. Electron J Bound Elem 2003;1(1):34–46.
[257] Watson JO. Singular boundary elements for the analysis of cracks in plane strain. Int J Numer Meth Eng 1995;38(14):2389–411.
[258] Wendland WL. Asymptotic accuracy and convergence. In: Brebbia CA, editor. Progress In Boundary Element Methods, vol 1. New York: John Wiley; 1981. p. 289–313.
[259] Whitley RJ, Hromadka II TV. Complex logarithms, Cauchy principal values, and the complex variable boundary element method. Appl Math Modell 1994;18(8):423–8.
[260] Whitley RJ, Hromadka II TV. Theoretical developments in the complex variable boundary element method. Eng Anal Bound Elem 2006;30(12): 1020–4.
[261] Wood MA, Ciejka CJ, Hromadka II TV. CVBEM error reduction using the approximate boundary method. 1993;11(3):233–7.
[262] Wrobel LC, Aliabadi MH. The boundary element method: applications in thermo-fluids and acoustics. John Wiley & Sons; 2002.
[263] Wrobel LC, Brebbia CA. Boundary element methods in heat transfer. CMP; 1992.
[264] Wrobel LC, Brebbia CA. The dual reciprocity boundary element formulation for nonlinear diffusion problems. Comp Meth Appl Mech Eng 1987;65(2): 147–64.
[265] Wrobel LC, Brebbia CA, Nardini D. Analysis of transient thermal problems in the BEASY system. In: Connor JJ, Brebbia CA, editors. BETECH 86. Southampton: CMP; 1986.
[266] Wrobel LC, Brebbia CA, Nardini D. The dual reciprocity boundary element formulation for transient heat conduction. In: Sa da Costa A, Melo Baptiste A, Gray WG, Brebbia CA, Pinder GF, editors. Finite elements in water resources VI. Berlin: Springer; 1986.

[267] Wrobel LC, Telles JCF, Brebbia CA. A dual reciprocity boundary element formulation for axisymmetric diffusion problems. In: Brebbia CA, Tanaka M, editors. Boundary elements VIII. Berlin: Springer; 1986.
[268] Xiao J, Tausch J, Cao Y, Wen L. Combined equivalent charge formulations and fast wavelet Galerkin BEM for 3-D electrostatic analysis. Int J Numer Meth Eng 2009;79(6):753–72.
[269] Xiao J, Tausch J, Hu Y. A-posteriori compression of wavelet—BEM matrices. Comput Mech 2009;44(5):705–15.
[270] Xiao J, Zhang D, Wen L. Fully discrete Alpert multiwavelet Galerkin BEM in 2D. Eng Anal Bound Elem 2008;32(2):91–9.
[271] Yamada T, Wrobel LC, Power H. On convergence of dual-reciprocity boundary-element method. Eng Anal Bound Elem 1994;13(3):291–8.
[272] Yeih W, Chen JT, Chen KH, Wong FC. A study on the multiple reciprocity method and complex-valued formulation for the Helmholtz equation. Adv Eng Software 1998;29(1):1–6.
[273] Yiotis AJ, Katsikadelis JT. The boundary element method for nonlinear analysis of shells. Proceedingof the 4th German–Greek–Polish Symposium on Advances on Mechanics. Warsaw-Pultusk; 2001.
[274] Yu GY. A symmetric boundary element method/finite element coupling procedure for two-dimensional elastodynamic problems. J Appl Mech 2003;70(3):451–4.
[275] Yu G, Mansur WJ, Carrer JAM, Gong L. Stability of Galerkin and collocation time domain boundary element methods as applied to the scalar wave equation. Comput Struct 2000;74(4):495–506.
[276] Yuan W, Zhao Z, Lie ST, Yu G. Numerical implementation of the symmetric galerkin boundary element method in 2D elastodynamics. Int J Numer Meth Eng. 2003;58:1049–60.
[277] Zhang Y, Zhu S. On the choice of interpolation functions used in the dual-reciprocity boundary-element method. Eng Anal Bound Elem 1994;13(4): 387–96.