Engineering Analysis with Boundary Elements ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Advanced quadrature methods and splitting extrapolation algorithms for first kind boundary integral equations of Laplace's equation with discontinuity solutions ☆

Jin Huang [a], Zi-Cai Li [b,c,d,*], I-Lin Chen [e], Alexander H.D. Cheng [f]

[a] College of Applied Mathematics, University of Electronic & Science Technology of China, ChengDu, China
[b] Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan
[c] Department of Computer Science and Engineering, National Sun Yat-sen University, Kaohsiung, Taiwan
[d] Department of Applied Mathematics, Chung Hua University, Hsin-Chu, Taiwan
[e] Department of Naval Architecture, National Kaohsiung Marine University, Taiwan
[f] Department of Civil Engineering, University of Mississippi, 203 Carrier Hall, University, MS 38677, USA

ARTICLE INFO

Article history:
Received 28 May 2010
Accepted 15 July 2010

Keywords:
Laplace's equation
Singularity
Discontinuity solutions
First-kind boundary integral equation
Advanced quadrature method
Splitting extrapolation

ABSTRACT

The Laplace equation can be transformed to the first kind boundary integral equations (BIEs), and the corner and discontinuity singularity of Laplace's equation can be studied by the open arcs of the BIEs. For the first kind BIEs, there exist the Galerkin method (GM) and the collocation method (CM); but they suffer in low convergence. The advanced (i.e., the mechanical) quadrature methods (AQMs) and the splitting extrapolation methods (SEMs) originated in [16,17] are proposed in Huang et al. [15] for first kind BIEs with open arcs, to achieve $O(h^3)$ or even $O(h^5)$ convergence, and the excellent stability with Cond. $= O(h^{-1})$, where h is the uniform meshspacing, accompanied with the strict analysis. Moreover, the algorithms of AQMs and the SEMs are simple without any integration computation. Hence the AQMs and the SEMs are superior to the existing methods, such as GM and CM. A challenging discontinuity model of Laplace's equation is proposed in Li et al. [20], and the collocation Trefftz method (CTM) is used to give highly accurate solutions. For the AQMs and the SEMs, the strict theoretical analysis is given in [15], and the discontinuity model [20] is dealt with in this paper, to also achieve highly accurate solutions. The numerical solutions in this paper display that the AQMs and the SEMs are significant not only to the first kind BIEs with the open arc singularity, but also to Laplace's equation with highly strong singularity such as *the discontinuity of solutions*. Moreover, the link of the first kind BIEs with open arcs and the Laplace equation with discontinuity solutions are explored clearly, to display the significance of the proposed algorithms; this paper strengthens the first kind BIEs and its engineering applications.

1. Introduction

The Laplace equation can be transformed to the first kind boundary integral equation (BIE), and the corner and discontinuity singularity of Laplace's equation can be studied by the open arcs of the BIE. However, there exist some difficulties to solve the boundary integral equation (BIE) of first kind, see Cheng and Cheng [10]. To overcome such a difficulty, several numerical methods have been proposed, such as the Galerkin method in Stephan and Wendland [35], Chandler [5] and Sloan and Spence [33], the collocation method in Costabel et al. [11], Elschner and Graham [13] and Yan [38], and the quadrature method in Sidi and Israrli [32], Saranen [28] and Saranen and Sloan [29]. A new quadrature method, called the advanced (also called the mechanical) quadrature methods (AQMs), is developed by Huang and Lü [16,17]. The AQMs have the $O(h^3)$ order of convergence rates, and the excellent stability with Cond. $= O(h^{-1})$, where h is the minimal meshspacing of the quadrature nodes, and Cond. is the traditional 2-norm condition number. Furthermore, the high order $O(h^5)$ can also be achieved for both smooth and singular solutions by the AQMs using the extrapolation and the splitting extrapolation algorithms (SEAs), see [16,17,40,41].

In the indirect boundary element method for the 2-dimensional Dirichlet problem, an unknown harmonic function u in a

☆This work is supported by the National Natural Science Foundation of China (10871034).

* Corresponding author at: Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan.
E-mail addresses: huangjin12345@163.com (J. Huang), zcli@math.nsysu.edu.tw (Z.-C. Li), ilchen@mail.nkmu.edu.tw (I.-L. Chen), acheng@olemiss.edu (A.H.D. Cheng).

0955-7997/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2010.07.004

domain Ω with boundary Γ can be represented as a single-layer potential of the form

$$u(y) = -\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|\,ds_x \quad (y\in\Omega), \tag{1.1}$$

where ds_x is the element of arclength at a point $x\in\Gamma$, $\Gamma=\bigcup_{m=1}^{d}\Gamma_m$ is the open contour or the closed polygonal boundary with edges $\Gamma_m, \Omega=R^2\backslash\Gamma$, and $|x-y|$ is the Euclidean distance. The reformulated problem then becomes the first kind boundary integral equation (BIE) with a logarithmic function as follows (see Sloan and Spence [33,34]):

$$-\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|\,ds_x = f(y), \quad y\in\Gamma, \tag{1.2}$$

or

$$\begin{cases} -\frac{1}{2\pi}\int_{\Gamma} v(x)\ln|x-y|\,ds_x + \omega = f(y) \quad (y\in\Gamma), \\ -\frac{1}{2\pi}\int_{\Gamma} v(x)\,ds_x = b, \end{cases} \tag{1.3}$$

where $f(y)$ is the known value on Γ, and b is a given real number. The function $v(x)=\partial u(x)/\partial \nu^- - \partial u(x)/\partial \nu^+$ and the scalar ω are unknowns to be sought,[1] where ν is the unit outward to Γ. From [1,2,6,18,25,33,34,39], when the logarithmic capacity (transfinite diameter) $C_\Gamma \neq 1$, there exists a unique solution of (1.2) and (1.3). As soon as $v(x)$ is solved from (1.2) and (1.3), the solution $u(y)$ $(y\in\Omega)$ can be calculated by (1.1).

A challenging discontinuity model of Laplace's equation is proposed in Li et al. [20], and the collocation Trefftz method is used to give the highly accurate solutions. In this paper, the AQMs and the SEMs are used to solve such a discontinuity model, to also give the highly accurate solutions. The numerical solutions in this paper display that the AQMs and the SEMs are significant not only to the first kind BIEs with singularity, but also to Laplace's equation with strong singularity (i.e., the discontinuity of solutions). Moreover, the linkage of the first kind BIEs with open arcs and the Laplace equation with discontinuity solutions is explored clearly, to display the significance of the proposed algorithms; this paper strengthens the first kind BIEs (Carstensen and Faermann [4], Schmidlin, et al. [30], and Vodicka [36]) and its engineering applications (Chen et al. [7–9], Martin [24], Poljak and Brebbia [26] and Ma et al. [23]).

This paper is organized as follows: In Section 2, the singularity of the integral kernels and the solutions is removed for the first kind BIEs by transformation. In Section 3, the AQMs and the SEAs are described. In Section 4, the discontinuity model of [20] is solved, and numerical results are reported, to display the significance of the algorithms in this paper. In the last section, a few concluding remarks are addressed.

2. Singularity of integral kernels and solutions

Let $\Gamma=\bigcup_{m=1}^{d}\Gamma_m (d>1)$ be an open contour or a closed polygonal Γ with $C_\Gamma\neq 1$, and $\Gamma_m (m=1,\ldots,d)$ be a piecewise smooth curve. Define the boundary integral operators on Γ_m,

$$(K_{qm}v_m)(y) = -\frac{1}{2\pi}\int_{\Gamma_m} v_m(x)\ln|y-x|\,ds_x, \quad y\in\Gamma_q \quad (m,q=1,\ldots,d). \tag{2.1}$$

Then Eqs. (1.2) and (1.3) can be converted to a matrix operator equation

$$Kv = F, \tag{2.2}$$

where $K=[K_{qm}]_{q,m=1}^{d}$, $v=(v_1(x),\ldots,v_d(x))^T$ and $F=(f_1(y),\ldots,f_d(y))^T$. Assume that Γ_m can be described by the parameter mapping: $x_m(s)=(x_{m1}(s), x_{m2}(s)) : [0,1]\to\Gamma_m$ with $\check{\mu}\geq|x_m'(s)| = [|x_{m1}'(s)|^2+|x_{m2}'(s)|^2]^{1/2}\geq\hat{\mu}>0$. Using the $\sin^p$-transformation [31]

$$s=\varphi_p(t) : [0,1]\to[0,1], \quad p\in N, \tag{2.3}$$

with $\varphi_p(t)=\vartheta_p(t)/\vartheta_p(1)$ and $\vartheta_p(t)=\int_0^t(\sin\pi\tau)^p\,d\tau$, the operators in (2.1) are converted to integral operators on [0,1]. Define

$$(A_{qq}w_q)(t) = \int_0^1 a_{qq}(t,\tau)w_q(\tau)\,d\tau, \quad t\in[0,1], \tag{2.4}$$

and

$$(B_{qm}w_m)(t) = \int_0^1 b_{qm}(t,\tau)w_m(\tau)\,d\tau, \quad t\in[0,1], \tag{2.5}$$

where $a_{qq}(t,\tau) = -(1/2\pi)\ln|2e^{-1/2}\sin\pi(t-\tau)|$, $w_m(t)=v_m(x_m(\varphi_p(t)))|x_m'(\varphi_p(t))|\varphi_p'(t)$,

$$b_{qm}(t,\tau) = \begin{cases} -\frac{1}{2\pi}\ln\left|\frac{x_q(t)-x_q(\tau)}{2e^{-1/2}\sin\pi(t-\tau)}\right| & \text{for } q=m, \\ -\frac{1}{2\pi}\ln|x_q(t)-x_m(\tau)| & \text{for } q\neq m, \end{cases}$$

$x_m(t)=(x_{m1}(\varphi_p(t)), x_{m2}(\varphi_p(t)))$ $(m=1,\ldots,d)$ and $|x_q(t)-x_m(\tau)| = [(x_{q1}(t)-x_{m1}(\tau))^2+(x_{q2}(t)-x_{m2}(\tau))^2]^{1/2}$. Then Eq. (2.2) becomes

$$(A+B)W = G, \tag{2.6}$$

where $A=\mathrm{diag}(A_{11},\ldots,A_{qq})$, $B=[B_{qm}]_{q,m=1}^{d}$, $W=(w_1,\ldots,w_d)^T$, and $G=(g_1,\ldots,g_d)^T$ with $g_m(t)=f_m(x_m(t))$.

Because the operator A_{mm} $(m=1,\ldots,d)$ is an isometry operator [2,3,33,39] from $H^s[0,1]$ to $H^{s+1}[0,1]$ for any real number s, A is also an isometry operator from $(H^s[0,1])^d$ to $(H^{s+1}[0,1])^d$. Hence Eq. (2.6) is equivalent to

$$(E+A^{-1}B)W = A^{-1}G = \tilde{G}. \tag{2.7}$$

Since $\varphi_p(t)\in C^\infty[0,1]$ increases monotonously on [0,1] with $\varphi_p(0)=0$ and $\varphi_p(1)=1$, the solutions of (2.6) are equivalent to those of (2.2), see [31].

Now let us study the solution singularity for (2.2). Denote the corner points $Q_1,\ldots,Q_d$ of Γ. If Γ is an open contour, we have $Q_1\neq Q_d$, where Q_1 and Q_d are endpoints of Γ. If Γ is a closed polygonal boundary, $Q_1=Q_d$. Denote $\chi_m\in[-1,1]$ by $\chi_1=\chi_d=-1$ for the endpoints Q_1 and Q_d, or $(1-\chi_m)\pi=\angle Q_{m-1}Q_mQ_{m+1}$ for the middle corner Q_m. Based on the potential theory [33,39], near the corner or the endpoint Q_m, the solution $v_m(x)=\partial u_m(x)/\partial\nu^- - \partial u_m(x)/\partial\nu^+$ has a singularity $O(|s-s_m|^{\beta_m})$, where $\beta_m = -|\chi_m|/(1+|\chi_m|)\geq-\frac{1}{2}$, and s $(=s_m)$ at Q_m is the arc parameter. If Γ is a closed polygon boundary [2,33,39], however, the solution singularity may become weaker. Since the singularity of v_m results from the singularities of the potential $u(y)$, if $u(y)$ is nonsingular in the exterior region, the singularity of v_m becomes $O(|s-s_m|^{\chi_m/(1-\chi_m)})$. Also if it is nonsingular in the interior region, the singularity of v_m becomes $O(|s-s_m|^{-\chi_m/(1+\chi_m)})$. Note that in the model in [20], the singularity of discontinuity solutions is strongest, and the numerical experiments in Section 4 display that the AQMs are very effective and efficient.

3. Algorithms of AQMs and SEAs

Let $h_m=1/n_m$ $(n_m\in N, m=1,\ldots,d)$ be mesh widths, and $t_j=\tau_j=(j-1/2)h_m$ $(j=1,\ldots,n_m)$ be nodes. By the trapezoidal or the midpoint rule [12] we construct the Nyström's approximate

[1] In [33,34], when $C_\Gamma\neq 1$, Eq. (1.3) can be converted to (1.2) by the parametric transformation. Hence, we only discuss (1.2) in this paper.

Please cite this article as: Huang J, et al. Advanced quadrature methods and splitting extrapolation algorithms for first kind boundary integral equations of Laplace's equation with discontinuity.... Eng Anal Bound Elem (2010), doi:10.1016/j.enganabound.2010.07.004

operator B^h_{qm} of the integral operator B_{qm}, defined by

$$(B^h_{qm}w_m)(t) = h_m \sum_{j=1}^{n_m} b_{qm}(t,\tau_j)w_m(\tau_j), \quad t \in [0,1], \ (q,m=1,\dots,d). \tag{3.1}$$

We have the error bounds [12]

$$(B_{qm}w_m)(t) - (B^h_{qm}w_m)(t) = O(h^{2l}) \quad \text{for } \Gamma_q = \Gamma_m \quad \text{or}$$

$\Gamma_q \cap \Gamma_m = \emptyset, \ l \in N$, and

$$(B_{qm}w_m)(t) - (B^h_{qm}w_m)(t) = O(h^{\omega}) \quad \text{for } \Gamma_q \cap \Gamma_m = Q \in \{Q_m\}, \tag{3.3}$$

where (see [31,15])

$$\omega = \begin{cases} \min((p+1)(\beta_m+1), & p+1), \quad p \text{ odd}, \\ \min((p+1)(\beta_m+1), 2p+2), & p \text{ even}. \end{cases} \tag{3.4}$$

For the logarithmically singular operators A_{mm}, by the quadrature formula [32], we can construct the Fredholm approximate operator A^h_{qq},

$$(A^h_{qq}w_q)(t_i) = -\frac{1}{2\pi}h_q \left\{ \sum_{j=1, t\neq\tau_j}^{n_m} \ln|2e^{-1/2}\sin\pi(t_i-\tau_j)|w_q(\tau_j) + \ln|2\pi e^{-1/2}h_q/(2\pi)|w_q(t_i) \right\}, \quad i=1,\dots,n_q. \tag{3.5}$$

Then we have the error bounds [32]

$$(A^h_{qq}w_q)(t) - (A_{qq}w_q)(t) = \frac{-2}{\pi}\sum_{\mu=1}^{2l-1} \frac{\zeta'(-2\mu)}{(2\mu)!}[w_q(t)]^{(2\mu)}h_q^{2\mu+1} + O(h_q^{2l}), \quad t \in \{t_i\}. \tag{3.6}$$

Setting $t=t_i$ $(i=1,\dots,n_q)$, we obtain the discrete equations of (2.6)

$$(A_h + B_h)W_h = G_h, \tag{3.7}$$

where

$$W_h = (w^h_1(t_1),\dots,w^h_1(t_{n_1}),\dots,w^h_d(t_1),\dots,w^h_d(t_{n_d}))^T,$$

$$A_h = \text{diag}(A^h_{11},\dots,A^h_{dd}), \quad A^h_{qq} = [a_{qq}(t_j,\tau_i)]^{n_q}_{j,i=1},$$

$$B_h = [B^h_{qm}]^d_{q,m=1}, \quad B^h_{qm} = [b_{qm}(t_j,\tau_i)]^{n_q,n_m}_{j,i=1},$$

$$G_h = (g_1(t_1),\dots,g_1(t_{n_1}),\dots,g_d(t_1),\dots,g_d(t_{n_d}))^T,$$

and

$$a_{qq}(t_j,\tau_i) = \begin{cases} -h_q \ln|2e^{-1/2}\sin\pi(t_i-\tau_j)|/(2\pi) & \text{as } i\neq j, \\ -h_q \ln|2\pi e^{-1/2}h_q/(2\pi)|/(2\pi) & \text{as } i=j. \end{cases} \tag{3.8}$$

Obviously, Eq. (3.7) is a system of linear equations with n $(=\Sigma^d_{m=1}n_m)$—unknowns. Once W_h is solved by (3.7), the solution $u(y)$ $(y\in\Omega)$ can be computed by

$$u^h(y) = \frac{-1}{2\pi}\sum_{m=1}^{d}\sum_{i=1}^{n_m} h_m \ln|y-x_m(t_i)||x_m'(t_i)|w^h_m(t_i). \tag{3.9}$$

From (2.4) and (3.8), we have

$$A^h_{mm} = -h_m/\pi \text{ circular } (\ln(e^{-1/2}h_m), \ln(2e^{-1/2}\sin(\pi h_m)), \dots, \ln(2e^{-1/2}\sin((n_m-1)\pi h_m))).$$

We provide a lemma and a theorem, whose proof appears in [15].

Lemma 3.1. *The eigenvalues λ_k $(k=1,\dots,n_m)$ of A^h_{mm} are positive, and there exist two positive constants c_1 and c_2 independent of $h=(h_1,\dots,h_d)^T$ such that $c_2 \ge \lambda_k \ge c_1 h_m$.*

Theorem 3.1. *Let the open contour or the closed polygonal $\Gamma = \bigcup^d_{m=1}\Gamma_m$ satisfy $C_\Gamma \neq 1$, and $\Gamma_m (m=1,\dots,d)$ be smooth curve. Suppose that $f_m \in C^4(\Gamma_m)$. Then when choose an appropriate number p in (2.3) such that $\omega>3$, there exists a vector function $\phi=(\phi_1,\dots,\phi_d)^T \in (C_0[0,1])^d$ independent of $h=(h_1,\dots,h_d)^T$ such that the following multi-parameter asymptotic expansions hold at nodes [15]*

$$W - \hat{W}^h = \text{diag}(h_1^3,\dots,h_d^3)\phi + o(h_0^3), \quad h_0 = \max_{1\le m\le d} h_m. \tag{3.10}$$

By means of the splitting extrapolation algorithms (SEAs) [14,17,21,22,27,37,34,40,41] for the multi-parameter asymptotic expansions (3.10), we can obtain the approximate solutions with a higher order accuracy $o(h^3)$ by solving some coarse grid discrete equations in parallel. The SEAs are described as follows.

Step 1. Choose $h^{(0)}=(h_1,\dots,h_d)$ and $h^{(m)}=(h_1,\dots,h_m/2,\dots,h_d)$, solve (3.7) for $h^{(m)}$ $(m=1,\dots,d)$ in parallel, and then obtain the solutions $W_{h^{(m)}}(t_i)$.

Step 2. Compute the h^3—Richardson extrapolation, based on the numerical solutions on the coarse grids:

$$W^*(t_i) = \frac{8}{7}\left[\sum_{m=1}^{d} W_{h^{(m)}}(t_i) - \left(d-\frac{7}{8}\right)W_{h^{(0)}}(t_i)\right], \tag{3.11}$$

and then obtain $u^h(y)$ $(y\in\Omega\backslash\Gamma)$ from (3.9).

Step 3. From (3.11) we have

$$\begin{aligned}\left|W(t_i) - \frac{1}{d}\sum_{m=1}^{d} W_{h^{(m)}}(t_i)\right| &\le \left|W(t_i) - \frac{8}{7}\left[\sum_{m=1}^{d} W_{h^{(m)}}(t_i) - \left(d-\frac{7}{8}\right)W_{h^{(0)}}(t_i)\right]\right| \\ &\quad + \left(\frac{8d-7}{7}\right)\left|\frac{1}{d}\sum_{m=1}^{d} W_{h^{(m)}}(t_i) - W_{h^{(0)}}(t_i)\right| \\ &\le \left(\frac{8d-7}{7}\right)\left|\frac{1}{d}\sum_{m=1}^{d} W_{h^{(m)}}(t_i) - W_{h^{(0)}}(t_i)\right| + o(h_0^3).\end{aligned} \tag{3.12}$$

Note that the most right side hand of (3.12) provides the a posteriori error estimates.

Suppose that A_h and B_h are discrete matrices defined by (3.1) and (3.5), respectively. Define the condition number

$$\text{Cond.}(K_h) = \frac{\max_{1\le i\le n}|\lambda_i(K_h)|}{\min_{1\le i\le n}|\lambda_i(K_h)|}, \tag{3.13}$$

where $\lambda_i(K_h)$ are the eigenvalues of $K_h = A_h + B_h$. Then there exists the bound

$$\text{Cond. } (K_h) = O(\overline{h}^{-1}), \quad \overline{h} = \min_{1\le m\le d} h_m. \tag{3.14}$$

When $B_h \equiv 0$, Eq. (3.14) is obtained directly from Lemma 3.1. When $B_h \neq 0$, Eq. (3.13) is observed from the numerical results in the next section.

4. Discontinuity model in [20]

We carry out the numerical experiments for singularity problems in Li et al. [20], by the AQMs, the h^3—Richardson extrapolations (REs) and the SEAs, to provide the excellent numerical solutions.

Let $\Gamma = \Sigma^8_{m=1}\Gamma_m$ denote the boundary of the rectangle in Fig. 1, where $\Gamma_1 = \{(x_1,-1): -2\le x_1\le -1\}$, $\Gamma_2 = \{(x_1,-1): -1\le x_1\le 1\}$, $\Gamma_3 = \{(x_1,-1): 1\le x_1\le 2\}$, $\Gamma_4 = \{(2,x_2): -1\le x_2\le 1\}$, $\Gamma_5 = \{(x_1,1): 1\le x_1\le 2\}$, $\Gamma_6 = \{(x_1,1): -1\le x_1\le 1\}$, $\Gamma_7 = \{(x_1,1): -2\le x_1\le -1\}$ and $\Gamma_8 = \{(-2,x_2): -1\le x_2\le 1\}$. The Dirichlet conditions are described as following

$$\begin{cases} u = 500 \text{ on } \Gamma_1 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_7 \cup \Gamma_8, \\ u = 0 \text{ on } \Gamma_2 \cup \Gamma_6. \end{cases} \quad (4.1)$$

This is the new discontinuity model proposed in [20], where the discontinuity singularities of the solutions occur at the points $(\pm 1, \pm 1)$, which are much stronger than the angular singularities $O(r^{\alpha})$, $0 < \alpha < 1$. The solution expansions on the subdomain $S = \{(x_1,x_2) |\ 0 \le x_1 \le 2,\ -1 \le x_2 \le 0\}$ in Fig. 2 are given by

$$w_L = 500(\pi - \theta)/\pi + \sum_{i=1}^{L} c_i r^i \sin i\theta, \quad (4.2)$$

where (r,θ) are the polar coordinates with the origin at $D(-1, 1)$. In the rest of the solution domain in Fig. 1, the solution expansions are obtained from (4.2) by symmetry. The very accurate coefficients with $L = 44$ are provided in [20]. In Table 1 we list the numerical results of the AQMs and the SEAs at $n_m = 256$ $(m = 1,\ldots,8)$, and (4.2). In Table 1 the second column lists the results of AQMs based on (3.7) and (3.9), the third column lists the results of the SEAs based on (3.9) and (3.11), and the fourth column the results from (4.2) by the CTM, where the coefficients c_i are given in [20]. Note that in Table 1, the significant digits of the solutions by the AQMs and the SEAs under lines indicate the same as those by (4.2).

From Table 1 we can see numerically that the solutions by the AQMs and the SEAs have at least 6 and 10 significant digits as the same as those by the CTM [20], respectively.

In Table 2, we list errors, $\partial u^h/\partial y_2 - \partial u/\partial y_2$, near $(1, -1)$ using φ_6, where $\partial u^h/\partial y_2$ is the numerical solutions by the AQMs for

$$\frac{\partial u}{\partial y_2} = -\frac{1}{2\pi}\int_{\Gamma} v(x)\frac{\partial}{\partial y_2}\ln|x-y|\, ds_x ((y_1,y_2) \in \Omega)$$

and the highly accurate derivatives from the CTM [20],

$$\frac{\partial w_L}{\partial y_2} = \frac{\partial}{\partial y_2}\left(500(\pi-\theta)/\pi + \sum_{i=1}^{L} c_i r^i \sin i\theta\right), \quad (4.3)$$

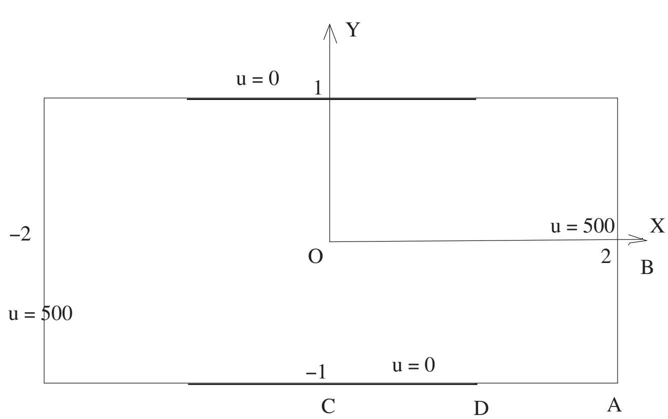

Fig. 1. The solution domain.

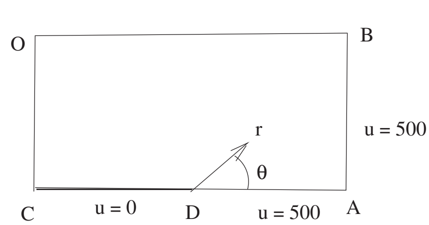

Fig. 2. A subdomain S of Fig. 1.

Table 1
The numerical results using φ_6.

(r,θ)	AQMs	SEAs	CTM [20]
$(10^{10},\pi/7)$	15785.8992349629	15785.8991523623	void
$(10^{10},3\pi/7)$	15785.8992348793	15785.8991522791	void
$(10^{5},2\pi/7)$	7892.94853198447	7892.94849068435	void
$(10,6\pi/7)$	1481.42193361461	1481.42192566293	void
$(1,2\pi/7)$	428.278604101757	428.278608666219	428.278608666166
$(1,5\pi/7)$	150.133575624116	150.133591819418	150.133591819509
$(10^{-1},\pi/7)$	430.707964181915	430.707948681748	430.707948681676
$(10^{-1},2\pi/7)$	360.834274326207	360.834271309190	360.834271309172
$(10^{-1},3\pi/7)$	290.049286939370	290.049297564220	290.049297564266
$(10^{-1},4\pi/7)$	218.348644067850	218.348668597170	218.348668597238
$(10^{-1},5\pi/7)$	145.938541804645	145.938579840071	145.938579840180
$(10^{-1},6\pi/7)$	73.0772501109721	73.0772519714319	73.0772519714259
$(10^{-2},6\pi/7)$	71.6136434702438	71.6136444558542	71.6136444556666
$(10^{-2},5\pi/7)$	143.191989684890	143.191965317376	143.191965318021
$(10^{-2},4\pi/7)$	214.705694173145	214.705651270992	214.705651271404
$(10^{-3},\pi/7)$	428.590246227442	428.5902029 05261	428.590202977022
$(10^{-3},2\pi/7)$	357.176663849000	357.176673866490	357.176673865291
$(10^{-4},2\pi/7)$	357.146231435877	357.1462360 88934	357.146236071379
$(10^{-4},3\pi/7)$	285.718457496044	285.7184989 46133	285.718498924804
$(10^{-4},4\pi/7)$	214.289950407389	214.289927244076	214.289927224960
$(10^{-5},3\pi/7)$	285.714605548281	285.7147070 17629	285.714707023132
$(10^{-5},4\pi/7)$	214.286173505822	214.286135500996	214.286135591847

Table 2
Errors of derivatives $\partial u^h/\partial y_2$ at points (y_1,y_2) near $(1, -1)$ using φ_6.

$(n_1,\ldots,n_8)$	$(1,-1+10^{-1})$	$(1,-1+10^{-2})$	$(1,-1+10^{-3})$	$(1,-1+10^{-4})$
$(2^8,\ldots,2^8)$	8.175E−5	1.022E−3	2.971E−2	8.444E−1
$(2^9,2^8,\ldots,2^8)$	8.134E−5	1.021E−3	2.971E−2	8.444E−1
$(2^8,2^9,\ldots,2^8)$	−7.809E−5	−2.796E−3	−7.610E−2	−2.042
…	1.662E−4	3.943E−3	1.095E−1	2.991
…	8.274E−5	1.023E−3	2.971E−2	8.444E−1
…	7.893E−5	1.019E−3	2.970E−2	8.444E−1
…	8.806E−5	1.028E−3	2.971E−2	8.444E−1
…	8.139E−5	1.021E−3	2.971E−2	8.444E−1
$(2^8,2^8,\ldots,2^9)$	8.182E−5	1.022E−3	2.971E−3	8.444E−1
SEA error	2.469E−10	1.512E−8	1.282E−6	2.708E−4

which can be regarded as the true derivatives due to high accuracy of the CTM. Note that the solution derivatives are unbounded near $(\pm 1, \pm 1)$. In the 3rd–10th rows of Table 2, the errors are computed by replacing h_i by $h_i/2$ on Γ_i. $i = 1,2,\ldots,8$.

We calculate the relative errors of derivatives $\partial u^h/\partial y_1$ near $(1,-1)$ using φ_6. In Tables 3 and 4, we list the relative errors: $er = |(\partial u^h/\partial y_1 - \partial u/\partial y_1)/\partial u/\partial y_1|$ near $(1,-1)$, where $\partial u^h/\partial y_1$ is the numerical solutions by the AQMs for

$$\frac{\partial u}{\partial y_1} = -\frac{1}{2\pi}\int_{\Gamma} v(x)\frac{\partial}{\partial y_1}\ln|x-y|\, ds_x ((y_1,y_2) \in \Omega),$$

and the highly accurate derivatives are obtained from the CTM [20],

$$\frac{\partial w_L}{\partial y_1} = \frac{\partial}{\partial y_1}\left(500(\pi-\theta)/\pi + \sum_{i=1}^{L} c_i r^i \sin i\theta\right). \quad (4.4)$$

In the last row of Tables 3 and 4, the huge derivatives near $(1,-1)$ are listed, which are computed from (4.3) and (4.4) based on the coefficients given in [20]. From Tables 3 and 4 we can see

Table 3
(A) Relative errors of derivatives $\partial u^h/\partial y_1$ at points (y_1, y_2) near $(1,-1)$ using φ_6.

$(n_1,\ldots,n_8)$	$(1+10^{-1},-1)$	$(1+10^{-2},-1)$	$(1+10^{-3},-1)$	$(1+10^{-4},-1)$
$(2^7,\ldots,2^7)$	1.554E−6	4.4986E−6	1.275E−5	3.534E−5
$(2^8, 2^7, \ldots, 2^7)$	1.554E−6	4.498E−6	1.275E−5	3.534E−5
$(2^7, 2^8, \ldots, 2^7)$	6.420E−7	2.055E−6	6.242E−6	1.752E−5
…	1.109E−6	3.005E−6	8.107E−6	2.223E−5
…	1.553E−6	4.498E−6	1.275E−5	3.534E−5
…	1.554E−6	4.498E−6	1.275E−5	3.534E−5
…	1.551E−6	4.498E−6	1.275E−5	3.534E−5
…	1.554E−6	4.498E−6	1.275E−5	3.534E−5
$(2^7, 2^7, \ldots, 2^8)$	1.554E−6	4.498E−6	1.275E−5	3.534E−5
SEA error	1.183E−11	3.081E−11	−2.133E−10	−3.068E−9
$(2^8,\ldots,2^8)$	1.942E−7	5.624E−7	1.595E−6	4.422E−6
$(2^9, 2^8, \ldots, 2^8)$	1.943E−7	5.624E−7	1.595E−6	4.422E−6
$(2^8, 2^9, \ldots, 2^8)$	8.026E−8	2.569E−7	7.806E−7	2.192E−6
…	1.386E−7	3.757E−7	1.014E−6	2.782E−6
…	1.941E−7	5.624E−7	1.595E−6	4.422E−6
…	1.942E−7	5.624E−7	1.595E−6	4.422E−6
…	1.939E−7	5.624E−7	1.595E−6	4.422E−6
…	1.943E−7	5.624E−7	1.595E−6	4.422E−6
$(2^8, 2^8, \ldots, 2^9)$	1.942E−7	5.624E−7	1.595E−6	4.422E−6
SEA error	6.434E−13	3.479E−12	1.391E−11	2.827E−11
True derivatives [20]	1.597E+3	1.591E+4	1.591E+5	1.591E+6

Table 4
(B) Relative errors of derivatives $\partial u/\partial y_1$ at points (y_1, y_2) near $(1,-1)$ using φ_6.

$(n_1,\ldots,n_8)$	$(1+10^{-5},-1)$	$(1+10^{-6},-1)$	$(1+10^{-7},-1)$	$(1+10^{-8},-1)$
$(2^7,\ldots,2^7)$	9.482E−5	3.924E−4	7.080E−4	1.339E−2
$(2^8, 2^7, \ldots, 2^7)$	9.482E−5	3.924E−4	7.080E−4	1.339E−2
$(2^7, 2^8, \ldots, 2^7)$	4.750E−5	1.430E−4	2.802E−4	1.602E−4
…	5.933E−5	2.815E−4	5.151E−4	1.391E−2
…	9.482E−5	3.924E−4	7.080E−4	1.339E−2
…	9.482E−5	3.924E4	7.080E−4	1.339E−2
…	9.482E−5	3.924E−4	7.080E−4	1.339E−2
…	9.482E−5	3.924E−4	7.080E−4	1.339E−2
$(2^7, 2^7, \ldots, 2^8)$	9.482E−5	3.924E−4	7.080E−4	1.339E−2
SEA error	1.824E−7	−1.920E−5	−1.372E−6	−1.133E−3
$(2^8,\ldots,2^8)$	1.206E−5	3.264E−5	9.080E−5	2.248E−4
$(2^9, 2^8, \ldots, 2^8)$	1.206E−5	3.264E−5	9.080E−5	2.248E−4
$(2^8, 2^9, \ldots, 2^8)$	5.999E−6	1.624E−5	4.431E−5	1.561E−4
…	7.576E−6	2.047E−5	5.745E−5	9.790E−5
…	1.206E−5	3.264E−5	9.080E−5	2.248E−4
…	1.206E−5	3.264E−5	9.080E−5	2.248E−4
…	1.206E−5	3.264E−5	9.080E−5	2.248E−4
…	1.206E−5	3.264E−5	9.080E−5	2.248E−4
$(2^8, 2^8, \ldots, 2^9)$	1.206E−5	3.264E−5	9.080E−5	2.248E−4
SEA error	−1.805E−10	−3.182E−9	−4.494E−7	1.279E−6
True derivatives [20]	1.591E+7	1.591E+8	1.591E+9	1.591E+10

numerically

$$\frac{er|_{(2^7,\ldots,2^7)}}{er|_{(2^8,\ldots,2^8)}} \approx 8 \quad \text{and} \quad \frac{er|_{(2^7,\ldots,2^8,\ldots,2^7)}}{er|_{(2^7,\ldots,2^9,\ldots,2^7)}} \approx 8, \tag{4.5}$$

to indicate the optimal convergence rate $O(h^3)$. When $y_1=1+0.1$, $er=O(10^{-13})$, and when $y_1=1+10^{-8}$, $er=O(10^{-6})$. Also from Tables 3 and 4, we can see that when $y_1 \to 1$, the relative errors of derivatives also retain the optimal $O(h^3)$. From Tables 1–4, the accuracy is greatly improved by the SEAs.

Next, we provide in Table 5 the condition number, $|\lambda_{\min}|$ and $|\lambda_{\max}|$ of eigenvalues λ_i $(i=1,\ldots,n)$ for the matrix $K^h=A^h+B^h$. From Table 5, we can see for $k=4,5,6,7$

$$\frac{\text{Cond.}|_{(2^{k+1},\ldots,2^{k+1})}}{\text{Cond.}|_{(2^k,\ldots,2^k)}} \approx 2, \tag{4.6}$$

to indicate (3.14). Eq. (4.6) shows that the AQMs have excellent stability for discontinuity singularity problems! The effective condition number defined in [19] is smaller than the condition number in Table 5; details appear elsewhere. In contrast, for the finite element method (FEM), the finite difference method (FDM), etc., the local refinements for singularity problems in partial differential equations are often used to retain the optimal

Table 5
The condition number using φ_6.

$(n_1,\ldots,n_8)$	$(2^4,\ldots,2^4)$	$(2^5,\ldots,2^5)$	$(2^6,\ldots,2^6)$	$(2^7,\ldots,2^7)$	$(2^8,\ldots,2^8)$
$\lvert\lambda_{\min}\rvert$	3.500E−3	1.799E−3	9.037E−4	4.519E−4	2.260E−4
$\lvert\lambda_{\max}\rvert$	5.028	5.031	5.031	5.031	5.031
Cond.	1.436E+3	2.796E+3	5.566E+3	1.114E+4	2.316E+4

convergence rates, but to greatly increase the condition number. This comparison displays a significance of the AQMs given in this paper.

5. Concluding remarks

To close this paper, let us make a few concluding remarks.

1. For the AQMs to solve singularity problems, there exist the advantages: (a) Evaluation on entries of discrete matrices is very simple and straightforward, without any singular integrals, and (b) the convergence rate is optimal as $O(h^3)$. A strict analysis is provided in [15]. The splitting extrapolation algorithms (SEAs) can be used, to raise again the accuracy of the solutions, see Tables 1–4.

2. The numerical experiments of the discontinuity singularity model of Laplace's equation in [20] are carried out. The errors of solutions and the relative errors of derivatives near singular points computed show the optimal $O(h^3)$. Also, the condition numbers in Table 5 show the optimal $O(h^{-1})$. These excellent numerical results display the significance of the AQMs and the SEMs, not only to the first kind BIEs with open arcs, but also to Laplace's equation with the solution singularity.

3. This paper explores the linkage of numerical solutions of first kind BIEs with numerical PDE. Although such a linkage is given in theory, see [33,34], there exist few numerical examples in the literature. In this paper, the challenging discontinuity model in [20] is studied to strengthen this linkage. Note that even for the corner and the discontinuity singularity, the Cond $= O(h^{-1})$ remains for the AQMs. This is a remarkable advantage over the FEM, FDM, etc. for PDEs. The algorithms and numerical examples in this paper strengthen the first kind BIEs (Carstensen and Faermann [4], Schmidlin et al. [30], and Vodicka [36]) and its engineering applications (Chen et al. [7–9], Martin [24], Poljak and Brebbia [26] and Ma et al. [23]).

Acknowledgement

We are grateful for reviewers for their valuable comments and suggestions.

References

[1] Atkinson K. A discrete Galerkin method for first-kind integral equations with a logarithmic kernel. J Integral Eqns Appl 1988;1:343–63.
[2] Atkinson K. The numerical solution of integral equations of the second kind. Cambridge University Press; 1997.
[3] Atkinson K, Sloan I. The numerical solution of the first kind logarithmic-kernel integral equations on smooth open arcs. Math Comput 1991;56:119–39.
[4] Carstensen C, Faermann B. Mathematical foundation of a posteriori error estimates and adaptive mesh-refining algorithms for boundary integral equations of the first kind. Eng Anal Bound Elem 2001;25:497–509.
[5] Chandler G. Galerkin's method for boundary integral equations on polygonal domains. I Aust Math Soc Ser B,V 1984;26:1–13.
[6] Chandler G. Superconvergent approximations to the solution of a boundary integral equation on polygonal domains. SIAM J Numer Anal 1986;23:1214–29.
[7] Chen JT, Chiu YP. On the pseudo-differential operators in the dual boundary integral equations using degenerate kernels and circulants. Eng Anal Bound Elem 2002;26:41–53.
[8] Chen JT, Hong HK. Review of dual boundary elements with emphasis on hypersingular integrals and divergent series. Appl Mech Rev ASME 1999;52:17–33.
[9] Chen JT, Wu CS, Chen KH. A study of free terms for plate problems in the dual boundary integral equations. Eng Anal Bound Elem 2005;29:435–46.
[10] Cheng AH-D, Cheng DT. Heritage and early development of boundary elements. Eng Anal Bound Elem 2005;29:268–302.
[11] Costabel M, Ervin VJ, Stephan EP. On the convergence of collocation methods for Symm's integral equation on open curves. Math Comput 1988;51: 167–79.
[12] Davis P. Methods of numerical integration. 2nd ed.. New York: Academic Press; 1984.
[13] Elschner J, Graham I. An optimal order collocation method for first kind boundary integral equations on polygons. Numer Math 1995;70:1–31.
[14] Graham I, Qun L, Rui-Feng X. Extrapolation of Nyström solutions of boundary integral equations on non-smooth domains. J Comput Math 1992;10:231–44.
[15] Huang J, Lü T, Li ZC. The mechanical quadrature methods and their splitting extrapolation for boundary integral equations of first kind on open contours. Appl Numer Math 2009;59:2908–22.
[16] Huang J, Lü T. The mechanical quadrature methods and their extrapolations for solving BIEs of Steklov eigenvalue problems. J Comput Math 2004;22: 719–26.
[17] Huang J, Lü T. Splitting extrapolations for solving boundary integral equations of linear elasticity Dirichlet problems on polygons by mechanical quadrature methods. J Comput Math 2006;1:9–18.
[18] Kress R. Linear integral equations. Springer-Verlag; 1989.
[19] Li ZC, Huang TH. Effective condition number for partial differential equations. Numer Linear Algebra Appl 2008;15:575–94.
[20] Li ZC, Lu TT, Hu HY, Cheng HD. Particular solutions of Laplace's equations on polygons and new models involving mild singularities. Eng Anal Bound Elem 2005;29:59–75.
[21] Lin Q, Lü T. Splitting extrapolation for multidimensional problem. J Comput Math 1983;1:376–83.
[22] Lin CB, Lü T, Shih TM. The splitting extrapolation method. World Scientific Singapore; 1995.
[23] Ma H, Yan C, Qin QH. Eigenstrain formulation of boundary integral equations for modeling particle-reinforced composites. Eng Anal Bound Elem 2009;33:410–9.
[24] Martin PA. On connection between boundary integral equations and T-matrix methods. Eng Anal Bound Elem 2003;27:771–7.
[25] Paris F, Cans J. Boundary element method. Oxford University Press; 1997.
[26] Poljak D, Brebbia CA. Indirect Galerkin–Bubnov boundary element method for solving integral equations in electromagnetic. Eng Anal Bound Elem 2004;28:771–7.
[27] Rüde U, Zhou A. Multi-parameter extrapolation methods for boundary integral equations. Adv Comput Math 1988;9:173–90.
[28] Saranen J. The modified quadrature method for logarithmic-kernel integral equations on closed curves. J Integral Eqns Appl 1991;3:575–600.
[29] Saranen J, Sloan I. Quadrature methods for logarithmic-kernel integral equations on closed curves. IMA J Numer Anal 1992;12:167–87.
[30] Schmidlin G, Lage Ch, Schwab Ch. Rapid solution of first kind boundary integral equations in R^3. Eng Anal Bound Elem 2003;27:469–90.
[31] Sidi A. A new variable transformation for numerical integration. Int Ser Numer Math 1993;112:359–73.
[32] Sidi A, Israrli M. Quadrature methods for periodic singular Fredholm integral equation. J Sci Comput 1988;3:201–31.
[33] Sloan IH, Spence A. The Galerkin method for integral equations of first-kind with logarithmic kernel: theory. IMA J Numer Anal 1988;8:105–22.
[34] Sloan IH, Spence A. The Galerkin method for integral equations of first-kind with logarithmic kernel: applications. IMA J Numer Anal 1988;8:123–40.
[35] Stephan EP, Wendland WL. An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. Appl Anal 1984;18:183–219.
[36] Vodicka R. The first-kind and the second-kind boundary integral equation systems for solution of frictionless contact problems. Eng Anal Bound Elem 2000;24:407–26.
[37] Xu YS, Zhao YH. An extrapolation method for a class of boundary integral equation. Math Comput 1990;18:139–54.
[38] Yan Y. The collocation method for first-kind boundary integral equations on polygonal regions. Math Comput 1990;54:139–54.
[39] Yan Y, Sloan I. On integral equation of the first kind with logarithmic kernels. J Integral Eqns Appl 1988;1:517–48.
[40] Yang R, Lu T. Mechanical quadrature methods and their extrapolations for solving boundary integral equations of the conductivity problem with discontinuous media. Eng Anal Bound Elem 2008;32:176–85.
[41] Zhu R, Huang J, Lu T. Mechanical quadrature methods and their extrapolations for solving boundary integral equations of axisymmetric Laplace mixed boundary value problems. Eng Anal Bound Elem 2006;30:391–8.