Engineering Analysis with Boundary Elements 35 (2011) 1060–1074

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Evaluation of free terms in hypersingular boundary integral equations

K. Davey *, A. Farooq

School of Mechanical, Aerospace and Civil Engineering, The University of Manchester, UK

ARTICLE INFO

Article history:
Received 5 November 2009
Accepted 14 April 2011

Keywords:
Free terms
Hypersingular integrals
Limit methods

SUMMARY

The accurate numerical solution of hypersingular boundary integral equations necessitates the precise evaluation of free terms, which are required to counter discontinuous and often unbounded behaviour of hypersingular integrals at a boundary. The common approach for the evaluation of free terms involves integration over a portion of a spherical shaped surface centred at a singularity and allowing the radius of the sphere to tend to zero.

In this paper two alternative methods, which are shape invariant, are proposed and investigated for the determination of free terms. One approach, the *point-limiting method*, involves moving a singularity towards a shrinking integration domain at a faster rate than the domain shrinks. Issues surrounding the choice of approach and shrinkage rates, and path dependency are examined. A related approach, the *boundary-limiting method*, involves moving an invariant but shrinking boundary toward the singularity again at a faster rate than the shrinkage of the domain. The latter method can be viewed as a vanishing exclusion zone approach but the actual boundary shape is used for the boundary of the exclusion zone. Both these methods are shown to provide consistent answers and can be shown to be directly related to the result obtained by moving a singularity towards a boundary, i.e. by comparison with the direct method. Unlike the spherical approach the two methods involve integration over the actual boundary shape and consequently shape dependency is not a concern. A particular highlight of the point limiting approach, as a result of field approximations being restricted to the boundary, is the ability to obtain free terms in a mixed formulation without reference to the underpinning constitutive equations, which is not available to the spherical method.

Focus in the paper is on the 2-D potential equation as this is shown to be sufficient to demonstrate the concepts involved.

1. Introduction

Research into the development and application of hypersingular boundary integral equations has been ongoing over the past decade. The approach presents an alternative to the general solution of thermal, elastostatic and elastodynamic problems. However, it is invariably more costly computationally than the standard integral equation formulation involving kernels with greater complexity and higher-order singularities. The method is often employed in dual formulations in combination with standard integral methods; a common usage is fracture mechanics for the prediction of stress intensity and crack propagation [1–5].

The hypersingular approach is formulated as the sum of free terms and singular integrals incorporating two-point kernels. A number of investigations have postulated the existence of additional free terms [6–9] associated with corners connected to adjacent curved boundary parts. Free terms and associated integrals typically exist in the sense of the Hadamard finite part whose determination involves the creation of a vanishing exclusion zone and asymptotic analysis [10–16]. A reasonable review of the analytical treatments proposed for the evaluation of hypersingular integrals is given in Ref. [9].

The "natural" shape of the exclusion zone is a ε-ball $B_\varepsilon(\mathbf{x})=\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\|_2<\varepsilon\}$ whose boundary is an ε-sphere $S_\varepsilon(\mathbf{x})=\{\mathbf{y}:\|\mathbf{y}-\mathbf{x}\|_2=\varepsilon\}$ oriented by the normal vector $\mathbf{n}(\mathbf{y})$ pointing to the centre $\mathbf{x}$ [10]. However, it is recognised that the Hadamard finite part is not unique and depends on the shape of the vanishing exclusion zone [10,11]. A question that immediately arises: does the non-uniqueness lead to incorrect free terms in the governing integral equation? Shape dependency is a feature of the individual integrals in an integral equation and for these individual integrals the application of the spherical approach can differ with the result obtained using a direct approach [11]. However, despite these differences, integral equations contain combinations of said integrals which are shape independent giving rise to correct free terms. This paper re-examines the issue of shape dependency with the introduction of two new limiting processes for the evaluation of free terms in the hypersingular boundary integral equation for the

* Corresponding author.
E-mail address: keith.davey@manchester.ac.uk (K. Davey).

0955-7997/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2011.04.002

potential problem. One method, denoted the *point-limiting method*, involves moving a singularity towards a shrinking integration domain at a faster rate than the domain shrinks. A particular concern and issue arising with this approach is path dependency. It is shown in the paper that individual integrals are path dependent in the sense that the angle at which a boundary is approached affects the limit. It is proved in this paper that path and shape dependency are intrinsically linked. Because no distortion of the boundary is involved a particular benefit of this approach is that boundary conditions can be directly incorporated into a free term. This option is not available to the spherical method which relies on constitutive relationships and the limiting process to arrive at the correct term. Although not a particular issue for the potential problem the use of constitutive equations in this way can be complex. It is worth highlighting at this point that boundary integral approaches can be viewed in the light of a strong variational method where different types of boundary condition can be approximated independently; a feature that is reinforced with the point-limiting approach.

A related approach, presented in the paper and denoted the *boundary-limiting method*, involves moving an invariant but shrinking boundary toward the singularity again at a faster rate than the shrinkage of the domain. The latter method can be viewed as a vanishing exclusion zone approach but the actual boundary shape is used for the boundary of the exclusion zone. Consistent results are achieved with the two methods.

In order to introduce the new subject matter, basic concepts are considered in Section 2 along with the potential hypersingular integral equation, which in this paper serves as a vehicle for illustrating the concepts involved. In Section 3, singularity annihilation, shape dependence, and the boundary and point-limiting approach are introduced. In Section 4 the theoretical aspects relating to path independence are examined in detail along with issues relating to the rate at which a boundary is approached. It is shown that shape and path independence are intrinsically linked as both properties stem from an integral identity of the form $\int_\Gamma g d\Gamma = 0$. In Sections 5–7 the point-limiting approach is applied to curved and planar boundaries, where it is established that analysis can be restricted to a local tangent plane for the least singular integral in the hypersingular equation. The limiting method is extended to a planar corner in Section 8 and a curved corner in Section 9. In Section 10, limiting exclusion zone methods are discussed along with the boundary-limiting approach. The free terms predicted by the various methods are compared, but also established in Section 10 is the equivalence of the boundary and point-limiting method. In the Section 11 the flexibility of the point limiting method is highlighted with its ability to cater for mixed boundary conditions without recourse to constitutive equations and thus particularly suited to the strong variational method. Finally in Section 12 a number of examples are considered to contrast the method against the direct approach.

2. Basic concept review

Consider the potential problem $\nabla^2 u = 0$ satisfied on the spatial domain Ω, where it is assumed that $u \in C^2(\Omega)$ (or $u \in C^\infty(\Omega)$). The boundary Γ for Ω is oriented by the outward pointing normal $\mathbf{n}$. Consider an arbitrary two-point function $w(\mathbf{x},\mathbf{y}) \in C^\infty$ for $\mathbf{x} \neq \mathbf{y}$ and application of Green's second identity gives

$$\int_\Omega w(\mathbf{x})\nabla^2 u d\Omega - \int_\Omega u \nabla^2 w(\mathbf{x}) d\Omega = \int_\Gamma w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma \qquad (1)$$

where it is understood the that $d\Omega = d\Omega(\mathbf{y})$, $w(\mathbf{x}) = w(\mathbf{x},\mathbf{y})$, etc., with $\mathbf{y}$ excluded for convenience.

Consider the definition $\nabla^2 w = -\delta(\mathbf{x},\mathbf{y})$, where δ is the Dirac delta distribution satisfying $\delta(\mathbf{x},\mathbf{y}) = 0$ if $\mathbf{x} \neq \mathbf{y}$ and $\int_\Omega \delta(\mathbf{x},\mathbf{y}) d\Omega(\mathbf{y}) = 1$ if $\mathbf{x} \in \Omega$. In 2-D, a solution to this equation is Green's function $w = -(2\pi)^{-1}\ln r$, where $r = \|\mathbf{y}-\mathbf{x}\|_2$, which is evidently singular at $\mathbf{x} = \mathbf{y}$. The traditional approach for dealing with a singularity in the domain Ω is to exclude it by creating a vanishing exclusion zone Ω_ε with boundary Γ_ε. Recognising that Green's Theorem applies to multi-connected domains gives

$$\int_{\Omega-\Omega_\varepsilon} w(\mathbf{x})\nabla^2 u d\Omega - \int_{\Omega-\Omega_\varepsilon} u \nabla^2 w(\mathbf{x}) d\Omega = \int_\Gamma w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma + \int_{\Gamma_\varepsilon} w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma - \int_{\Gamma_\varepsilon} u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma \qquad (2)$$

Since $\nabla^2 u = 0$ and $\nabla^2 w = 0$ on $\Omega - \Omega_\varepsilon$ it follows that:

$$\int_{\Gamma_\varepsilon} u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma - \int_{\Gamma_\varepsilon} w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma = \int_\Gamma w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma \qquad (3)$$

which confers shape independency on the expression $w\partial u/\partial n - u\partial w/\partial n$ for potential functions w and u.

The issue of shape dependence is recognised not to be an issue for the integrals appearing in Eq. (3) since these are at worse Cauchy Principal Value singular. Note however that Eq. (3) is of the form $\int_{\Gamma\varepsilon} g d\Gamma = \int_\Gamma g d\Gamma$, where $g = u\partial w/\partial n - w\partial u/\partial n$ with $\underline{n}$ suitably redefined as $-\underline{n}$ on Γ_ε. This demonstrates that the limit $\lim_{\epsilon \to 0} \int_{\Gamma_\varepsilon} g d\Gamma$ does not depend on the shape of Γ_ε. This is an issue that is particularly important for hypersingular integral equations. Any free terms present are generated on the left-hand side of Eq. (3) in the limit $\varepsilon \to 0$. In taking this limit, knowledge about the behaviour of the function u and its normal derivative is required. It is common to assume for hypersingular integral equations that $u \in C^{1,\alpha}$ where $0 < \alpha \le 1$, then $u(\mathbf{y}) = u(\mathbf{x}) + (y_i - x_i)\partial u/\partial y_i(\mathbf{x}) + O(r^\alpha)$ and $\partial u/\partial y_i(\mathbf{y}) = \partial u/\partial y_i(\mathbf{x}) + O(r^{1-\alpha})$ as $r \to 0$. It should be appreciated however that for internal points $u \in \Gamma_\varepsilon$ is simply a restriction denoted $u_{\Gamma\varepsilon}$. Thus $u_{\Gamma\varepsilon}$ takes on the properties of u in Ω, i.e. $u \in C^2(\Omega)$. In principle there is nothing preventing the repeated differentiation of Eq. (2) and it could be argued that $u \in C^\infty$ on Ω; a property transferred to $u_{\Gamma\varepsilon}$.

Since the shape of the Ω_ε is not an issue for the integrals on the left-hand side of Eq. (3); it is commonplace to set $\Omega_\varepsilon(\mathbf{x}) = B_\varepsilon(\mathbf{x}) = \{\mathbf{y} : \|\mathbf{y}-\mathbf{x}\|_2 < \varepsilon\}$. Thus on setting $d\Gamma = \varepsilon d\theta$ gives

$$\int_{\Gamma_\varepsilon} u(\mathbf{y}) \frac{\partial w}{\partial n}(\mathbf{x}) d\Gamma(\mathbf{y}) = \int_0^{2\pi} (u(\mathbf{x}) + O(\varepsilon)) \frac{\partial}{\partial r}\left(\frac{\ln r}{2\pi}\right)\Big|_{r=\varepsilon} \varepsilon d\theta = u(\mathbf{x}) + O(\varepsilon) \qquad (4)$$

and

$$\int_{\Gamma_\varepsilon} w(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma = \int_0^{2\pi} \frac{\ln\varepsilon}{2\pi}\left(\frac{\partial u}{\partial r}(\mathbf{x}) + O(\varepsilon)\right)\varepsilon d\theta = O(\varepsilon \ln\varepsilon) \qquad (5)$$

as $\varepsilon \to 0$.

Although it is usual to take the limit $\varepsilon \to 0$ prior to differentiation of Eq. (3) it is of interest to immediately differentiate Eq. (3) prior to taking the limit, i.e.

$$\int_{\Gamma_\varepsilon} u \frac{\partial w_i}{\partial n}(\mathbf{x}) d\Gamma - \int_{\Gamma_\varepsilon} w_i(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma = \int_\Gamma w_i(\mathbf{x}) \frac{\partial u}{\partial n} d\Gamma - \int_\Gamma u \frac{\partial w_i}{\partial n}(\mathbf{x}) d\Gamma \qquad (6)$$

where

$$w_i = \frac{\partial w}{\partial x_i} = \frac{\partial r}{\partial x_i}\frac{dw}{dr} = \frac{\partial r}{\partial x_i}\frac{d}{dr}\left(-\frac{\ln r}{2\pi}\right) = \left(-\frac{y_i - x_i}{r}\right)\left(-\frac{1}{2\pi r}\right) = \frac{r_i}{2\pi r^2} \qquad (7)$$

and

$$\frac{\partial w_i}{\partial n} = n_j \frac{\partial w_i}{\partial y_j} = n_j \frac{\partial}{\partial y_j}\left(\frac{r_i}{2\pi r^2}\right) = n_j\left(\frac{\delta_{ij} - 2r_i r_j/r^2}{2\pi r^2}\right) \qquad (8)$$

where $r_i = y_i - x_i$.

Note that Eq. (6) immediately confers shape independency on the expression $w_i\partial u/\partial n - u\partial w_i/\partial n$ for potential functions w and u, since it is of the form $\int_{\Gamma\varepsilon} g_i d\Gamma = \int_\Gamma g_i d\Gamma$, where $g_i = w_i\partial u/\partial n - u\partial w_i/\partial n$ with

$\underline{n}$ suitably redefined as $-\underline{n}$ on Γ_ε, which means that integration of this term over an arbitrary closed boundary is invariant and independent of the boundary shape. This does not necessarily infer that the integrals associated with the individual terms $w_i\partial u/\partial n$ and $u\partial w_i/\partial n$ are shape invariant.

3. Singularity annihilation and shape dependence

Consider setting u equal to a constant function in Eq. (6), which reduces to $\int_{\Gamma_\varepsilon}\partial w_i/\partial n d\Gamma = -\int_\Gamma \partial w_i/\partial n d\Gamma$, which consequently confirms that integration of the derivative $\partial w_i/\partial n$ is shape independent. Thus it is permissible to utilise the ε-ball $\Omega_\varepsilon(\mathbf{x})=B_\varepsilon(\mathbf{x})$ with boundary $\Gamma_\varepsilon(\mathbf{x})=S_\varepsilon(\mathbf{x})$ to give

$$\int_{\Gamma_\varepsilon}\frac{\partial w_i}{\partial n}d\Gamma = \int_0^{2\pi} n_j\left(\frac{\delta_{ij}-(2r_ir_j/r^2)}{2\pi r^2}\right)_{r=\varepsilon}\varepsilon d\theta = \int_0^{2\pi}\frac{(n_i-2(r_i/r)(\mathbf{r}\cdot\mathbf{n}/r))_{r=\varepsilon}}{2\pi\varepsilon^2}\varepsilon d\theta = -\frac{1}{2\pi\varepsilon}\int_0^{2\pi} n_i d\theta = 0 \tag{9}$$

It follows that for an internal point $\mathbf{x}$, $u(\mathbf{x})$ does not arise as a free term, since u can be replaced by $u-u_0$ for arbitrary constant u_0 without affect on Eq. (6). Consider further setting $u(\mathbf{y})=u_0+(\mathbf{y}-\mathbf{z})\bullet\nabla u$, where ∇u is a constant vector and z is a fixed coordinate. Substitution into Eq. (6) gives

$$-\int_{\Gamma_\varepsilon}\left[w_i(\mathbf{x})n_j-(y_j-z_j)\frac{\partial w_i}{\partial n}(\mathbf{x})\right]d\Gamma = \int_\Gamma\left[w_i(\mathbf{x})n_j-(y_j-z_j)\frac{\partial w_i}{\partial n}(\mathbf{x})\right]d\Gamma \tag{10}$$

from which it can be deduced that integration on a closed boundary of $w_in_j-(y_j-z_j)\partial w_i/\partial n$ for potential function w, is shape independent. Thus it is permissible to utilise the ε-sphere $\Gamma_\varepsilon(\mathbf{x})=S_\varepsilon(\mathbf{x})$ to give

$$\begin{aligned}-\int_{\Gamma_\varepsilon}&\left[w_i(\mathbf{x})n_j-(y_j-z_j)\frac{\partial w_i}{\partial n}(\mathbf{x})\right]d\Gamma\\ &=\int_0^{2\pi}\left[\frac{n_i}{2\pi\varepsilon}n_j-((y_j-x_j)-(z_j-x_j))\frac{n_i}{2\pi\varepsilon^2}\right]\varepsilon d\theta\\ &=\int_0^{2\pi}\left[\frac{n_i}{2\pi\varepsilon}n_j\pm n_j\frac{n_i}{2\pi\varepsilon}\right]\varepsilon d\theta=\begin{cases}0 & \text{if } \mathbf{x}\notin\overline{\Omega}\\ \delta_{ij} & \text{if } \mathbf{x}\in\Omega\end{cases}\end{aligned} \tag{11}$$

where it is recognised that $y_j-x_j=-\varepsilon n_j$ for $\mathbf{x}\in\Omega$ and $y_j-x_j=\varepsilon n_j$ for $\mathbf{x}\notin\overline{\Omega}=\Omega\cup\Gamma$ giving the result δ_{ij} for an internal point and zero for an external point.

Results (9) and (11) are particularly useful in evaluation and annihilation of unbounded terms in the hypersingular integral equation. However, prior to examining issues surrounding the evaluation of hypersingular integrals on boundary Γ it is useful first to consider evaluation of said integrals on a Lipschitz continuous orientable boundary $\tilde{\Gamma}$ contained in the domain Ω, i.e., $\tilde{\Gamma}\subset\Omega$. It is assumed that $\tilde{\Gamma}$ is the boundary for the open domain $\tilde{\Omega}\subset\Omega$. The field variable u for $u\in\tilde{\Gamma}$ is simply a restriction and denoted $u_{\tilde{\Gamma}}$, which takes on the properties of u in Ω, i.e. $u\in C^2$ (or $u\in C^\infty$). This approach circumvents some of the issues that arise with discontinuous boundary conditions and corners on Γ, which are discussed in Sections 11 and 12. Consider the hypersingular integral equation

$$c_{\tilde{\Omega}}\frac{\partial u}{\partial x_i}(\mathbf{x})+\int_{\tilde{\Gamma}}u\frac{\partial w_i}{\partial n}(\mathbf{x})d\Gamma=\int_{\tilde{\Gamma}}w_i(\mathbf{x})\frac{\partial u}{\partial n}d\Gamma \tag{12}$$

where $c_{\tilde{\Omega}}\in\{0,\ 1\}$ being equal to 1 for $\mathbf{x}\in\tilde{\Omega}$ and 0 for $\mathbf{x}\notin\tilde{\Omega}\cup\tilde{\Gamma}$.

Eq. (12) is defined for $\mathbf{x}\notin\tilde{\Gamma}$ but special provision is required to account for $\mathbf{x}=\mathbf{x}_c$ with $\mathbf{x}_c\in\tilde{\Gamma}$. One approach is to consider the removal of a constant $u(\mathbf{x}_c)=u^c$ and $\partial u/\partial x_j(\mathbf{x}_c)=u_j^c$ times Eq. (6) with $\mathbf{z}=\mathbf{x}_c$, which gives

$$\begin{aligned}c_{\tilde{\Omega}}&\left[\frac{\partial u}{\partial x_i}(\mathbf{x})-\frac{\partial u^c}{\partial x_i}\right]+\int_{\tilde{\Gamma}}\left[u-u^c-(y_j-x_j^c)\frac{\partial u^c}{\partial x_j}\right]\frac{\partial w_i}{\partial n}(\mathbf{x})d\Gamma\\ &=\int_{\tilde{\Gamma}}w_i(\mathrm{x})\left[\frac{\partial u}{\partial n}-\frac{\partial u^c}{\partial x_j}n_j\right]d\Gamma\end{aligned} \tag{13}$$

where repeating suffices of j indicate summation.

Eq. (13) is well defined for $\mathbf{x}\notin\tilde{\Gamma}$ and is in fact identical to Eq. (12) but in addition is well defined at $\mathbf{x}=\mathbf{x}_c\in\tilde{\Gamma}$. This follows because as $\mathbf{x}\to\mathbf{x}_c$ $[u-u^c-(y_j-x_j^c)u_j^c]\partial w_i/\partial n=O(1)$ and $w_i[\partial u/\partial n-u_j^c n_j]=O(1)$. Unfortunately the form of Eq. (13) is not computationally efficient since additional integrals are required to be evaluated for each point $\mathbf{x}_c$. An alternative strategy adopted by many researchers is to consider an exclusion zone $\tilde{\Omega}_\varepsilon$ centred at $\mathbf{x}_c$ of diameter $O(\varepsilon)$ with boundary $\hat{\Gamma}_\varepsilon$ and consider

$$\begin{aligned}c_{\tilde{\Omega}}&\frac{\partial u}{\partial x_i}(\mathbf{x}_c)+\int_{\tilde{\Gamma}-\hat{\Gamma}_\varepsilon}u\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma-\int_{\tilde{\Gamma}-\hat{\Gamma}_\varepsilon}w_i(\mathbf{x}_c)\frac{\partial u}{\partial n}d\Gamma\\ &+\int_{\hat{\Gamma}_\varepsilon}\left[u-u^c-(y_j-x_j^c)\frac{\partial u^c}{\partial x_j}\right]\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma\\ &-\int_{\hat{\Gamma}_\varepsilon}w_i(\mathbf{x}_c)\left[\frac{\partial u}{\partial n}-\frac{\partial u^c}{\partial x_j}n_j\right]d\Gamma-\int_{\hat{\Gamma}_\varepsilon}w_i(\mathbf{x}_c)\frac{\partial u^c}{\partial x_j}n_jd\Gamma\\ &+\int_{\hat{\Gamma}_\varepsilon}(y_j-x_j^c)\frac{\partial u^c}{\partial x_j}\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma+\int_{\hat{\Gamma}_\varepsilon}u^c\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma=0\end{aligned} \tag{14}$$

which is classified as a conversion method in Ref. [9].

It is common practice to define $\tilde{\Omega}_\varepsilon$ to be a fraction of a ε-ball as depicted in Fig. 1 but alternative shapes are of course permissible. The exclusion zone can in fact be selected to include the point $\mathbf{x}_c$ although in this case $c_{\tilde{\Omega}}=1$.

The exclusion-zone procedure of particular interest in this paper involves translating a portion of the original boundary shape of diameter $O(\varepsilon)$ into position at a faster rate (e.g. $O(\varepsilon^\gamma)$ with $\gamma>1$) than the boundary collapses as $\varepsilon\to 0$, as depicted in Fig. 1. This approach is denoted the boundary-limiting method. In the

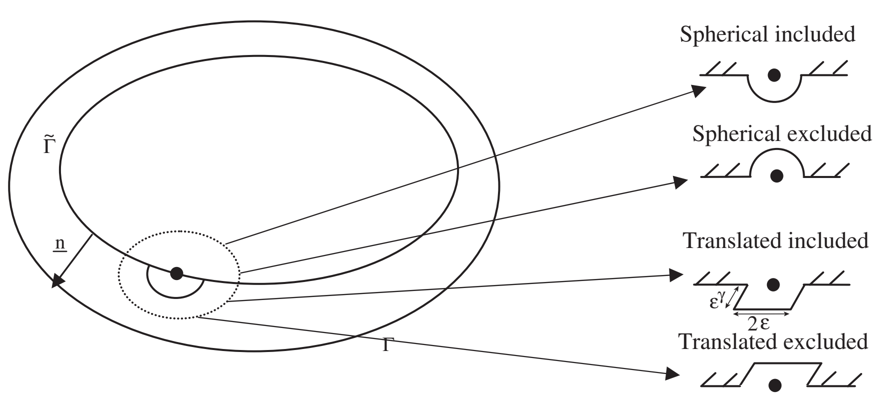

Fig. 1. Spherical and translated exclusion zones.

limit $\varepsilon\to 0$ the following is obtained:

$$c_{\tilde{\Omega}}\frac{\partial u}{\partial x_i}(\mathbf{x}_c)+\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} u\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma-\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} w_i(\mathbf{x}_c)\frac{\partial u}{\partial n}d\Gamma$$
$$-\frac{\partial u^c}{\partial x_j}\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon}\left[w_i(\mathbf{x}_c)n_j-(y_j-x_j^c)\frac{\partial w_i}{\partial n}(\mathbf{x}_c)\right]d\Gamma$$
$$+u^c\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon}\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma=0 \tag{15}$$

where the first term and the last two terms in Eq. (15) generate the free terms.

In addition, in the limit, it is understood that only finite parts are retained and the last term in Eq. (15) is often neglected as it is recognised to have a zero finite part on smooth surfaces [7] (a result reaffirmed in this paper). Observe also that the last two integrands in Eq. (15) are precisely those that that appear in Eqs. (9) and (11). It can be deduced therefore that these integrals are shape independent. The property stems from the identity $\int_{\hat{\Gamma}}gd\Gamma=0$, (usually applies when $x\notin\tilde{\Omega}\cup\tilde{\Gamma}$) which is satisfied by the particular integrands in Eq. (15). For the case of an internal point, $\int_{\tilde{\Gamma}}gd\Gamma=\int_{\hat{\Gamma}_\varepsilon}gd\Gamma$ irrespective of the internal boundary shape $\hat{\Gamma}_\varepsilon$, hence the limit is unique, i.e. $\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon}gd\Gamma=\int\tilde{\Gamma}gd\Gamma$. For the situation $\mathbf{x}_c\in\tilde{\Gamma}$ however the identity $\int_{\hat{\Gamma}}gd\Gamma=0$ gives $\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}gd\Gamma=-\int_{\hat{\Gamma}_\varepsilon}gd\Gamma$. Consider then for a particular ε arbitrary shapes for $\hat{\Gamma}_\varepsilon$ with $\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon$ invariant (in the sense that it is unaffected by a change in $\hat{\Gamma}_\varepsilon$) as depicted in Fig. 2. It is clear that the limit $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}gd\Gamma$ is unaffected by the shape of $\hat{\Gamma}_\varepsilon$ and it immediately follows that $\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon}gd\Gamma$ is shape invariant, i.e. does not depend on the shape of $\hat{\Gamma}_\varepsilon$. If the limits $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}gd\Gamma$ and $\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon}gd\Gamma$ are unbounded, then the asymptotic identity $\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}gd\Gamma\sim\int_{\hat{\Gamma}_\varepsilon}gd\Gamma$ holds as $\varepsilon\to 0$, from which a finite part can be extracted. In the light of shape independence and in view of the relative ease in which integration can be performed on portions of the ε-sphere $S_\varepsilon(\mathbf{x}_c)$ it is readily apparent why the research community generally adopt the ε-sphere approach. The advantage of the boundary limiting method however is that, it can be directly related to the point limiting method which offers the advantage of restricting field approximations to the boundary. Unlike the exclusion strategies the point limiting approach considers an interval $\tilde{\Gamma}_\varepsilon$ centred on $\mathbf{x}_c$ of diameter $O(\varepsilon)$ but contained in $\tilde{\Gamma}$ as depicted in Fig. 3. As illustrated in the figure the point $\mathbf{x}$ can approach $\mathbf{x}_c$ from within or external to the domain $\tilde{\Omega}$. The integral equation of interest is similar in form to Eq. (14), i.e.

$$c_{\tilde{\Omega}}\frac{\partial u}{\partial x_i}(\mathbf{x})+\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} u\frac{\partial w_i}{\partial n}(\mathbf{x})d\Gamma-\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} w_i(\mathbf{x})\frac{\partial u}{\partial n}d\Gamma$$
$$+\int_{\tilde{\Gamma}_\varepsilon}\left[u-u^c-(y_j-x_j^c)\frac{\partial u^c}{\partial x_j}\right]\frac{\partial w_i}{\partial n}(\mathbf{x})d\Gamma$$

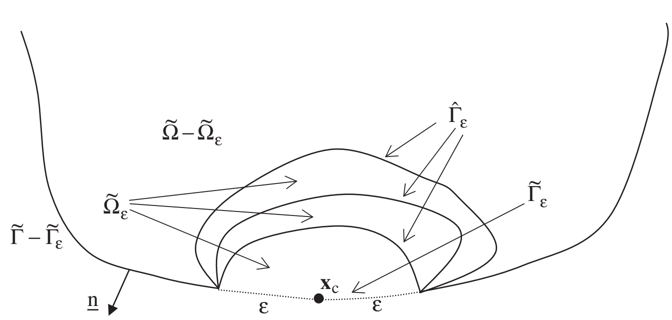

Fig. 2. Various exclusion zones $\tilde{\Omega}_\varepsilon$ where $\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon$ is invariant.

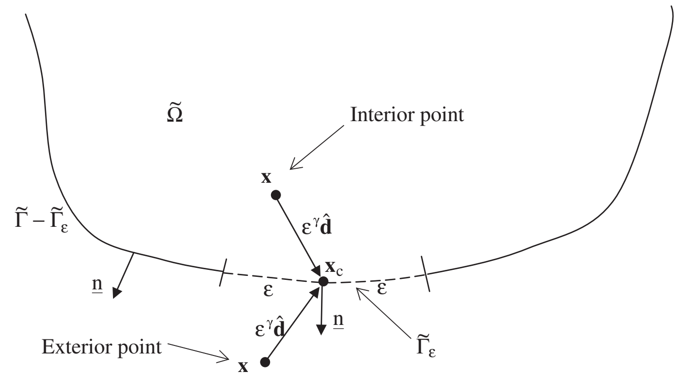

Fig. 3. Point limiting method involves source point movement towards a shrinking boundary $\tilde{\Gamma}_\varepsilon$.

$$-\int_{\tilde{\Gamma}_\varepsilon} w_i(\mathbf{x})\left[\frac{\partial u}{\partial n}-\frac{\partial u^c}{\partial x_j}n_j\right]d\Gamma-\int_{\tilde{\Gamma}_\varepsilon} w_i(\mathbf{x})\frac{\partial u^c}{\partial x_j}n_jd\Gamma$$
$$+\int_{\tilde{\Gamma}_\varepsilon}(y_j-x_j^c)\frac{\partial u^c}{\partial x_j}\frac{\partial w_i}{\partial n}(\mathbf{x})d\Gamma+\int_{\tilde{\Gamma}_\varepsilon}u^c\frac{\partial w_i}{\partial n}(\mathbf{x})d\Gamma=0 \tag{16}$$

although in this case $\mathbf{x}\neq\mathbf{x}_c$ and observe that all field variables are defined on the original unchanged boundary $\tilde{\Gamma}$.

Consider then $\mathbf{x}\to\mathbf{x}_c$ in the limit $\varepsilon\to 0$ which for sufficiently high γ (proved in Section 4) yields

$$c_{\tilde{\Omega}}\frac{\partial u}{\partial x_i}(\mathbf{x}_c)+\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} u\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma-\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} w_i(\mathbf{x}_c)\frac{\partial u}{\partial n}d\Gamma$$
$$-\frac{\partial u^c}{\partial x_j}\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon}\left[w_i(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}})n_j-(y_j-x_j^c)\frac{\partial w_i}{\partial n}(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}})\right]d\Gamma$$
$$+u^c\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon}\frac{\partial w_i}{\partial n}(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}})d\Gamma=0 \tag{17}$$

where $\hat{\mathbf{d}}$ is a unit vector which dictates the manner (i.e. direction) in which the boundary point $\mathbf{x}_c$ is approached by $\mathbf{x}=\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}}$ as $\varepsilon\to 0$.

The unit vector $\hat{\mathbf{d}}$ is assumed to be fixed here although could be made dependent on ε for a curved approach path. Note that it is assumed here that γ is sufficiently high to ensure that

$$\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_c} u\frac{\partial w_i}{\partial n}(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}})d\Gamma=\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_c} u\frac{\partial w_i}{\partial n}(\mathbf{x}_c)d\Gamma \tag{18}$$

and

$$\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_c} w_i(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}})\frac{\partial u}{\partial n}d\Gamma=\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_c} w_i(\mathbf{x}_c)\frac{\partial u}{\partial n}d\Gamma \tag{19}$$

which is a property established in the next section.

The use of Eqs. (18) and (19) is a particular advantage of the approach when contrasted against the direct method [9], as analytical solutions are not required for source points off the boundary. Moreover, these integrals are typically evaluated using analytical or semi-analytical methods as there is a requirement to remove singular terms to obtain a finite part (see Ref. [17] for example). The right-hand integral in Eq. (18) is recognised to be hypersingular however in order to avoid the introduction of unnecessary notation the limits in this paper are assumed to refer to finite parts if unbounded. Observe that limits on the right-hand side of Eqs. (18) and (19) infer path independency on the integrals involved, i.e. the limits are not dependent on $\hat{\mathbf{d}}$.

4. Path dependence

It can be readily shown that shape and path dependency are intrinsically linked. Recall that for singular $g(\mathbf{x},\mathbf{y})$ shape dependency infers that the limit $\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon}g(\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})$ depends on the shape $\hat{\Gamma}_\varepsilon$ (see Fig. 2), which is a property associated with hypersingular integrals. Path dependency on the other hand arises in a situation where the limit $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c+\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})$ for a prescribed $\tilde{\Gamma}_\varepsilon$ (as in Fig. 3) is dependent on $\hat{\mathbf{d}}$.

Theorem 4.1. *If the smooth singular integrand* $g(\mathbf{x},\mathbf{y})$ (*singular for* $\mathbf{x}=\mathbf{y}$) *satisfies the integral identity* $\int_{\tilde{\Gamma}} g(\mathbf{x},\mathbf{y})d\Gamma(\mathbf{y})=0$ for $\mathrm{x}\notin\tilde{\Omega}\cup\tilde{\Gamma}$, *then the limit* $\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon} g(\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})$ *is shape independent and also the limit* $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon} g\left(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y}\right)d\Gamma(\mathbf{y})$ *with* $\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}}\notin\tilde{\Gamma}$ *is path independent for sufficiently high* $\gamma>0$.

Proof. As demonstrated in Section 3 an integrand $g(\mathbf{x},\mathbf{y})$ satisfying the property $\int_{\tilde{\Gamma}} g(\mathbf{x},\mathbf{y})d\Gamma(\mathbf{y})=0$ is shape independent in the sense that $\lim_{\varepsilon\to 0}\int_{\hat{\Gamma}_\varepsilon} g(\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})$ does not depend on the shape of $\hat{\Gamma}_\varepsilon$.

For path independency, consider then the identity $\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})=-\int_{\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})$, which is readily derived from $\int\tilde{\Gamma} g(\mathbf{x},\mathbf{y})d\Gamma(\mathbf{y})=0$ where, as depicted in Fig. 3, $\tilde{\Gamma}_\varepsilon$ is centred on $\mathbf{x}_c$ and $\tilde{\Gamma}_\varepsilon\subset\tilde{\Gamma}$. For sufficiently high $\gamma>0$ it can be shown (shown below) that $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})=\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g\ (\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})$. Path independency immediately follows from this since $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})=-\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})$ and the right-hand limit is independent of $\hat{\mathbf{d}}$.

The issue that a sufficiently high $\gamma>0$ exists to ensure that Eqs. (18) and (19) are satisfied is pivotal to the point limiting approach. However, the precise value of γ is unimportant and not generally required. Consider again the integral $\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}})d\Gamma$, which can be written as

$$\int_{\tilde{\Gamma}-\tilde{\Gamma}_c} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})=\int_{\tilde{\Gamma}-\tilde{\Gamma}_c} g(\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})+\int_{\tilde{\Gamma}-\tilde{\Gamma}_c}(g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})-g(\mathbf{x}_c,\mathbf{y}))d\Gamma(\mathbf{y}) \quad (20)$$

The limit $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})=\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_c,\mathbf{y})d\Gamma(\mathbf{y})$ is achieved if the limit $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_c}(g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})-g(\mathbf{x}_c,\mathbf{y}))d\Gamma(\mathbf{y})=0$. Evidently for $\mathbf{y}\neq\mathbf{x}_c$ and for $\gamma>0$, $\lim_{\varepsilon\to 0}[g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})-g(\mathbf{x}_c,\mathbf{y})]=0$; note from Taylor's Theorem that

$$g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})=g(\mathbf{x}_c,\mathbf{y})-\varepsilon^\gamma\hat{d}_i\frac{\partial g}{\partial x_i}(\mathbf{x}_c-\eta^\gamma\hat{\mathbf{d}},\mathbf{y}) \quad (21)$$

where $0\le\eta\le\varepsilon$ with summation over repeating subscripts.

Note also that $\mathbf{y}-\mathbf{x}=\mathbf{y}-\mathbf{x}_c+\mathbf{x}_c-\mathbf{x}=\mathbf{y}-\mathbf{x}_c+\varepsilon^\gamma\hat{\mathbf{d}}$, $\inf_{\mathrm{y}\in\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}\|\mathbf{y}-\mathbf{x}_c\|_2=O(\varepsilon)$, and the inequalities $\left|\|\mathbf{y}-\mathbf{x}_c\|_2-\varepsilon^\gamma\right|\le\|\mathbf{y}-\mathbf{x}\|_2\le\|\mathbf{y}-\mathbf{x}_c\|_2+\varepsilon^\gamma$; from which it can be deduced for $\gamma>1$ that $\inf_{\mathrm{y}\in\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}\|\mathbf{y}-\mathbf{x}\|_2=O(\varepsilon)$ and it follows, depending on the singular order of g, that $\sup_{\mathrm{y}\in\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}\left|\frac{\partial g}{\partial x_i}(\mathbf{x}_c-\eta^\gamma\hat{\mathbf{d}},\mathbf{y})\right|=O(\varepsilon^{-\beta})$ for $\mathbf{y}\in\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon$, with β independent of γ. Hence, because γ is arbitrary, it can be specified sufficiently high to ensure that the limit $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_c}[g(\mathbf{x}_c-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})-g(\mathbf{x}_c,\mathbf{y})]d\Gamma(\mathbf{y})=0$. □

Theorem 4.1 can be extended to cater for corners where g is discontinuous.

Corollary 4.1. *If* $g(\mathbf{x},\mathbf{y})$ *is a piecewise smooth singular integrand, possessing finite discontinuities at a finite set of points* $\{\mathbf{x}_d\}\in\tilde{\Gamma}$, *satisfies the integral identity* $\int\tilde{\Gamma} g(\mathbf{x},\mathbf{y})d\Gamma(\mathbf{y})=0$ for $\mathrm{x}\notin\tilde{\Omega}\cup\tilde{\Gamma}$, *then the limit* $\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon} g(\mathbf{x}_d-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{y})d\Gamma(\mathbf{y})$ *with* $\mathrm{x}_c-\varepsilon^\gamma\hat{\mathrm{d}}\notin\tilde{\Gamma}$ *is path independent for sufficiently high* $\gamma>0$.

Proof. The proof follows almost identically to that of Theorem 4.1 apart from the representation of $\tilde{\Gamma}_\varepsilon$ in two parts, i.e. $\tilde{\Gamma}_\varepsilon=\tilde{\Gamma}_\varepsilon^\ell\cup\tilde{\Gamma}_\varepsilon^b$ where $\mathbf{x}_d\in\tilde{\Gamma}_\varepsilon^\ell\cap\tilde{\Gamma}_\varepsilon^b$. A Taylor expansion of the form in Eq. (21) applies to each part of $\tilde{\Gamma}_\varepsilon$ taking into consideration the discontinuity at $\mathbf{x}_d$ and the proof follows.

Prior to examining the specific details in applying the point-limiting approach it is of interest to consider the influence of curvature at a point in $\tilde{\Gamma}$. In the theory that follows a smooth boundary point is denoted $\mathbf{x}_c$ whilst a corner boundary point is denoted $\mathbf{x}_d$.

5. Free terms at a smooth boundary point

With reference to Fig. 4 consider the source point $\mathbf{x}$ a distance ε^γ from the point $\mathbf{x}_c$. Let $\mathbf{y}\in\tilde{\Gamma}$ and observe that $\mathbf{r}=\mathbf{y}-\mathbf{x}=\mathbf{y}-\mathbf{x}_c+\mathbf{x}_c-\mathbf{x}$, where $\mathbf{x}_c-\mathbf{x}=\varepsilon^\gamma\hat{\mathbf{d}}$ and consider the asymptotic approximation of $\mathbf{y}(t_1)$ on the boundary at $\mathbf{x}_c$, i.e.

$$\mathrm{y}(t_1)-\mathrm{x}_c=t_1\mathrm{y}'(0)+\frac{t_1^2}{2}\mathbf{y}''(0)+\frac{t_1^3}{6}\mathbf{y}'''(0)+O(t_1^4)=t_1\mathbf{t}_1^c-\kappa_c\frac{t_1^2}{2}\mathbf{n}^c-\frac{t_1^3}{6}(\kappa_c'\mathbf{n}^c+\kappa_c^2\mathbf{t}_1^c)+O(t_1^4) \quad (22)$$

where curvature of $\tilde{\Gamma}$ at $\mathbf{x}_c$ is $\kappa_c=\kappa(0)$, $\kappa_c'=\kappa'(0)$ and $\mathbf{n}^c=\mathbf{t}_2^c$ although for definiteness $\mathbf{n}^c$ is assumed to be in the direction of the outward pointing normal (κ_c is set to $-\kappa_c$ if this is not the case).

It follows that:

$$\mathbf{r}=t_1\mathbf{t}_1^c-\frac{1}{2}\kappa_c t_1^2\mathbf{n}^c-\frac{1}{6}t_1^3(\kappa_c'\mathbf{n}^c+\kappa_c^2\mathbf{t}_1^c)+\varepsilon^\gamma\hat{\mathbf{d}}+O\left(t_1^4\right) \quad (23)$$

and

$$\mathbf{t}_1=\mathbf{t}_1^c-\kappa_c t_1\mathbf{n}^c-\frac{t_1^2}{2}(\kappa_c'\mathbf{n}^c+\kappa_c^2\mathbf{t}_1^c)+O(t_1^3) \quad (24)$$

and

$$\mathbf{n}=\mathbf{n}^c+\kappa_c t_1\mathbf{t}_1^c+\frac{t_1^2}{2}(\kappa_c'\mathbf{t}_1^c-\kappa_c^2\mathbf{n}^c)+O(t_1^3) \quad (25)$$

One of integrals of interest in Eq. (17) is

$$\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon} n_j w_i d\Gamma=\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon} n_j^c w_i d\Gamma+\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}(n_j-n_j^c)w_i d\Gamma=\frac{\partial u^c}{\partial n^c}\int_{\tilde{\Gamma}_\varepsilon} w_i d\Gamma+\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}(n_j-n_j^c)w_i d\Gamma \quad (26)$$

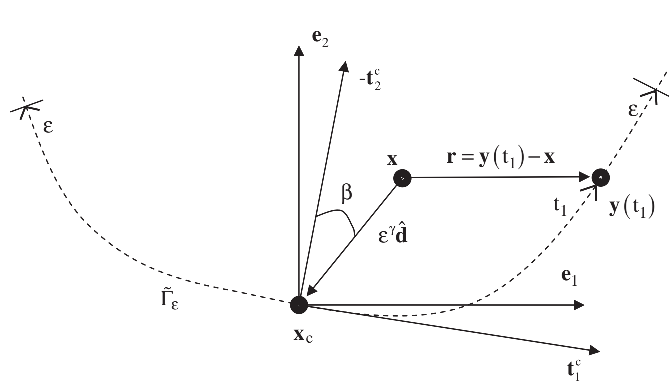

Fig. 4. Tangential coordinate system at a smooth boundary point.

which on application of Eq. (25) gives

$$\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon} n_j w_i d\Gamma \sim \frac{\partial u^c}{\partial n^c}\int_{\tilde{\Gamma}_\varepsilon} w_i d\Gamma + \frac{\partial u^c}{\partial t_1^c}\kappa_c \int_{\tilde{\Gamma}_\varepsilon} t_1 w_i d\Gamma \quad (27)$$

and similarly

$$\int_{\tilde{\Gamma}_\varepsilon} (y_j - x_j^c)\frac{\partial u^c}{\partial x_j}\frac{\partial w_i}{\partial n} d\Gamma \sim \frac{\partial u^c}{\partial t_1^c}\int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial n} d\Gamma - \frac{\kappa_c}{2}\frac{\partial u^c}{\partial n^c}\int_{\tilde{\Gamma}_\varepsilon} t_1^2 \frac{\partial w_i}{\partial n} d\Gamma \quad (28)$$

Note however that the integrals $\int_{\tilde{\Gamma}_\varepsilon} t_1 w_i d\Gamma$ and $\int_{\tilde{\Gamma}_\varepsilon} t_1^2 \partial w_i/\partial n d\Gamma$ are identically zero in the limit $\varepsilon \to 0$. The result follows because $t_1 w_i = O(1)$, $t_1^2 \partial w_i/\partial n = O(1)$ and $dia\{\tilde{\Gamma}_\varepsilon\} = O(\varepsilon)$ as $\varepsilon \to 0$, so $\int_{\tilde{\Gamma}_\varepsilon} t_1 w_i d\Gamma = O(\varepsilon)$ and $\int_{\tilde{\Gamma}_\varepsilon} t_1^2 \partial w_i/\partial n d\Gamma = O(\varepsilon)$. Thus, on subtraction of Eq. (28) from (27) gives

$$\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j w_i - (y_j - x_j^c)\frac{\partial w_i}{\partial n}\right] d\Gamma \sim \frac{\partial u^c}{\partial n^c}\int_{\tilde{\Gamma}_\varepsilon} w_i d\Gamma - \frac{\partial u^c}{\partial t_1^c}\int_{\tilde{\Gamma}_c} t_1 \frac{\partial w_i}{\partial n} d\Gamma \quad (29)$$

Moreover, the approximation $\mathbf{n} \sim \mathbf{n}^c - \kappa_c t_1 \mathbf{t}_1^c$ can be applied to the derivative in the last integral on the right-hand side of Eq. (29) to give

$$\int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial n} d\Gamma \sim \int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial n^c} d\Gamma + \kappa_c \int_{\tilde{\Gamma}_\varepsilon} t_1^2 \frac{\partial w_i}{\partial t_1^c} d\Gamma \sim \int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial n^c} d\Gamma \quad (30)$$

In addition, note the identity $d\tilde{\Gamma}^p = \mathbf{t}_1^c \cdot \mathbf{t}_1 d\tilde{\Gamma}$ where $d\Gamma^p$ is the elemental area measured in the planar tangent at x_c as depicted in Fig. 5. It follows that:

$$d\tilde{\Gamma} = \frac{d\tilde{\Gamma}^p}{\mathbf{t}_1^c \cdot \mathbf{t}_1} = \frac{d\tilde{\Gamma}^p}{\mathbf{t}_1^c \cdot (\mathbf{t}_1^c - \kappa_c t_1 \mathbf{n}^c + O(t_1^2))} = \frac{d\tilde{\Gamma}^p}{(1 + O(t_1^2))} = (1 + O(t_1^2))d\tilde{\Gamma}^p \quad (31)$$

suggesting that the approximation of a curved surface with an appropriate planar surface $\mathbf{y}^p(t_1) - \mathbf{x}_c = t_1 \mathbf{t}_1^c$ with $\mathbf{r}^p = t_1 \mathbf{t}_1^b + \varepsilon^\gamma \hat{\mathbf{d}}$ is possibly sufficient to obtain the correct limiting value.

This suggestion is confirmed on analysis of $w_i - w_i^p$ and $(\partial w_i/\partial n^c) - (\partial w_i^p/\partial n^c)$ where w_i^p and $\partial w_i^p/\partial n^c$ denote w_i and $\partial w_i/\partial n^c$ evaluated on the tangent plane. Note that Eq. (23) gives $\mathbf{r} = \mathbf{r}^p + O(t_1^2)$ which yields $r^2 = (r^p)^2 + O(t_1^2)$ and $r = r^p + O(t_1^2)$. Direct substitution of these expressions into Eqs. (7) and (8) gives $w_i = w_i^p + O(t_1^2)$ and $\partial w_i/\partial n^c = \partial w_i^p/\partial n^c + O(t_1^2)$, which on substitution into Eq. (29) along with Eq. (31) gives

$$\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j w_i - (y_j - x_j^c)\frac{\partial w_i}{\partial n}\right] d\Gamma \sim \frac{\partial u^c}{\partial n^c}\int_{\tilde{\Gamma}_\varepsilon^p} w_i d\Gamma^p - \frac{\partial u^c}{\partial t_1^c}\int_{\tilde{\Gamma}_c^p} t_1 \frac{\partial w_i}{\partial n^c} d\Gamma^p \quad (32)$$

with integration performed on the planar surface denoted $\tilde{\Gamma}_\varepsilon^p$.

The right-hand side of Eq. (32) confirms that at a smooth point that curvature does not affect the left-hand side of this equation and that integration can be performed on a planar tangent at $\mathbf{x}_c$.

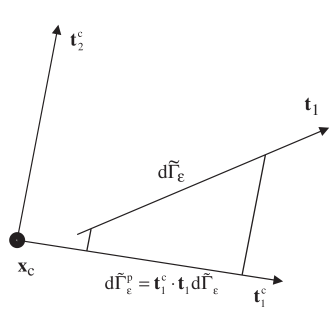

Fig. 5. Elemental areas on tangent plane and boundary.

Consider the application of $\mathbf{n} \sim \mathbf{n}^c + \kappa_c t_1 \mathbf{t}_1^c$ to the other free-term integral appearing in Eq. (17), i.e. $u^c \int_{\tilde{\Gamma}_c} (\partial w_i/\partial n) d\Gamma$, to give

$$u^c \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial w_i}{\partial n}(\mathbf{x}_c - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma \sim u^c \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial w_i}{\partial n^c}(\mathbf{x}_c - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma + u^c \kappa_c \int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial t_1^c}(\mathbf{x}_c - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma \quad (33)$$

indicating that the first integral on the right-hand side of Eq. (33) is potentially unbounded.

The appearance of a curvature term suggests that performing integration on a plane in this case will not yield a sufficiently precise asymptotic approximation. A circular boundary $\mathbf{y}^p(t_1) - \mathbf{x}^c = t_1 \mathbf{t}_1^c - 0.5\kappa_c t_1^2 \mathbf{n}^c$ with $\mathbf{r} = t_1 \mathbf{t}_1^c - 0.5\kappa_c t_1^2 \mathbf{n}^c + \varepsilon^\gamma \hat{\mathbf{d}}$ being of the same order as the singular integrand is sufficient to obtain the correct limiting value up to order-one accuracy. It transpires that the representation $\mathbf{n} = (\mathbf{n} \cdot \mathbf{n}^c)\mathbf{n}^c + (\mathbf{n} \cdot \mathbf{t}_1^c)\mathbf{t}_1^c$ provides for a closed form solution with Eq. (33) replaced by

$$u^c \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial w_i}{\partial n}(\mathbf{x}_c - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma \sim u^c \int_{\tilde{\Gamma}_\varepsilon^p} \mathbf{n} \cdot \mathbf{n}^c \frac{\partial w_i}{\partial n^c}(\mathbf{x}_c - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma^p + u^c \int_{\tilde{\Gamma}_\varepsilon^p} \mathbf{n} \cdot \mathbf{t}_1^c \frac{\partial^2 w_i}{\partial t_1^c}(\mathbf{x}_c - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma^p \quad (34)$$

where $\tilde{\Gamma}_\varepsilon^p$ represents a circular boundary in Eq. (34).

If the surface happens to be planar, then the second integral on the right-hand side of Eq. (34) is identically zero arising from $\mathbf{n} \cdot \mathbf{t}_1^c = 0$.

Proposition 5.1. *At a smooth point $\mathbf{x}_c \in \tilde{\Gamma}_\varepsilon$ the derivative free term can be determined using the point-limiting method with integration performed on the tangent plane at $\mathbf{x}_c$ using Eq. (32).*

Proof. The proof for Proposition 5.1 is performed above but it is important to realise that it is founded on the validity of asymptotic approximation $\mathbf{r} = \mathbf{r}^p + O(t_1^2)$ as $t_1 \to 0$ with $\mathbf{r}^p = t_1 \mathbf{t}_1^b + \varepsilon^\gamma \hat{\mathbf{d}}$. The manner in which the point-limiting method is applied is in the limit $\varepsilon \to 0$ with $t_1 = O(\varepsilon)$ and with γ set sufficiently high but certainly greater than 1. Moreover, the unit vector $\hat{\mathbf{d}}$ does not belong to the tangent plane (typically chosen to be perpendicular in fact), so it follows that $\mathbf{r}^p = O(t_1)$ which confirms the validity of the asymptotic approximation $\mathbf{r} = \mathbf{r}^p + O(t_1^2)$. □

6. General properties at a planar smooth boundary point

This section is concerned with the evaluation of the integrals on the right-hand side of Eqs. (32) and (34) on a planar surface. However, prior to this evaluation it is useful at this stage to obtain relationships between the derivatives in the $\mathbf{e}_i$ directions and those in the $\mathbf{t}_i$ directions as depicted in Fig. 4. Let $\mathbf{t}_1$ be the unit tangent and $\mathbf{t}_2$ be the unit normal which are related to the $\mathbf{e}_i$ direction by $\mathbf{t}_i = \alpha_{ij}\mathbf{e}_j$, where $\alpha_{ij} = \mathbf{t}_i \cdot \mathbf{e}_j$ are the direction cosines for $\mathbf{t}_i$. Note that $\mathbf{dx} = \mathbf{e}_i dx_i = \mathbf{e}_i(\partial x_i/\partial t_j)dt_j$ and $\mathbf{dt} = \mathbf{t}_i dt_i = \mathbf{t}_i(\partial t_i/\partial x_j)dx_j$, and equating these gives $\partial x_i/\partial t_j = \alpha_{ji}$ and $\partial t_i/\partial x_j = \alpha_{ij}$. Thus

$$\frac{\partial w}{\partial x_i} = \frac{\partial w}{\partial t_j}\frac{\partial t_j}{\partial x_i} = \alpha_{ji}\frac{\partial w}{\partial t_j} \quad (35)$$

which can be used to obtain general solutions from solutions involving only tangential and normal derivatives.

To facilitate the evaluation it is expedient to establish some general properties for the derivatives of the radial function w.

Proposition 6.1. *The following properties apply to radial function $w(\mathbf{x},\mathbf{y})$:*

(i) $w(\mathbf{x},\mathbf{x} + \alpha\mathbf{e}_1 + \beta\mathbf{e}_2) = w(\mathbf{x},\mathbf{x} \pm \alpha\mathbf{e}_1 \pm \beta\mathbf{e}_2)$

(ii) $w_1(\mathbf{x},\mathbf{x} + \alpha\mathbf{e}_1 + \beta\mathbf{e}_2) = -w_1(\mathbf{x},\mathbf{x} - \alpha\mathbf{e}_1 + \beta\mathbf{e}_2)$

and
$w_1(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = -w_1(\mathbf{x},\mathbf{x}-\alpha\mathbf{e}_1+\beta\mathbf{e}_2)$

and similarly for w_2.

(iii) $w_{12}(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = -w_{12}(\mathbf{x},\mathbf{x}-\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = -w_{12}(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1-\beta\mathbf{e}_2)$

(iv) $w_{11}(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = w_{11}(\mathbf{x},\mathbf{x}\pm\alpha\mathbf{e}_1\pm\beta\mathbf{e}_2)$ and similarly for w_{22}.

(v) $\int_{-\varepsilon}^{\varepsilon} w_1(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1+\beta\mathbf{e}_2)d\alpha = 0$

(vi) $\int_{-\varepsilon}^{\varepsilon} w_{12}(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1+\beta\mathbf{e}_2)d\alpha = \int_{-\varepsilon}^{\varepsilon} w_{12}(\mathbf{x},\mathbf{x}+\alpha\mathbf{e}_1+\beta\mathbf{e}_2)d\beta = 0.$

with α and β not simultaneously equal to zero and $w_{ij} = \partial^2 w/\partial y_i \partial y_j$

Proof. The proofs are elementary, so are performed in brief.

Property (i) applies because w is a radial function centred on $\mathbf{x}$ and $r = \| \pm\alpha\mathbf{e}_1 \pm \beta\mathbf{e}_2\|_2 = \sqrt{\alpha^2+\beta^2}$. For (ii) $w_1 = \partial w/\partial y_1 = r_1(w'/r)$ and w'/r being a radial function centred on $\mathbf{x}$ has property (i) but $r_1(\mathbf{x},\mathbf{x}\pm\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = \pm\alpha$; hence the result. For (iii) $w_{12} = \partial w_1/\partial y_2 = \partial(r_1 w'/r)/\partial y_2 = r_1 r_2 (w'/r)'$ where $(w'/r)'$ being radial has property (i) and the result flows from $r_1(\mathbf{x},\mathbf{x}\pm\alpha\mathbf{e}_1\pm\beta\mathbf{e}_2)r_2(\mathbf{x},\mathbf{x}\pm\alpha\mathbf{e}_1\pm\beta\mathbf{e}_2) = \pm\alpha\times\pm\beta$. Similarly for (iv) $w_{11} = \partial w_1/\partial y_1 = \partial(r_1(w'/r))/\partial y_1 = w'/r + r_1^2(w'/r)'$ and w'/r and $r_1^2(w'/r)'$ possess property (i) and $r_1^2(\mathbf{x},\mathbf{x}\pm\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = \alpha^2$ and the result follows. Properties (v) and (vi) follow trivially from (ii) and (iii) and represent the behaviour of an asymmetric (odd) function. □

Note that, Proposition 6.1 also applies with $\mathbf{e}_i$ replaced by $\mathbf{t}_i$ along with the corresponding changes in the direction of the derivatives. This proposition along with Theorem 4.1 allows for the determination of general properties and simplification of the integrals found in Eqs. (32) and (34). Theorem 4.1 confers path independence on the integrals on the left-hand side of Eqs. (32) and (34) and the question arises whether the integrals on the right-hand side are also path independent. One possible means of determining this is by direct evaluation for an arbitrary direction vector $\hat{\mathbf{d}}$ as depicted in Fig. 4. However, this can be avoided for Eq. (32) or equivalently Eq. (29) by direct substitution of $\partial u/\partial n^c = n_j^c(\partial u/\partial x_j)$ and $\partial u/\partial t_1^c = \alpha_{1j}^c(\partial u/\partial x_j)$ to give

$$\int_{\tilde{\Gamma}_\varepsilon}\left[n_j w_i - (y_j - x_j^c)\frac{\partial w_i}{\partial n}\right]d\Gamma \sim n_j^c \int_{\tilde{\Gamma}_\varepsilon} w_i d\Gamma - \alpha_{1j}^c \int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial n} d\Gamma \tag{36}$$

where the right-hand side is a somewhat expected asymptotic approximation for the left although obtained in a somewhat indirect manner.

Multiplication of Eq. (36) by n_j^c (summed over j) gives

$$\int_{\tilde{\Gamma}_\varepsilon} w_i d\Gamma \sim n_j^c \int_{\tilde{\Gamma}_\varepsilon}\left[n_j w_i - (y_j - x_j^c)\frac{\partial w_i}{\partial n}\right]d\Gamma \tag{37}$$

and similarly for α_{1j}^c gives

$$\int_{\tilde{\Gamma}_\varepsilon} t_1 \frac{\partial w_i}{\partial n} d\Gamma \sim \alpha_{1j}^c \int_{\tilde{\Gamma}_\varepsilon}\left[n_j w_i - (y_j - x_j^c)\frac{\partial w_i}{\partial n}\right]d\Gamma \tag{38}$$

where it is immediately deduced that the integrals on the left-hand side of Eqs. (37) and (38) are path independent as a direct consequence of the right-hand sides being formed by the linear combination of path independent integrals.

On a planar surface Eq. (34) reduces to one integral which by default is path independent.

Proposition 6.2. *For a smooth surface the integrals appearing on the right-hand sides of* Eq. (32) *are path independent.*

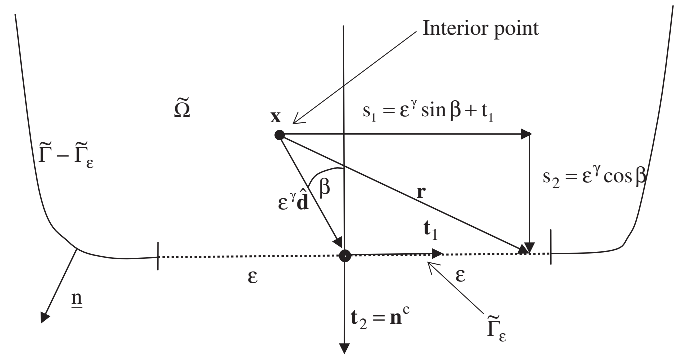

Fig. 6. Integration on a tangent plane.

6.1. Simplification of integrals using Proposition 6.1

Prior to considering the right-hand side of Eqs. (32) and (34) for an arbitrary direction derivative $\mathbf{e}_i$ it is insightful to consider the situations where $\mathbf{e}_i = \mathbf{t}_1^c$ as well as $\mathbf{e}_i = \mathbf{n}^c = \mathbf{t}_2^c$. It is also convenient to select $\hat{\mathbf{d}}$ to be perpendicular to the tangent plane $\tilde{\Gamma}_\varepsilon^p$ (i.e. $\hat{\mathbf{d}} = \mathbf{t}_2^c$) which is permissible since the integrals are path independent. Consider first $\mathbf{e}_i = \mathbf{t}_1^c$ and Proposition 6.1 with point on the tangent plane $\mathbf{y}^p(t_1) = \mathbf{x} + t_1\mathbf{t}_1^c + \varepsilon^\gamma \mathbf{t}_2^c$ (see Fig. 6); then Eqs. (32) and (34) become

$$\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j \frac{\partial w}{\partial t_1} - (y_j - x_j^c)\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial t_1}\right)\right]d\Gamma \sim -\frac{\partial u^c}{\partial t_1^c}\int_{-\varepsilon}^{\varepsilon} t_1 \frac{\partial}{\partial t_2}\left(\frac{\partial w}{\partial t_1}\right)dt_1 \tag{39}$$

and

$$u^c \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial}{\partial t_2}\frac{\partial w}{\partial t_1} d\Gamma \sim 0 \tag{40}$$

confirming that at a smooth point, in the case of a tangential derivative, a free term contribution arises for the tangential derivative term only.

Similarly for $\mathbf{e}_i = \mathbf{n}^c = \mathbf{t}_2^c$

$$\frac{\partial u^c}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j \frac{\partial w}{\partial n^c} - (y_j - x_j^c)\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial n^c}\right)\right]d\Gamma \sim \frac{\partial u^c}{\partial n^c}\int_{-\varepsilon}^{\varepsilon} \frac{\partial w}{\partial t_2} dt_1 \tag{41}$$

and

$$u^c \int_{\tilde{\Gamma}_\varepsilon} \frac{\partial}{\partial n}\left(\frac{\partial w}{\partial n^c}\right) d\Gamma \sim u^c \int_{-\varepsilon}^{\varepsilon} \frac{\partial}{\partial t_2}\left(\frac{\partial w}{\partial t_2}\right) dt_1 \tag{42}$$

The integrands on the right-hand side of Eqs. (39), (41) and (42) are even, permitting the doubling of the integrand and the integration being performed over the domain [0, ε] rather than [−ε, ε].

6.2. Integral evaluation for point limiting method

This section is concerned with the evaluation of the integrals that appear on the right-hand side of Eqs. (39)–(42). To achieve this, consider the point $\mathbf{x}$ approaching $\mathbf{x}_c$ as depicted in Fig. 6 where the approach is at an angle β to the surface normal. It is recognised of course from Theorem 4.1 that the angle β cannot influence the outcome here although it is of interest to demonstrate this explicitly. However, to demonstrate the relative ease of this approach β is initially set to zero. In this case $r^2 = s_2^2 + t_1^2$ where $s_2 = \varepsilon^\gamma$ on the tangent plane $\tilde{\Gamma}_\varepsilon^p$ and

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial w}{\partial t_2} dt_1 = \int_{-\varepsilon}^{\varepsilon} \frac{dw}{dr}\frac{\partial r}{\partial t_2} dt_1 = \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{s_2}{r^2} dt_1 = \frac{1}{\pi}\int_0^{\varepsilon} \frac{s_2}{s_2^2 + t_1^2} dt_1$$
$$= \frac{1}{\pi}\int_0^{\varepsilon/s_2} \frac{1}{1+s^2} ds = \frac{\tan^{-1}(\varepsilon^{1-\gamma})}{\pi} \tag{43}$$

where for $\gamma > 1$, in the limit $\varepsilon \to 0$, the right-hand side of Eq. (41) yields 0.5.

Recall that the parameter γ is required to be set above a certain value (whose existence is assured by Theorem 4.1) to obtain a unique limit for the integral, which is achieved here with $\gamma > 1$. In a similar fashion

$$\int_{-\varepsilon}^{\varepsilon} t_1 \frac{\partial}{\partial t_2}\left(\frac{\partial w}{\partial t_1}\right) dt_1 = \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} t_1 \frac{\partial}{\partial t_2}\left(\frac{t_1}{r^2}\right) dt_1 = -\frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \frac{t_1^2 s_2}{r^4} dt_1$$
$$= -\frac{2}{\pi}\int_0^{\varepsilon} \frac{t_1^2 s_2}{s_2^2+t_1^2} dt_1 = -\frac{2}{\pi}\int_0^{\varepsilon/s_2} \frac{s^2}{(1+s^2)^4} ds$$
$$= -\frac{1}{\pi}\left(\tan^{-1}(\varepsilon^{1-\gamma}) - \frac{\varepsilon^{1-\gamma}}{1+\varepsilon^{2(1-\gamma)}}\right) \tag{44}$$

where for $\gamma > 1$, in the limit $\varepsilon \to 0$, the right-hand side of Eq. (39) yields -0.5.

Repeating the process for the integral in Eq. (42) gives

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial}{\partial t_2}\left(\frac{\partial w}{\partial t_2}\right) dt_1 = \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \frac{\partial}{\partial t_2}\left(\frac{s_2}{r^2}\right) dt_1 = \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \frac{t_1^2 - s_2^2}{r^4} dt_1$$
$$= \frac{2}{\pi}\int_0^{\varepsilon} \frac{t_1^2 - s_2^2}{s_2^2+t_1^2} dt_1 = \frac{2}{\pi}\int_0^{\varepsilon/s_2} \frac{s^2-1}{(1+s^2)^4} ds$$
$$= -\frac{2}{\pi}\frac{s}{1+s^2}\bigg|_0^{\varepsilon/s_2} = -\frac{2}{\pi}\frac{\varepsilon^{1-\gamma}}{1+\varepsilon^{2(1-\gamma)}} \tag{45}$$

where for $\gamma > 1$, in the limit $\varepsilon \to 0$, the right-hand side of Eq. (42) yields no contribution.

Utilisation of Eq. (35) gives

$$\frac{\partial u^c}{\partial x_j}\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j\frac{\partial w}{\partial x_i} - (y_j - x_j^c)\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial x_i}\right)\right] d\Gamma = \frac{1}{2}\left(\frac{\partial u^c}{\partial n^c}\alpha_{2i} + \frac{\partial u^c}{\partial t_1^c}\alpha_{1i}\right) \tag{46}$$

and

$$u^c \lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon} \frac{\partial}{\partial n}\left(\frac{\partial w}{\partial x_i}\right) d\Gamma \sim 0 \tag{47}$$

The right-hand side of Eq. (46) is equal to $0.5(\partial u^c/\partial x_i)$, which is as expected since the free term external and internal to $\tilde{\Omega}$ is zero and $\partial u^c/\partial x_i$, respectively. It is of interest to explicitly confirm the path independence of the integrals appearing on the right-hand side of Eqs. (32) and (34). The integrals are presented in Appendix I, where the limiting values for Eqs. (43)–(45) are obtained with $\beta \neq 0$, thus demonstrating explicitly path independence for these integrals.

7. Point limiting method at a smooth boundary point

The analysis pertaining to Eq. (32) performed on a planar surface is applicable to an arbitrary smooth surface as a consequence of Proposition 5.1. However, this is not the case for Eq. (34) because the severity of the singularity requires a more accurate description of the surface geometry. It is of interest to note from Eq. (22) that $\mathbf{y}^p(t_1) - \mathbf{x}^c = t_1\mathbf{t}_1^c - 0.5\kappa_c t_1^2 \mathbf{n}^c$ represents the next order of approximation and is a constant curvature geometry which is of course commonly used in the vanishing exclusion method. It is reasonable to question what order of accuracy is required and this is linked to achieving up to order-one accuracy for the integral $\int_{\tilde{\Gamma}_\varepsilon} g(\mathbf{x} - \varepsilon^\gamma \hat{\mathbf{d}}) d\Gamma(\mathbf{y})$ as $\varepsilon \to 0$. The representation of this integral by $\int_{\tilde{\Gamma}_\varepsilon^p} g(\mathbf{x} - \varepsilon^\gamma \hat{\mathbf{d}}, \mathbf{y}) d\Gamma(\mathbf{y}^p)$ is sufficient to order-one accuracy if for example $g \sim \varepsilon^{-2}$ and $d\Gamma - d\Gamma^p = O(\varepsilon^3)$ as $\varepsilon \to 0$, as is the case for Eq. (34).

As for the planar case consider the situations where $\mathbf{e}_i = \mathbf{t}_1^c$ as well as $\mathbf{e}_i = \mathbf{n}^c = \mathbf{t}_2^c$ with $\hat{\mathbf{d}}$ perpendicular to the tangent plane (i.e. $\hat{\mathbf{d}} = \mathbf{t}_2^c$). With reference to Fig. 7 it can be seen that $s_2 = \kappa_c^{-1}(\cos\theta - 1) + \varepsilon^\gamma$, $s_1 = \kappa_c^{-1}\sin\theta$, $\mathbf{n}\cdot\mathbf{t}_1^c = \sin\theta$, $\mathbf{n}\cdot\mathbf{t}_2^c = \cos\theta$ and $r^2 = s_2^2 + s_1^2$; Eq. (34) for $\mathbf{e}_i = \mathbf{t}_1^c$ becomes

$$u^d \int_\Gamma \frac{\partial^2 w}{\partial n \partial t_1^c} d\Gamma \sim -\frac{1}{2\kappa_c \pi}\int_{-\kappa_c\varepsilon}^{\kappa_c\varepsilon} \cos\theta \frac{s_1 s_2}{(s_1^2+s_2^2)^2} d\theta + \frac{1}{\pi\kappa_c}\int_{-\kappa_c\varepsilon}^{\kappa_c\varepsilon} \sin\theta \frac{s_2^2 - s_1^2}{(s_1^2+s_2^2)^2} d\theta = 0 \tag{48}$$

which is identically zero because the integrands are odd functions.

Similarly for $\mathbf{e}_i = \mathbf{t}_2^c$ Eq. (34) gives

$$u^d \int_\Gamma \frac{\partial^2 w}{\partial n \partial t_2^c} d\Gamma \sim \frac{1}{2\pi\kappa_c}\int_{-\kappa_c\varepsilon}^{\kappa_c\varepsilon} \cos\theta \frac{s_1^2 - s_2^2}{(s_1^2+s_2^2)^2} d\theta - \frac{1}{\pi\kappa_c}\int_{-\kappa_c\varepsilon}^{\kappa_c\varepsilon} \sin\theta \frac{s_1 s_2}{(s_1^2+s_2^2)^2} d\theta$$
$$= -\frac{2}{\pi\kappa_c}\frac{\tan\left(\frac{\varepsilon\kappa_c}{2}\right)}{\varepsilon^{2\gamma}\left(1+\tan\left(\frac{\varepsilon\kappa_c}{2}\right)\right) - 4\kappa_c^{-1}\varepsilon^\gamma \tan^2\left(\frac{\varepsilon\kappa_c}{2}\right) + 4\kappa_c^{-2}\tan^2\left(\frac{\varepsilon\kappa_c}{2}\right)} \sim -\frac{1}{\pi}\varepsilon^{-1} \tag{49}$$

which has zero finite part and hence confirming that there is no free term associated with u at a smooth boundary point.

8. Point limiting method at a planar corner

Consider next the point $\mathbf{x}$ approaching a corner point $\mathbf{x}_d$ as depicted in Fig. 8 where the angle of the corner is 2α. The evaluation of a free term at a corner does not significantly change the proposed methodology and many of the results discussed to this point apply equally to a corner. The significant change is that the free-term integrals in Eq. (17) evaluated on $\tilde{\Gamma}_\varepsilon$ are required to be determined as the sum of contributions from $\tilde{\Gamma}_\varepsilon^\ell$ and $\tilde{\Gamma}_\varepsilon^b$ where $\tilde{\Gamma}_\varepsilon = \tilde{\Gamma}_\varepsilon^\ell \cup \tilde{\Gamma}_\varepsilon^b$. In addition the tangent and normal vectors at $\mathbf{x}_d$ are not uniquely defined although it is advantageous to utilise

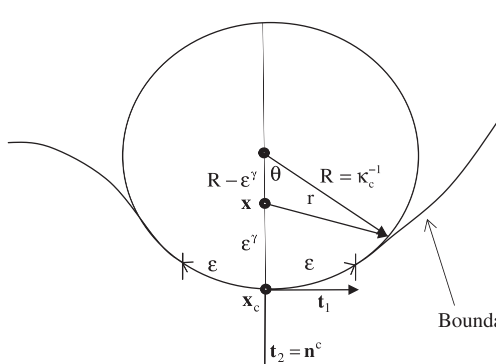

Fig. 7. Integration over a smooth boundary with local curvature κ_c.

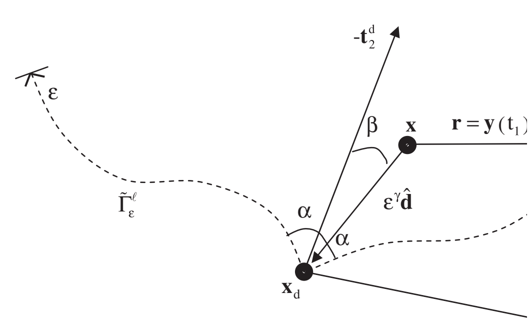

Fig. 8. Point limiting at a corner boundary point.

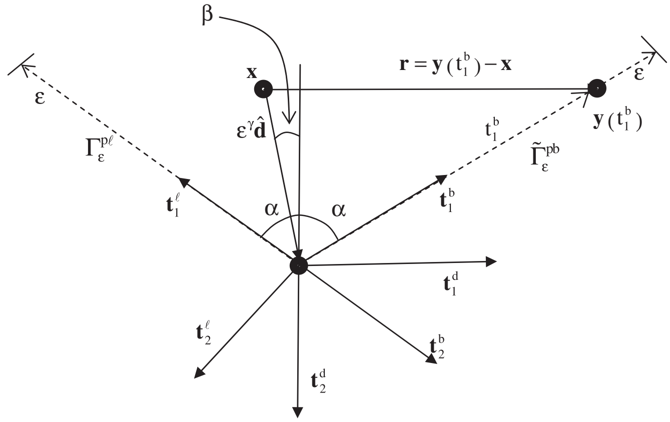

Fig. 9. Coordinate systems at a corner point.

symmetry to stipulate these as $\{\mathbf{t}_1^d,\mathbf{t}_2^d\}$ as depicted in Fig. 9. Limiting values of tangent vectors are assumed to exist for $\tilde{\Gamma}_\varepsilon^\ell$ and $\tilde{\Gamma}_\varepsilon^b$ at $\mathbf{x}_d$ and these are denoted as $\{\mathbf{t}_1^\ell,\mathbf{t}_2^\ell\}$ and $\{\mathbf{t}_1^b,\mathbf{t}_2^b\}$. Thus, along with $\{e_1,e_2\}$, a corner point has four coordinate frames associated with it; hence things are somewhat more involved. Theorem 4.1 is applicable although it is important to appreciate that $\mathbf{x}_d$ must lie internal to the boundary $\tilde{\Gamma}_\varepsilon$ to ensure that $\inf_{y\in\tilde{\Gamma}-\tilde{\Gamma}_\varepsilon}\|\mathbf{y}-\mathbf{x}\|_2 = O(\varepsilon)$, which would not occur if $\mathbf{x}_d$ was at the edge of $\tilde{\Gamma}_\varepsilon$. In addition, the approximations on a tangent plane discussed in Section 5 apply equally to a corner where Eqs. (32) and (34) give

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon^\ell}\left[n_j w_i-(y_j-x_j^d)\frac{\partial w_i}{\partial n}\right]d\Gamma \sim \frac{\partial u^d}{\partial t_2^\ell}\int_{\tilde{\Gamma}_\varepsilon^{p\ell}} w_i d\Gamma^p-\frac{\partial u^d}{\partial t_1^\ell}\int_{\tilde{\Gamma}_\varepsilon^{p\ell}} t_1^\ell\frac{\partial w_i}{\partial t_2^\ell}d\Gamma^p \tag{50}$$

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon^b}\left[n_j w_i-(y_j-x_j^d)\frac{\partial w_i}{\partial n}\right]d\Gamma \sim \frac{\partial u^d}{\partial t_2^b}\int_{\tilde{\Gamma}_\varepsilon^{pb}} w_i d\Gamma^p-\frac{\partial u^d}{\partial t_1^b}\int_{\tilde{\Gamma}_\varepsilon^{pb}} t_1^b\frac{\partial w_i}{\partial t_2^b}d\Gamma^p \tag{51}$$

and

$$u^d\int_{\tilde{\Gamma}_\varepsilon^\ell}\frac{\partial w_i}{\partial n}d\Gamma \sim u^d\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial w_i}{\partial t_2^\ell}d\Gamma^p \tag{52}$$

$$u^d\int_{\tilde{\Gamma}_\varepsilon^b}\frac{\partial w_i}{\partial n}d\Gamma \sim u^d\int_{\tilde{\Gamma}_\varepsilon^{pb}}\frac{\partial w_i}{\partial t_2^b}d\Gamma^p \tag{53}$$

where for convenience $\mathbf{t}_2^\ell$ and $\mathbf{t}_2^b$ are assumed to be outward pointing and where the free terms are obtained on addition of Eqs. (50)–(51) and Eqs. (52)–(53).

Note that the superscripts $p\ell$ and pb denote the two tangent planes at $\mathbf{x}_d$ whilst κ_d^ℓ and κ_d^ℓ are the limiting curvatures for $\tilde{\Gamma}_\varepsilon^\ell$ and $\tilde{\Gamma}_\varepsilon^b$ evaluated at $\mathbf{x}_d$. The various tangent systems are depicted in Fig. 9 along with the corner angle 2α and the approach angle β measure between $\hat{\mathbf{d}}$ and $\mathbf{t}_2^d$. It should be appreciated that the integrals formed on addition of (50) and (51), and (52) and (53), i.e. integration performed over Γ_ε, are path independent although this property does not necessarily extend to $\tilde{\Gamma}_\varepsilon^\ell$ and $\tilde{\Gamma}_\varepsilon^b$. In other words, integrals formed on $\tilde{\Gamma}_\varepsilon^\ell$ and $\tilde{\Gamma}_\varepsilon^b$ can depend on β but this dependence vanishes on the appropriate addition of said integrals as $\varepsilon\to 0$. This property is investigated explicitly for particular integrals in Appendix II whilst in this section all calculations are performed with β set equal to zero.

8.1. Coordinate frames at a corner

The identification of symmetries for simplification is achieved by expressing $\mathbf{t}_i^d$ as a linear combination of suitable linear independent pairings of $\mathbf{t}_i^\ell$ and $\mathbf{t}_i^b$, $i=1$, 2. With reference to Fig. 9 the relationships are: $\mathbf{t}_1^d=-\mathbf{t}_1^\ell\sin\alpha-\mathbf{t}_2^\ell\cos\alpha$, $\mathbf{t}_2^d=-\mathbf{t}_1^\ell\cos\alpha+\mathbf{t}_2^\ell\sin\alpha$ and $\mathbf{t}_1^d=\mathbf{t}_1^b\sin\alpha+\mathbf{t}_2^b\cos\alpha$, $\mathbf{t}_2^d=-\mathbf{t}_1^b\cos\alpha+\mathbf{t}_2^b\sin\alpha$, which can be succinctly written as $\mathbf{t}_i^d=\alpha_{ij}^\ell\mathbf{t}_j^\ell$ and $\mathbf{t}_i^d=\alpha_{ij}^b\mathbf{t}_j^b$. With this choice $\mathbf{t}_2^d$ lies on the symmetric bisector at a corner and $\mathbf{t}_1^d$ being perpendicular is a natural choice for a tangential derivative at a corner mirroring what occurs at a smooth point. The following derivative relationships are assumed to exist: $\partial/\partial t_i^\ell\equiv\alpha_{ji}^\ell(\partial/\partial t_i^d)$ and $\partial/\partial t_i^b\equiv\alpha_{ji}^b(\partial/\partial t_i^b)$. Similarly, $\mathbf{t}_i^d$ is related to the global system $\mathbf{e}_i$ via $\mathbf{t}_i^d=\alpha_{ij}^d\mathbf{e}_j$ giving $\partial/\partial x_i\equiv\alpha_{ji}^d(\partial/\partial t_j^d)$, where $\alpha_{ij}^d=\mathbf{t}_i^d\cdot\mathbf{e}_j$. It is convenient at this stage to define a shorthand notation for the integrals involved, i.e.

$$\beta_i^\ell=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial w}{\partial t_i^\ell}d\Gamma^p,\quad \beta_{ij}^\ell=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}t_1^\ell\frac{\partial w}{\partial t_i^\ell\partial t_j^\ell}d\Gamma^p \text{ and } \chi_{ij}^\ell=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial w}{\partial t_i^\ell\partial t_j^\ell}d\Gamma^p \tag{54}$$

and likewise on $\tilde{\Gamma}_\varepsilon^{pb}$, i.e. β_i^b, β_{ij}^b and χ_{ij}^b, where it is understood that one derivative in the integrals is with respect to coordinate variable $\mathbf{x}$ with others with respect to coordinate variable y for the two point function $w(\mathbf{x},\mathbf{y})$.

A particular property which provides for significant simplification is that for $\beta=0$ (see Fig. 9) the following identities apply: $\beta_i^\ell=\beta_i^b$, $\beta_{ij}^\ell=\beta_{ij}^b$ and $\chi_{ij}^\ell=\chi_{ij}^b$, and it is convenient to adopt the notation β_i, β_{ij} and χ_{ij} in this case.

Proposition 8.1. For $\beta=0$: $\beta_i^\ell=\beta_i^b$, $\beta_{ij}^\ell=\beta_{ij}^b$ and $\chi_{ij}^\ell=\chi_{ij}^b$.

Proof. On $\tilde{\Gamma}_\varepsilon^{p\ell}$, $\mathbf{r}=\varepsilon^\gamma\hat{\mathbf{d}}+t_1^\ell\mathbf{t}_1^\ell=\varepsilon^\gamma\mathbf{t}_2^d+t_1^\ell\mathbf{t}_1^\ell$, so $\mathbf{r}\cdot\mathbf{t}_1^\ell=\varepsilon^\gamma\mathbf{t}_2^d\cdot\mathbf{t}_1^\ell+t_1^\ell=-\varepsilon^\gamma\cos\alpha+t_1^\ell$ and $\mathbf{r}\cdot\mathbf{t}_2^\ell=\varepsilon^\gamma\mathbf{t}_2^d\cdot\mathbf{t}_2^\ell=\varepsilon^\gamma\sin\alpha$; and similarly on $\tilde{\Gamma}_\varepsilon^{pb}$, $\mathbf{r}\cdot\mathbf{t}_1^b=\varepsilon^\gamma\mathbf{t}_2^d\hat{N}\mathbf{t}_1^b+t_1^b=-\varepsilon^\gamma\cos\alpha+t_1^b$ and $\mathbf{r}\cdot\mathbf{t}_2^b=\varepsilon^\gamma\mathbf{t}_2^d\cdot\mathbf{t}_2^b=\varepsilon^\gamma\sin\alpha$. The proof essentially stems from these relationships since for example

$$\begin{aligned}\beta_1^\ell&=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial w}{\partial t_1^\ell}d\Gamma^p=\int_0^\varepsilon w'\frac{\partial r}{\partial t_1^\ell}dt_1^\ell=\int_0^\varepsilon\frac{w'}{r}(\mathbf{r}\cdot\mathbf{t}_1^\ell)dt_1^\ell\\&=\int_0^\varepsilon\frac{w'}{r}(-\varepsilon^\gamma\cos\alpha+t_1^\ell)dt_1^\ell\\&=\int_0^\varepsilon\frac{w'}{r}(-\varepsilon^\gamma\cos\alpha+t_1^b)dt_1^b=\int_0^\varepsilon\frac{w'}{r}(\mathbf{r}\cdot\mathbf{t}_1^b)dt_1^b\\&=\int_0^\varepsilon w'\frac{\partial r}{\partial t_1^b}dt_1^b=\int_{\tilde{\Gamma}_\varepsilon^{pb}}\frac{\partial w}{\partial t_1^b}d\Gamma^p=\beta_i^b\end{aligned} \tag{55}$$

and

$$\begin{aligned}\beta_{12}^\ell&=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}t_1^\ell\frac{\partial w}{\partial t_i^\ell\partial t_j^\ell}d\Gamma^p=\int_0^\varepsilon t_1^\ell\frac{\partial}{\partial t_1^\ell}\left(w'\frac{\partial r}{\partial t_2^\ell}\right)dt_1^\ell\\&=\int_0^\varepsilon t_1^\ell\frac{\partial}{\partial t_1^\ell}\left(\frac{w'}{r}\mathbf{r}\cdot\mathbf{t}_2^\ell\right)dt_1^\ell=\int_0^\varepsilon t_1^\ell(\mathbf{r}\cdot\mathbf{t}_2^\ell)(\mathbf{r}\cdot\mathbf{t}_1^\ell)\frac{1}{r}\left(\frac{w'}{r}\right)'dt_1^\ell\\&=\int_0^\varepsilon t_1^\ell(-\varepsilon^\gamma\cos\alpha+t_1^\ell)(\varepsilon^\gamma\sin\alpha)\frac{1}{r}\left(\frac{w'}{r}\right)'dt_1^\ell\\&=\int_0^\varepsilon t_1^b(-\varepsilon^\gamma\cos\alpha+t_1^b)(\varepsilon^\gamma\sin\alpha)\frac{1}{r}\left(\frac{w'}{r}\right)'dt_1^b\\&=\int_0^\varepsilon t_1^b(\mathbf{r}\cdot\mathbf{t}_2^b)(\mathbf{r}\cdot\mathbf{t}_1^b)\frac{1}{r}\left(\frac{w'}{r}\right)'dt_1^\ell=\int_{\tilde{\Gamma}_\varepsilon^{pb}}t_1^b\frac{\partial w}{\partial t_1^b\partial t_2^b}d\Gamma^p=\beta_{12}^b\end{aligned} \tag{56}$$

and the remaining integrals follow in a similar fashion. □

8.2. Simplification of integrals using Proposition 8.1

For $\beta=0$ it is instructive to consider integrals (50)–(53) and to consider the situations where $\mathbf{e}_i=\mathbf{t}_1^d$ as well as $\mathbf{e}_i=\mathbf{t}_2^d$. For the case

$\mathbf{e}_i = \mathbf{t}_1^d$, Eqs. (50) and (51) give

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon^\ell}\left[n_j\frac{\partial w}{\partial t_1^d}-(y_j-x_j^c)\frac{\partial^2 w}{\partial n\partial t_1^d}\right]d\Gamma\,\frac{\partial u^d}{\partial t_2^\ell}\alpha_{1j}^\ell\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial w}{\partial t_j^\ell}d\Gamma^p$$
$$-\frac{\partial u^d}{\partial t_1^\ell}\alpha_{1j}^\ell\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}t_1^\ell\frac{\partial^2 w}{\partial t_2^\ell\partial t_j^\ell}d\Gamma^p=\alpha_{k2}^\ell\frac{\partial u^d}{\partial t_k^d}\alpha_{1j}^\ell\beta_j^\ell-\alpha_{k1}^\ell\frac{\partial u^d}{\partial t_k^d}\alpha_{1j}^\ell\beta_{2j}^\ell$$
$$=(\alpha_{k2}^\ell\alpha_{1j}^\ell\beta_j^\ell-\alpha_{k1}^\ell\alpha_{1j}^\ell\beta_{2j}^\ell)\frac{\partial u^d}{\partial t_k^d} \tag{57}$$

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon^b}\left[n_j\frac{\partial w}{\partial t_1^d}-(y_j-x_j^c)\frac{\partial^2 w}{\partial n\partial t_1^d}\right]d\Gamma\sim\frac{\partial u^d}{\partial t_2^b}\alpha_{1j}^b\int_{\tilde{\Gamma}_\varepsilon^{pb}}\frac{\partial w}{\partial t_j^b}d\Gamma^p$$
$$-\frac{\partial u^d}{\partial t_1^b}\alpha_{1j}^b\int_{\tilde{\Gamma}_\varepsilon^{pb}}t_1^b\frac{\partial^2 w}{\partial t_2^b\partial t_j^b}d\Gamma^p=\alpha_{k2}^b\frac{\partial u^d}{\partial t_k^d}\alpha_{1j}^b\beta_j^b-\alpha_{k1}^b\frac{\partial u^d}{\partial t_k^d}\alpha_{1j}^b\beta_{2j}^b$$
$$=(\alpha_{k2}^b\alpha_{1j}^b\beta_j^b-\alpha_{k1}^b\alpha_{1j}^b\beta_{2j}^b)\frac{\partial u^d}{\partial t_k^d} \tag{58}$$

which add to give

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j\frac{\partial w}{\partial t_1^d}-(y_j-x_j^c)\frac{\partial^2 w}{\partial n\partial t_1^d}\right]d\Gamma\sim((\alpha_{k2}^\ell\alpha_{1j}^\ell+\alpha_{k2}^b\alpha_{1j}^b)\beta_j$$
$$-(\alpha_{k1}^\ell\alpha_{1j}^\ell+\alpha_{k1}^b\alpha_{1j}^b)\beta_{2j})\frac{\partial u^d}{\partial t_k^d}=((\alpha_{k2}^\ell-\alpha_{k2}^b)\alpha_{1j}^\ell\beta_j-(\alpha_{k1}^\ell-\alpha_{k1}^b)\alpha_{1j}^\ell\beta_{2j})\frac{\partial u^d}{\partial t_k^d}$$
$$=2\alpha_{1j}^\ell(\alpha_{12}^\ell\beta_j-\alpha_{11}^\ell\beta_{2j})\frac{\partial u^d}{\partial t_1^d} \tag{59}$$

where use is made of $\alpha_{1j}^\ell=-\alpha_{1j}^b$ and $\alpha_{2j}^\ell=\alpha_{2j}^b$ along with Proposition 8.1.

Eq. (59) confirms that for $e_i = t_1^d$ the free term is solely associated with the tangential derivative with no normal derivative component. Consider now Eqs. (52) and (53) for the case $e_i = t_1^d$ which gives

$$u^d\int_{\tilde{\Gamma}_\varepsilon^\ell}\frac{\partial^2 w}{\partial n\partial t_1^d}d\Gamma\sim u^d\alpha_{1j}^\ell\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial^2 w}{\partial t_2^\ell\partial t_j^\ell}d\Gamma^p=u^d\alpha_{1j}^\ell\chi_{2j}^\ell \tag{60}$$

$$u^d\int_{\tilde{\Gamma}_\varepsilon^b}\frac{\partial^2 w}{\partial n\partial t_1^d}d\Gamma\sim u^d\alpha_{1j}^b\int_{\tilde{\Gamma}_\varepsilon^{pb}}\frac{\partial^2 w}{\partial t_2^b\partial t_j^b}d\Gamma^p=u^d\alpha_{1j}^b\chi_{2j}^b \tag{61}$$

which on addition gives

$$u^d\int_{\tilde{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_1^d}d\Gamma\sim u^d\alpha_{1j}^\ell(\chi_{2j}^\ell-\chi_{2j}^d)=0 \tag{62}$$

Repeating the process for $\mathbf{e}_i = \mathbf{t}_2^d$ yields

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j\frac{\partial w}{\partial t_2^d}-(y_j-x_j^c)\frac{\partial^2 w}{\partial n\partial t_2^d}\right]d\Gamma\sim((\alpha_{k2}^\ell\alpha_{2j}^\ell+\alpha_{k2}^b\alpha_{2j}^b)\beta_j$$
$$-(\alpha_{k1}^\ell\alpha_{2j}^\ell+\alpha_{k1}^b\alpha_{2j}^b)\beta_{2j})\frac{\partial u^d}{\partial t_k^d}=((\alpha_{k2}^\ell+\alpha_{k2}^b)\alpha_{2j}^\ell\beta_j-(\alpha_{k1}^\ell+\alpha_{k1}^b)\alpha_{2j}^\ell\beta_{2j})\frac{\partial u^d}{\partial t_k^d}$$
$$=2\alpha_{2j}^\ell(\alpha_{22}^\ell\beta_j-\alpha_{12}^\ell\beta_{2j})\frac{\partial u^d}{\partial t_2^d} \tag{63}$$

$$u^d\int_{\tilde{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_2^d}d\Gamma\sim u^d\alpha_{2j}^\ell(\chi_{2j}^\ell+\chi_{2j}^d)=-2u^d\alpha_{2j}^\ell\chi_{2j} \tag{64}$$

8.3. Evaluation of integrals for a corner

To evaluate the expressions (59), (63) and (64) requires the asymptotic expansions for β_i, β_{ij} and χ_{2j}, typically obtained using a standard symbolic tool. However, the integrals involved are relatively simple for the potential problem enabling closed formed solutions to be directly obtained. With reference to Fig. 9 the following relationships can be defined/deduced: $s_2=\varepsilon^\gamma\sin\alpha$, $s_1=t_1^b-\varepsilon^\gamma\cos\alpha=t_1^b-s_2\cot\alpha$ and $r^2=s_2^2+s_1^2$, where $\gamma>1$ and $\alpha\in(0,\pi)$. Consider then

$$\beta_i=\int_{\tilde{\Gamma}_\varepsilon^p}\frac{\partial w}{\partial t_i}d\Gamma^p=\frac{1}{2\pi}\int_0^\varepsilon\frac{\mathbf{r}\cdot\mathbf{t}_i}{r^2}dt_1=\frac{1}{2\pi}\int_0^\varepsilon\frac{s_i}{s_2^2+s_1^2}dt_1$$
$$=\frac{1}{2\pi}\int_{-s_2\cot\alpha}^{\varepsilon-s_2\cot\alpha}\frac{s_i}{s_2^2+s_1^2}ds_1 \tag{65}$$

Hence

$$\beta_1=\frac{1}{2\pi}\int_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/s_2}\frac{s}{1+s^2}ds=\frac{1}{4\pi}\ln(1+s^2)\Big|_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/s_2}\sim\frac{1}{2\pi}\ln(\varepsilon^{1-\gamma}) \tag{66}$$

$$\beta_2=\frac{1}{2\pi}\int_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/s_2}\frac{ds}{1+s^2}=\frac{1}{2\pi}\tan^{-1}(s)\Big|_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/s_2}\sim\frac{\pi-\alpha}{2\pi} \tag{67}$$

where it is observed that β_1 is unbounded as $\varepsilon\to 0$ with no non-zero finite part.

Similarly for $i\neq j$

$$\beta_{ij}=\int_{\tilde{\Gamma}_\varepsilon^p}t_1\frac{\partial w}{\partial t_i\partial t_j}d\Gamma^p=-\frac{1}{\pi}\int_0^\varepsilon t_1\frac{s_is_j}{r^4}dt_1$$
$$=-\frac{1}{\pi}\int_{-s_2\cot(\alpha)}^{\varepsilon-s_2\cot(\alpha)}(s_1+s_2\cot\alpha)\frac{s_is_j}{(s_1^2+s_2^2)^2}ds_1 \tag{68}$$

Hence

$$\beta_{12}=\beta_{21}=-\frac{1}{\pi}\int_{-\cot(\alpha)}^{(\varepsilon-s_2\cot(\alpha))/s_2}\frac{s(s+\cot\alpha)}{(1+s^2)^2}ds$$
$$=-\frac{1}{2\pi}\left(\tan^{-1}s-\frac{s+\cot\alpha}{1+s^2}\right)\Big|_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/s_2}\sim-\frac{\pi-\alpha}{2\pi} \tag{69}$$

and

$$\beta_{11}=-\beta_{22}=\int_{\tilde{\Gamma}_\varepsilon^p}t_1\frac{\partial^2 w}{\partial t_1\partial t_1}d\Gamma^p=\frac{1}{2\pi}\int_0^\varepsilon t_1\frac{s_2^2-s_1^2}{(s_2^2+s_1^2)^2}dt_1$$
$$=-\frac{1}{2\pi}\left(\frac{1-s\cot\alpha}{1+s^2}+\frac{1}{2}\ln(1+s^2)\right)\Big|_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/s_2}\sim-\frac{1}{2\pi}\left[\ln(\varepsilon^{1-\gamma})-1\right] \tag{70}$$

Likewise

$$\chi_{12}=\chi_{21}=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial^2 w}{\partial t_1^\ell\partial t_2^\ell}d\Gamma^p=-\frac{1}{\pi}\int_0^\varepsilon\frac{s_1s_2}{r^4}dt_1$$
$$=-\frac{s_2^{-1}}{\pi}\int_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)/r_2}\frac{s}{(1+s^2)^2}ds$$
$$=\frac{s_2^{-1}}{2\pi}\frac{1}{1+s^2}\Big|_{-\cot\alpha}^{(\varepsilon-s_2\cot\alpha)r_2}\sim-\frac{\sin\alpha}{2\pi}\varepsilon^{-\gamma} \tag{71}$$

and

$$\chi_{11}=-\chi_{22}=\int_{\tilde{\Gamma}_\varepsilon^{p\ell}}\frac{\partial^2 w}{\partial t_1^\ell\partial t_1^\ell}d\Gamma^p=\frac{1}{2\pi}\int_0^\varepsilon\frac{s_2^2-s_1^2}{(s_2^1+s_1^2)^2}dt_1$$
$$=-\frac{s_2^{-1}}{2\pi}\int_{-\cot\alpha}^{(\varepsilon-r_2\cot\alpha)/r_2}\frac{s^2-1}{(1+s^2)^2}ds\sim-\frac{\cos\alpha}{2\pi}\varepsilon^{-\gamma} \tag{72}$$

Thus in summary the order one asymptotic terms are: $-\beta_{12}=-\beta_{21}=\beta_2=(\pi-\alpha)/2\pi$, with singular contributions from β_1, β_{ii} and χ_{ij} with non-zero finite parts limited to β_{ii}.

8.4. Determination of corner free terms

Substitution of the asymptotic expansions into (59), (63) and (64) gives

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j\frac{\partial w}{\partial t_1^d}-(y_j-x_j^c)\frac{\partial^2 w}{\partial n\partial t_1^d}\right]d\Gamma\sim2\left[\alpha_{11}^\ell(\alpha_{12}^\ell\beta_1-\alpha_{11}^\ell\beta_{21})\right.$$
$$\left.+\alpha_{12}^\ell(\alpha_{12}^\ell\beta_2-\alpha_{11}^\ell\beta_{22})\right]\frac{\partial u^d}{\partial t_1^d}$$

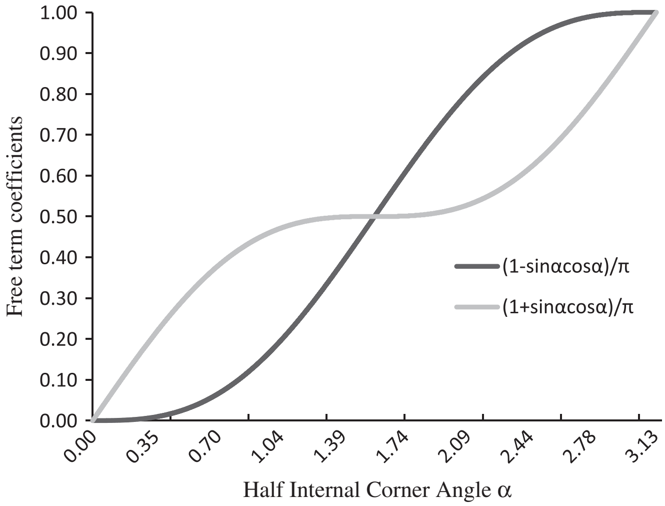

Fig. 10. Free term as a function of half internal corner angle.

$$=2\left[\left[(\alpha_{11}^{\ell})^2+(\alpha_{12}^{\ell})^2\right]\beta_2+\alpha_{12}^{\ell}\alpha_{11}^{\ell}(\beta_1-\beta_{22})\right]\frac{\partial u^d}{\partial t_1^d}$$
$$=\left[\frac{\pi-\alpha}{\pi}+\frac{1}{\pi}\cos\alpha\sin\alpha\right]\frac{\partial u^d}{\partial t_1^d} \tag{73}$$

$$\frac{\partial u^d}{\partial x_j}\int_{\tilde{\Gamma}_\varepsilon}\left[n_j\frac{\partial w}{\partial t_2^d}-(y_j-x_j^c)\frac{\partial^2 w}{\partial n\partial t_2^d}\right]d\Gamma\sim 2\left[-\alpha_{21}^{\ell}\alpha_{12}^{\ell}\beta_{21}+\alpha_{22}^{\ell}\alpha_{22}^{\ell}\beta_2+\alpha_{21}^{\ell}\alpha_{22}^{\ell}\beta_1-\alpha_{22}^{\ell}\alpha_{12}^{\ell}\beta_{22}\right]\frac{\partial u^d}{\partial t_2^d}=\left[\frac{\pi-\alpha}{\pi}-\frac{1}{\pi}\cos\alpha\sin\alpha\right]\frac{\partial u^d}{\partial t_2^d} \tag{74}$$

$$u^d\int_{\tilde{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_2^d}d\Gamma\sim -2u^d(\alpha_{21}^{\ell}\chi_{21}+\alpha_{22}^{\ell}\chi_{22})=0 \tag{75}$$

Although free terms have been established for local nodal tangents it is a relatively simple matter to relate these to global derivatives. Note also that Eqs. (73) and (74) are the contributions to the free terms arising when the boundary is approached with an internal point. The actual free terms are obtained on subtraction (73) and (74) from $\partial u^d/\partial t_1^d$ and $\partial u^d/\partial t_2^d$, respectively. The free terms obtained from Eqs. (73) and (74) are presented in Fig. 10 as a function of α.

9. Point limiting method at a curved corner

The analysis pertaining to Eq. (32) performed on a planar corner is applicable to a curved corner but the more singular integrand in Eq. (34) requires a higher order representation of the surfaces involved. With reference to Fig. 11 the element $\tilde{\Gamma}_\varepsilon=\tilde{\Gamma}_\varepsilon^{\ell}\cup\tilde{\Gamma}_\varepsilon^{b}$ is approximated by the union spherical elements $\tilde{\Gamma}_\varepsilon^{p\ell}$ an $\tilde{\Gamma}_\varepsilon^{pb}$. Since Eq. (34) is path independent it is convenient to approach the corner point $\mathbf{x}_d$ along the bisector $\hat{\mathbf{d}}=\mathbf{t}_2^d$. Consider integration over $\tilde{\Gamma}_\varepsilon^{pb}$ and with reference to Fig. 10 it can be seen that

$$\mathbf{r}=\varepsilon^{\gamma}\hat{\mathbf{d}}-(\kappa_d^b)^{-1}(1-\cos\theta)\mathbf{t}_2^{bd}+(\kappa_d^b)^{-1}\sin\theta\,\mathbf{t}_1^{bd} \tag{76}$$

and $s_2^b=\mathbf{r}\cdot\mathbf{t}_2^{pd}=\varepsilon^{\gamma}\sin\alpha+(\kappa_d^b)^{-1}(\cos\theta-1)$, $s_1^b=\mathbf{r}\cdot\mathbf{t}_1^{pd}=-\varepsilon^{\gamma}\cos\alpha+(\kappa_d^b)^{-1}\sin\theta$, $\mathbf{n}\cdot\mathbf{t}_1^d=\sin\theta$, $\mathbf{n}\cdot\mathbf{t}_2^d=\cos\theta$ and $r^2=s_2^2+s_1^2$; Eq. (34) for $\mathbf{e}_i=\mathbf{t}_1^{bd}$ becomes

$$\int_{\tilde{\Gamma}_\varepsilon^{pb}}\frac{\partial^2 w}{\partial n\partial t_1^{bd}}d\Gamma=\int_{\tilde{\Gamma}_\varepsilon^{pb}}\left(\mathbf{n}\cdot\mathbf{t}_2^{bd}\frac{\partial^2 w}{\partial t_2^{bd}\partial t_1^{bd}}+\mathbf{n}\cdot\mathbf{t}_1^{bd}\frac{\partial^2 w}{\partial t_1^{bd}\partial t_1^{bd}}\right)d\Gamma$$

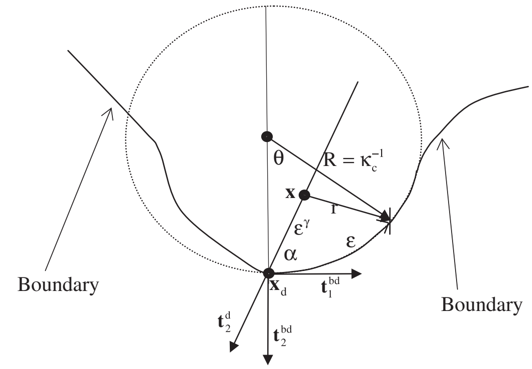

Fig. 11. Integration over a smooth boundary with local curvature κ_c.

$$=\frac{1}{2\kappa_d^b\pi}\int_0^{\kappa_d^b\varepsilon}\frac{-2s_1s_2\cos\theta+(s_2^2-s_1^2)\sin\theta}{(s_1^2+s_2^2)^2}d\theta \tag{77}$$

and for $\mathbf{e}_i=\mathbf{t}_2^{bd}$

$$\int_{\tilde{\Gamma}_\varepsilon^{pb}}\frac{\partial^2 w}{\partial n\partial t_2^{bd}}d\Gamma=\int_{\tilde{\Gamma}_\varepsilon^{pb}}\left(\mathbf{n}\cdot\mathbf{t}_2^{b}\frac{\partial^2 w}{\partial t_2^{bd}\partial t_2^{bd}}+\mathbf{n}\cdot\mathbf{t}_1^{b}\frac{\partial^2 w}{\partial t_1^{bd}\partial t_2^{bd}}\right)d\Gamma$$
$$=\frac{1}{2\kappa_d^b\pi}\int_0^{\kappa_d^b\varepsilon}\frac{(s_1^2-s_2^2)\cos\theta-2s_1s_2\sin\theta}{(s_1^2+s_2^2)^2}d\theta \tag{78}$$

Eqs. (86) and (87) are readily evaluated and have finite parts $(-\kappa_d^b+2\sin\alpha)/4\pi$ and $\cos\alpha/2\pi$, respectively. It follows on application of the coordinate transformation that:

$$\int_{\tilde{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_1^d}d\Gamma\underset{\text{finite part}}{\rightarrow}-\frac{\alpha_{11}^{\ell}}{4\pi}(\kappa_d^{\ell}-\kappa_d^b)=\frac{\sin\alpha}{4\pi}(\kappa_d^{\ell}-\kappa_d^b) \tag{79}$$

$$\int_{\tilde{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_2^d}d\Gamma\underset{\text{finite part}}{\rightarrow}-\frac{\alpha_{21}^{\ell}}{4\pi}(\kappa_d^{\ell}+\kappa_d^b-4\sin\alpha)+\frac{\alpha_{22}^{\ell}}{\pi}\cos\alpha=\frac{\cos\alpha}{4\pi}(\kappa_d^{\ell}+\kappa_d^b) \tag{80}$$

which is consistent with the result for a smooth boundary, i.e. gives zero for $\alpha=\pi/2$ and $\kappa_d^{\ell}=\kappa_d^b$.

10. The boundary limiting approach

The boundary limiting approach is an exclusion zone approach that involves sliding apportion of the boundary without shape distortion as depicted in Fig. 12. Shown in the figure is a corner with internal angle 2α and approach angle β. The similarities with the point limiting approach are evident where in this case the boundary approaches the singularity whilst in the point limiting case the singularity approaches the boundary. If no contribution is provided to the integrals from the side edges of length ε^{γ} (achieved by specification of sufficiently high value of γ) then the boundary limiting approach can be expected to give identical results to the point limiting approach. Prior to examining the similarities in detail it is useful to obtain expressions using the circular exclusion zone.

10.1. Circular exclusion zone

Recall that the integrals of interest for an exclusion zone are defined in Eq. (15), i.e.

$$\frac{\partial u^c}{\partial t_j^d}\int_{\tilde{\Gamma}_c}\left[\frac{\partial w}{\partial t_i^d}(\mathbf{x}_d)n_j-(y_j-x_j^d)\frac{\partial^2 w}{\partial n\partial t_i^d}(\mathbf{x}_d)\right]d\Gamma\ \ \textit{and}\ \ u^d\int_{\tilde{\Gamma}_c}\frac{\partial^2 w}{\partial n\partial t_i^d}(\mathbf{x}_d)d\Gamma \tag{81}$$

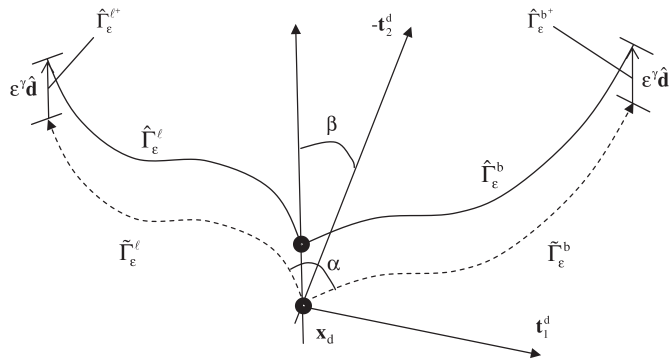

Fig. 12. Boundary limiting method at a corner boundary point.

where for convenience the nodal tangential derivatives are considered.

Note that for a circular exclusion zone or radius ε: i.e. with $\partial w/\partial t_1^d = \cos\theta/2\pi\varepsilon$, $\partial w/\partial t_2^d = -\sin\theta/2\pi\varepsilon$ and $\partial^2 w/\partial n\partial t_1^b = \cos\theta/2\pi\varepsilon^2$ and $\partial^2 w/\partial n\partial t_2^b = -\sin\theta/2\pi\varepsilon^2$, where θ is the angle measured between $\mathbf{t}_1^d$ and $\mathbf{r}$. Since it is recognised that that curvature does not impact on the first integrals in Eq. (87) these can be evaluated in a straightforward manner, i.e.

$$\int_{\hat{\Gamma}_c}\left[\frac{\partial w}{\partial t_1^d}n_1-(y_1-x_1^d)\frac{\partial^2 w}{\partial n\partial t_1^d}\right]d\Gamma$$
$$=\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\left[\frac{\cos\theta}{2\pi\varepsilon}(-\cos\theta)-\varepsilon\cos\theta\frac{\cos\theta}{2\pi\varepsilon^2}\right]\varepsilon d\theta$$
$$=-\frac{1}{\pi}\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\cos^2\theta d\theta=-\frac{\alpha-\sin\alpha\cos\alpha}{\pi} \quad (82)$$

$$\int_{\hat{\Gamma}_c}\left[\frac{\partial w}{\partial t_1^d}n_2-(y_2-x_2^d)\frac{\partial^2 w}{\partial n\partial t_1^d}\right]d\Gamma$$
$$=\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\left[\frac{\cos\theta}{2\pi\varepsilon}(\sin\theta)-(-\varepsilon\sin\theta)\frac{\cos\theta}{2\pi\varepsilon^2}\right]\varepsilon d\theta=0 \quad (83)$$

$$\int_{\hat{\Gamma}_c}\left[\frac{\partial w}{\partial t_2^d}n_1-(y_1-x_1^d)\frac{\partial^2 w}{\partial n\partial t_2^d}\right]d\Gamma$$
$$=\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\left[\frac{-\sin\theta}{2\pi\varepsilon}(-\cos\theta)-\varepsilon\cos\theta\frac{-\sin\theta}{2\pi\varepsilon^2}\right]\varepsilon d\theta=0 \quad (84)$$

$$\int_{\hat{\Gamma}_c}\left[\frac{\partial w}{\partial t_2^d}n_2-(y_2-x_2^d)\frac{\partial^2 w}{\partial n\partial t_2^d}\right]d\Gamma$$
$$=\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\left[\frac{-\sin\theta}{2\pi\varepsilon}(\sin\theta)-(-\varepsilon\sin\theta)\frac{-\sin\theta}{2\pi\varepsilon^2}\right]\varepsilon d\theta$$
$$=-\frac{1}{\pi}\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\sin^2\theta d\theta=-\frac{\alpha+\sin\alpha\cos\alpha}{\pi} \quad (85)$$

giving matching free terms to Eqs. (73) and (74) after accounting for the fact that here the source point is external to the domain.

The evaluation of the free term for the integral on the right-hand side of Eq. (81) using the spherical method is more involved and discussed in detail in Ref. [7]. The solutions arrived at in Ref. [7] are

$$u^d\int_{\hat{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_1^d}d\Gamma=u^d\int_{\theta_1(\varepsilon)}^{\theta_1(\varepsilon)}\frac{\cos\theta}{2\pi\varepsilon^2}\varepsilon d\theta\sim u^d\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\frac{\cos\theta}{2\pi\varepsilon^2}\varepsilon d\theta$$
$$-\frac{u^d}{4\pi}\left(\kappa_d^\ell\cos\left(\frac{\pi}{2}+\alpha\right)+\kappa_d^b\cos\left(\frac{\pi}{2}-\alpha\right)\right)\sim u^d\frac{\sin(\alpha)}{4\pi}(\kappa_d^\ell-\kappa_d^b) \quad (86)$$

$$u^d\int_{\hat{\Gamma}_\varepsilon}\frac{\partial^2 w}{\partial n\partial t_2^d}d\Gamma=-u^d\int_{\theta_1(\varepsilon)}^{\theta_1(\varepsilon)}\frac{\sin\theta}{2\pi\varepsilon^2}\varepsilon d\theta\sim -u^d\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}\frac{\sin\theta}{2\pi\varepsilon^2}\varepsilon d\theta$$
$$+\frac{u^d}{4\pi}\left(\kappa_d^\ell\sin\left(\frac{\pi}{2}+\alpha\right)+\kappa_d^b\sin\left(\frac{\pi}{2}-\alpha\right)\right)\sim u^d\frac{\sin(\alpha)}{2\pi\varepsilon}+u^d\frac{\cos(\alpha)}{4\pi}(\kappa_d^\ell+\kappa_d^b) \quad (87)$$

which match identically with results produced by the point-limiting method, i.e. Eqs. (79) and (80).

10.2. General concepts for the boundary limiting method

Consider next the boundary-limiting approach and the exclusion zone depicted in Fig. 12, i.e. $\hat{\Gamma}_\varepsilon=\hat{\Gamma}_\varepsilon^\ell\cup\hat{\Gamma}_\varepsilon^{\ell^+}\cup\hat{\Gamma}_\varepsilon^b\cup\hat{\Gamma}_\varepsilon^{b^+}$, where focus here is on the integral $\int_{\hat{\Gamma}_\varepsilon}g(\mathbf{x}_d,\mathbf{y})d\Gamma(\mathbf{y})$. On $\hat{\Gamma}_\varepsilon^\ell\cup\hat{\Gamma}_\varepsilon^b$ this integral can be contrasted with the point limiting approach with an exterior point approaching in direction $\varepsilon^\gamma\mathbf{d}$. The integral of interest in this case is $\int_{\tilde{\Gamma}_\varepsilon}g(\mathbf{x}_d-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{z})d\Gamma(\mathbf{z})$ and if $g(\mathbf{x},\mathbf{z})=g(\mathbf{z}-\mathbf{x})$, then this is identical to $\int_{\tilde{\Gamma}_\varepsilon}g(\mathbf{x}_d,\mathbf{z}+\varepsilon^\gamma d)d\Gamma(\mathbf{z})$. Note that for $\mathbf{z}\in\tilde{\Gamma}_\varepsilon^\ell\cup\tilde{\Gamma}_\varepsilon^b$ the translation property gives $\mathbf{y}=\mathbf{z}+\varepsilon^\gamma\hat{\mathbf{d}}\in\hat{\Gamma}_\varepsilon^\ell\cup\hat{\Gamma}_\varepsilon^b$ which implies that

$$\int_{\hat{\Gamma}_\varepsilon^\ell\cup\hat{\Gamma}_\varepsilon^b}g(\mathbf{x}_d,\mathbf{y})d\Gamma(\mathbf{y})=\int_{\tilde{\Gamma}_\varepsilon}g(\mathbf{x}_d-\varepsilon^\gamma\hat{\mathbf{d}},\mathbf{z})d\Gamma(\mathbf{z}) \quad (88)$$

confirming that any contribution arising on $\hat{\Gamma}_\varepsilon^\ell\cup\hat{\Gamma}_\varepsilon^b$ for the boundary-limiting approach is identical to that obtained for the point limiting approach.

For point limiting and boundary limiting approaches to provide identical results the contributions from the boundary $\hat{\Gamma}_\varepsilon^{\ell^+}\cup\hat{\Gamma}_\varepsilon^{b^+}$ are required to be identically zero. Consider then the integral $\int_{\hat{\Gamma}_\varepsilon^{b^+}}g(\mathbf{x}_d,\mathbf{y})d\Gamma(\mathbf{y})$ and let $\mathbf{y}_b\in\hat{\Gamma}_\varepsilon^b\cap\hat{\Gamma}_\varepsilon^{b^+}$, then $\mathbf{y}=\mathbf{y}_b+t\varepsilon^\gamma\hat{\mathbf{d}}\in\Gamma_\varepsilon^{b^+}$ for $t\in[0,1]$ which on substitution into the integral gives $\int_0^1 g(\mathbf{x}_d,\mathbf{y}_b+t\varepsilon^\gamma\hat{\mathbf{d}})\varepsilon^\gamma dt$. Note that

$$\int_0^1 g(\mathbf{x}_d,\mathbf{y}_b+t\varepsilon^\gamma\hat{\mathbf{d}})\varepsilon^\gamma dt\le\varepsilon^\gamma\sup_{0\le\eta\le\varepsilon}\left|g(\mathbf{x}_d,\mathbf{y}_b+\eta^\gamma\hat{\mathbf{d}})\right| \quad (89)$$

but following similar arguments deduced in Section 4 it is clear for $\gamma>1$ that $\|\mathbf{x}_d-(\mathbf{y}_b+\eta^\gamma\hat{\mathbf{d}})\|=O(\varepsilon)$ and it follows that $\sup_{0\le\eta\le\varepsilon}\left|g(\mathbf{x}_d,\mathbf{y}_b+\eta^\gamma\hat{\mathbf{d}})\right|\sim\varepsilon^{-\beta}$ in the limit $\varepsilon\to0$, where β is the order of the singularity of g. It is evident therefore that γ can be set sufficiently high to ensure that $\lim_{\varepsilon\to0}\int_0^1 g(\mathbf{x}_d,\mathbf{y}_b+t\varepsilon^\gamma\hat{\mathbf{d}})\varepsilon^\gamma dt=0$. The following proposition immediately follows.

Proposition 10.1. *For the conditions applicable in* Corollary 4.1 *and with the integrands involved satisfying* $g(\mathbf{x},\mathbf{z})=g(\mathbf{z}-\mathbf{x})$, *then for a sufficiently high value of* $\gamma>1$ *the point limiting and boundary limiting methods produce identical free terms.*

11. Free terms for a strong variational method

Consider the application of boundary conditions $\overline{u}$ on Γ_u and the normal derivative $\overline{q}$ on Γ_q, where $\Gamma=\Gamma_u\cup\Gamma_q$. The classic solution to $\nabla^2 u=0$ satisfies the boundary conditions $u=\overline{u}$ on Γ_u and $\partial u/\partial n=\overline{q}$ almost everywhere. It is useful to make the distinction between functions defined on the boundary and those which are simply restrictions from an open domain. In the previous sections an internal boundary $\tilde{\Gamma}$ is defined to reinforce the point that function restrictions are being applied. It is useful however to consider the implications on the integral equations, both singular and hypersingular, of applying boundary conditions. The applied boundary conditions are assumed here to be piecewise smooth on a Lipschitz continuous orientable boundary Γ. It is recognised that the derivative of $u(\mathbf{x})$ can be unbounded for $\mathbf{x}\in\Omega$ in the limit $\mathbf{x}\to\mathbf{x}_d$ where $\mathbf{x}_d\in\Gamma$ is a corner point. Similar concerns apply to point $\mathbf{x}_c\in\Gamma$ on a smooth boundary where a discontinuity is present in the boundary condition. These issues are noted but not considered further here as on the boundary the functions are assumed to be bounded and piecewise smooth.

One of the disadvantages of the exclusion approach in the determination of free terms is its inability to cater for mixed boundary conditions without recourse to the constitutive relationships. This arises because the vanishing exclusion zone is moved off the boundary where boundary conditions are undefined. This is not an issue for the point-limiting approach. However, prior to considering the issue involved it is of interest to re-derive Eq. (17). Consider and exterior point with $\mathbf{x}=\mathbf{x}_d-\varepsilon^{\gamma}\hat{\mathbf{d}}$ and observe that

$$\begin{aligned}\int_{\Gamma} u\frac{\partial w_i}{\partial n}d\Gamma &= \int_{\Gamma-\Gamma_\varepsilon} u\frac{\partial w_i}{\partial n}d\Gamma+\int_{\Gamma_\varepsilon} u\frac{\partial w_i}{\partial n}d\Gamma=\int_{\Gamma-\Gamma_\varepsilon} u\frac{\partial w_i}{\partial n}d\Gamma\\ &+\int_{\Gamma_\varepsilon}\left(u-u^d-[x_j-y_j]\frac{\partial u^d}{\partial x_j}\right)\frac{\partial w_i}{\partial n}d\Gamma\\ &+\int_{\Gamma_\varepsilon} u^d\frac{\partial w_i}{\partial n}d\Gamma+\int_{\Gamma_\varepsilon}\frac{\partial u^d}{\partial x_j}[x_j-y_j]\frac{\partial w_i}{\partial n}d\Gamma \end{aligned}\tag{90}$$

and similarly

$$\begin{aligned}\int_{\Gamma}\frac{\partial u}{\partial n}w_i d\Gamma &= \int_{\Gamma-\Gamma_\varepsilon}\frac{\partial u}{\partial n}w_i d\Gamma+\int_{\Gamma_\varepsilon}\frac{\partial u}{\partial n}w_i d\Gamma=\int_{\Gamma-\Gamma_\varepsilon}\frac{\partial u}{\partial n}w_i d\Gamma\\ &+\int_{\Gamma_\varepsilon}\left(\frac{\partial u}{\partial n}-n_j\frac{\partial u^d}{\partial x_j}\right)w_i d\Gamma+\int_{\Gamma_\varepsilon}\frac{\partial u^d}{\partial x_j}n_j w_i d\Gamma\end{aligned}\tag{91}$$

where it is clear that substitution of Eqs. (90) and (91) into Eq. (12) yields Eq. (16) which on letting $\varepsilon\to 0$ gives (17).

Consider further the application of this more direct approach to

$$\int_{\Gamma_q} u\frac{\partial w_i}{\partial n}d\Gamma+\int_{\Gamma_u}\overline{u}\frac{\partial w_i}{\partial n}d\Gamma=\int_{\Gamma_u} w_i q d\Gamma+\int_{\Gamma_q} w_i\overline{q}d\Gamma \tag{92}$$

which is essentially Eq. (12) with Γ replacing $\tilde{\Gamma}$, where $q=\partial u/\partial n$. The following cases are of interest: $\mathbf{x}_d\in\Gamma_u$, $\mathbf{x}_d\in\Gamma_q$ and $\mathbf{x}_d\in\Gamma_u\cap\Gamma_q$ but attention here is restricted to the latter case. Much of the analysis presented thus far applies with approximations pertaining to a corner being applicable. Consider then $\mathbf{x}_d\in\Gamma_u\cap\Gamma_q$ and assume $\Gamma_\varepsilon^{\ell}\subset\Gamma_u$ and $\Gamma_\varepsilon^{b}\subset\Gamma_q$; Eqs. (50)–(53) become

$$\int_{\Gamma_\varepsilon^{\ell}}\frac{\partial u^d}{\partial x_j}\left[n_j w_i-(y_j-x_j^d)\frac{\partial w_i}{\partial n}\right]d\Gamma\sim q^b\int_{\Gamma_\varepsilon^{p\ell}} w_i d\Gamma^p-\frac{\partial u^d}{\partial t_1^{\ell}}\int_{\Gamma_\varepsilon^{p\ell}} t_1^{\ell}\frac{\partial w_i}{\partial t_2^{\ell}}d\Gamma^p \tag{93}$$

$$\int_{\Gamma_\varepsilon^{b}}\frac{\partial u^d}{\partial x_j}\left[n_j w_i-(y_j-x_j^d)\frac{\partial w_i}{\partial n}\right]d\Gamma\sim \overline{q}^b\int_{\Gamma_\varepsilon^{pb}} w_i d\Gamma^p-\frac{\partial u^d}{\partial t_1^{b}}\int_{\Gamma_\varepsilon^{pb}} t_1^{b}\frac{\partial w_i}{\partial t_2^{b}}d\Gamma^p \tag{94}$$

and

$$\int_{\Gamma_\varepsilon^{\ell}} u^d\frac{\partial w_i}{\partial n}d\Gamma\sim\overline{u}^d\int_{\Gamma_\varepsilon^{p\ell}}\mathbf{n}\cdot\mathbf{t}_2^{\ell}\frac{\partial w_i}{\partial t_2^{\ell}}d\Gamma^p+\overline{u}^d\int_{\Gamma_\varepsilon^{p\ell}}\mathbf{n}\cdot\mathbf{t}_1^{\ell}\frac{\partial w_i}{\partial t_1^{\ell}}d\Gamma^p \tag{95}$$

$$\int_{\Gamma_\varepsilon^{b}} u^d\frac{\partial w_i}{\partial n}d\Gamma\sim u^d\int_{\Gamma_\varepsilon^{pb}}\mathbf{n}\cdot\mathbf{t}_2^{b}\frac{\partial w_i}{\partial t_2^{b}}d\Gamma^p+u^d\int_{\Gamma_\varepsilon^{p\ell}}\mathbf{n}\cdot\mathbf{t}_2^{b}\frac{\partial w_i}{\partial t_1^{b}}d\Gamma^p \tag{96}$$

which can be incorporated into Eq. (17) to facilitate mixed boundary conditions.

It should be appreciated that discontinuous behaviour at points on the boundary is permitted by Corollary 4.1.

12. Direct determination of free terms

It is of interest to contrast the point limiting and spherical method against the direct method for some simple examples.

12.1. Planar boundary with mixed boundary conditions

Consider the planar example depicted in Fig. 13 consisting of a point approaching the domain $[-1, 1]$. Consider first $\mathbf{x}=\mathbf{x}_d-h\hat{\mathbf{d}}$ and the integrals related to $\int_{[-11]}q(\partial w/\partial t_1 dt_1)$, where $s_2=h$, $s_1=t_1$ and $r^2=s_2^2+s_1^2$, i.e.

$$\int_{-1}^{1} q\frac{\partial w}{\partial t_1}dt_1=\frac{1}{2\pi}\int_{-1}^{1} q\frac{s_1}{s_2^2+s_1^2}dt_1=\frac{\overline{q}-q}{2\pi}\left[\tan^{-1}(h^{-1})\right] \tag{97}$$

$$\int_{-1}^{-\varepsilon} q\frac{\partial w}{\partial t_1}dt_1+\int_{\varepsilon}^{1}\overline{q}\frac{\partial w}{\partial t_1}dt_1=\frac{\overline{q}-q}{2\pi}\left[\tan^{-1}(h^{-1})-\tan^{-1}(\varepsilon h^{-1})\right] \tag{98}$$

and for $h=0$

$$\int_{-1}^{-\varepsilon} q\frac{\partial w}{\partial t_1}dt_1+\int_{\varepsilon}^{1}\overline{q}\frac{\partial w}{\partial t_1}dt_1=\frac{\overline{q}-q}{2\pi}\left[\ln(1)-\ln(\varepsilon)\right] \tag{99}$$

Setting $h=\varepsilon^{\gamma}$ in Eq. (98) with $\gamma>1$ and on letting $\varepsilon\to 0$ gives zero which is also obtained from the principle part of Eq. (99). Note that letting $h\to 0$ in Eq. (97) gives $(\overline{q}-q)/4$ which can be contrasted against

$$\int_{-\varepsilon}^{0} q\frac{\partial w}{\partial t_1}dt_1+\int_{0}^{\varepsilon}\overline{q}\frac{\partial w}{\partial t_1}dt_1=\frac{\overline{q}-q}{2\pi}\left[\tan^{-1}(\varepsilon^{1-\gamma})\right]\sim\frac{\overline{q}-q}{4} \tag{100}$$

for $\gamma>1$ as required providing agreement between the point limiting and the direct method.

12.2. A smooth boundary with constant curvature

Consider the circular boundary depicted in Fig. 14 and the integral $\int_{\Gamma}u(\partial^2 w/\partial t_2\partial n d\Gamma)$, where $u=u^c$ is constant over Γ. With reference to Fig. 14 $s_2=\kappa_c^{-1}(\cos\theta-1)+h$, $s_2=\kappa_c^{-1}\sin\theta$ and $r^2=s_2^2+s_1^2$; the integral becomes

$$u^d\int_{\Gamma}\frac{\partial^2 w}{\partial n\partial t_2}d\Gamma=\int_{-\theta_1}^{\theta_1}\sin\theta\frac{\partial^2 w}{\partial t_2\partial t_1}\kappa_c^{-1}d\theta+\int_{-\theta_1}^{\theta_1}\cos\theta\frac{\partial^2 w}{\partial t_2^2}\kappa_c^{-1}d\theta$$

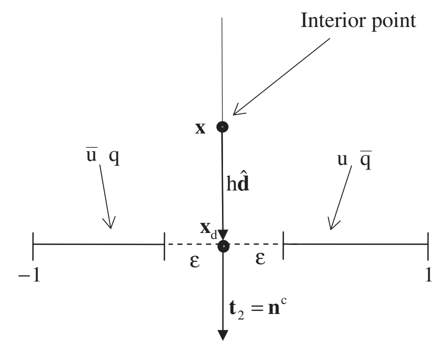

Fig. 13. Integration over the domain $[-1, 1]$.

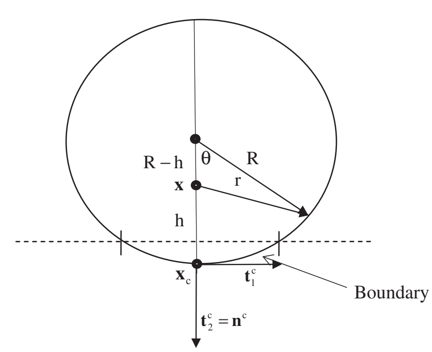

Fig. 14. Integration over a circular boundary of curvature R^{-1}.

$$= -\frac{\kappa_c^{-1}}{\pi}\int_{-\theta_1}^{\theta_1}\sin\theta\frac{s_1 s_2}{(s_1^2+s_2^2)^2}d\theta + \frac{\kappa_c^{-1}}{2\pi}\int_{-\theta_1}^{\theta_1}\cos\theta\frac{s_1^2-s_2^2}{(s_1^2+s_2^2)^2}d\theta \tag{101}$$

In the limit $h \to 0$ with $\theta_1 = 1$ (say) Eq. (101) gives $-(2\pi\kappa_c)^{-1}$ cot(0.5). Similarly, performing the integral with $h = 0$ but on the integral domain $[-\theta_1, -\kappa_c^{-1}\varepsilon] \cup [\kappa_c^{-1}\varepsilon, \theta_1]$ and on letting $\varepsilon \to 0$ yields the identical finite part $-(2\pi\kappa_c)^{-1}\cot(0.5)$. Thus no jump terms exist as predicted by the point limiting method Eq. (49).

12.3. Corner with curved surfaces

Consider the circular boundary with two distinct curvatures as depicted in Fig. 11 and the integral $\int_\Gamma u(\partial^2 w/\partial t_2 \partial n d\Gamma)$, where $u = u^d$ is constant over Γ. The integral of interest is essentially that given in Eq. (78) but considered here in two parts over a finite boundary, i.e.

$$\begin{aligned}\int_\Gamma \frac{\partial^2 w}{\partial n \partial t_2^d}d\Gamma = & \frac{\alpha_{21}^\ell}{2\kappa_d^\ell \pi}\int_0^{\theta_1}\frac{(\tilde{s}_2^2-\tilde{s}_1^2)\sin\theta - 2\tilde{s}_1\tilde{s}_2\cos\theta}{(\tilde{s}_1^2+\tilde{s}_2^2)^2}d\theta \\ & + \frac{\alpha_{21}^b}{2\kappa_d^b \pi}\int_0^{\theta_2}\frac{(s_2^2-s_1^2)\sin\theta - 2s_1 s_2\cos\theta}{(s_1^2+s_2^2)^2}d\theta \\ & + \frac{\alpha_{22}^\ell}{2\kappa_d^\ell \pi}\int_0^{\theta_1}\frac{(\tilde{s}_1^2-\tilde{s}_2^2)\cos\theta - 2\tilde{s}_1\tilde{s}_2\sin\theta}{(\tilde{s}_1^2+\tilde{s}_2^2)^2}d\theta \\ & + \frac{\alpha_{22}^b}{2\kappa_d^b \pi}\int_0^{\theta_2}\frac{(s_1^2-s_2^2)\cos\theta - 2s_1 s_2\sin\theta}{(s_1^2+s_2^2)^2}d\theta \end{aligned} \tag{102}$$

where $\theta_1 = \theta_2 = \pi/2$ (say) $\tilde{s}_2 = h\sin\alpha + (\kappa_d^\ell)^{-1}(\cos\theta - 1)$ and $\tilde{s}_1 = -h\cos\alpha + (\kappa_d^\ell)^{-1}\sin\theta$ and $s_2 = h\sin\alpha + (\kappa_d^b)^{-1}(\cos\theta - 1)$ and $s_1 = -h\cos\alpha + (\kappa_d^b)^{-1}\sin\theta$.

In the limit $h \to 0$ Eq. (102) gives

$$\int_{[-(\pi/2),(\pi/2)]}\frac{\partial^2 w}{\partial n \partial t_2^d}d\Gamma \underset{\text{finite part}}{\rightarrow} \frac{\kappa_d^b + \kappa_d^\ell}{4\pi}(\cos\alpha - \sin\alpha) \tag{103}$$

Similarly, performing the integral with $h = 0$ but on the integral domain $[-\pi/2, -\kappa_d^{\ell-1}\varepsilon] \cup [\kappa_d^{b-1}\varepsilon, \pi/2]$ and on letting $\varepsilon \to 0$ yields

$$\int_{\Gamma/\Gamma_\varepsilon}\frac{\partial^2 w}{\partial n \partial t_2^d}d\Gamma \underset{\text{finite part}}{\rightarrow} -\frac{\kappa_d^b + \kappa_d^\ell}{4\pi}\sin\alpha \tag{104}$$

The difference between Eq. (103) and (104) is identical to that predicted by Eq. (80).

13. Conclusions

This paper is concerned with two alternative approaches for the evaluation of free terms in hypersingular boundary integral equations. The point-limiting method involves a source point approaching a point centred on boundary interval, where the rate of approach is faster than the contraction rate of the interval. The boundary-limiting method is a vanishing exclusion zone method that involves moving a boundary interval into position at a faster rate than the interval contracts. The following conclusions can be drawn from the paper:

(1) The point-limiting method and the boundary-limiting method provide consistent results.
(2) The boundary and point-limiting methods can suffer from path dependency although if integrand g satisfies the identity $\int_\Gamma g d\Gamma = 0$ for a closed boundary Γ, then the integrals involved are both path and shape independent.
(3) The point and boundary-limiting methods have the advantage of requiring only asymptotic integral approximations on a shrinking boundary, thus avoiding the need for solutions in closed form.
(4) The point-limiting methods cater for mixed boundary conditions without need for reference to the constitutive equations.

Appendix I

This appendix is concerned the evaluation of the integrals that appear on the right-hand side of Eqs. (43)–(45) using the point-limiting method. Of particular interest is an explicit demonstration of path independence. Consider the point $\mathbf{x}$ approaching $\mathbf{x}_c$ as depicted in Fig. 6 where the approach is at an angle β to the surface normal. The radial vector $\mathbf{r} = \mathbf{y} - \mathbf{x}$ is conveniently represented as $\mathbf{r} = s_1\mathbf{t}_1 + s_2\mathbf{t}_2$, where as depicted in Fig. 6, $s_1 = \varepsilon^\gamma \sin\beta + t_1$ and $s_2 = \varepsilon^\gamma\cos\beta$. Note that $r^2 = s_1^2 + s_2^2$ and $s_1 = s_2\tan\beta + t_1$ and

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}\frac{\partial w}{\partial t_2}dt_1 &= \int_{-\varepsilon}^{\varepsilon}\frac{dw}{dr}\frac{\partial r}{\partial t_2}dt_1 = \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{s_2}{r^2}dt_1 = \frac{1}{2\pi}\int_{-\varepsilon+s_2\tan\beta}^{\varepsilon+s_2\tan\beta}\frac{s_2}{s_2^2+s_1^2}ds_1 \\ &= \frac{1}{2\pi}\int_{(-\varepsilon+s_2\tan\beta/s_2)}^{(\varepsilon+s_2\tan\beta/s_2)}\frac{1}{1+s^2}ds = \frac{\tan^{-1}(s)}{2\pi}\Big|_{(-\varepsilon+s_2\tan\beta/s_2)}^{(\varepsilon+s_2\tan\beta/s_2)}\end{aligned} \tag{I1}$$

where for $\gamma > 1$, in the limit $\varepsilon \to 0$, the upper and lower limits behave like $\varepsilon^{1-\gamma}/\cos\beta$ and $-\varepsilon^{1-\gamma}/\cos\beta$, and since $\cos\beta > 0$ the right-hand side of Eq. (43) yields 0.5.

This result matches that of Eq. (43) and explicitly confirms the path independency of the limit. In a similar fashion

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon} t_1\frac{\partial}{\partial t_2}\left(\frac{\partial w}{\partial t_1}\right)dt_1 &= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} t_1\frac{\partial}{\partial t_2}\left(\frac{s_1}{r^2}\right)dt_1 = -\frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} t_1\frac{s_2 s_1}{r^4}dt_1 \\ &= -\frac{1}{\pi}\int_{-\varepsilon+s_2\tan\beta}^{\varepsilon+s_2\tan\beta}\frac{(s_1-s_2\tan\beta)s_1 s_2}{(s_2^2+s_1^2)^4}ds_1 \\ &= -\frac{1}{\pi}\int_{(-\varepsilon+s_2\tan\beta/s_2)}^{(\varepsilon+s_2\tan\beta/s_2)}\frac{(s-\tan\beta)s}{(1+s^2)^2}ds \\ &= -\frac{1}{\pi}\left(\tan^{-1}(s) - \frac{s-\tan\beta}{1+s^2}\right)\Big|_{(-\varepsilon+s_2\tan\beta/s_2)}^{(\varepsilon+s_2\tan\beta/s_2)}\end{aligned} \tag{I2}$$

where for $\gamma > 1$, in the limit $\varepsilon \to 0$, the right-hand side of Eq. (44) yields -0.5 in agreement with Eq. (44).

Repeating the process for the integral in Eq. (45) gives

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}\frac{\partial}{\partial t_2}\left(\frac{\partial w}{\partial t_2}\right)dt_1 &= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\frac{\partial}{\partial t_2}\left(\frac{s_2}{r^2}\right)dt_1 = \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon}\frac{s_1^2-s_2^2}{r^4}dt_1 \\ &= \frac{2}{\pi}\int_{-\varepsilon+s_2\tan\beta}^{\varepsilon+s_2\tan\beta}\frac{s_1^2-s_2^2}{(s_2^2+s_1^2)^4}ds_1 = \frac{1}{\pi}\int_{(-\varepsilon+s_2\tan\beta/s_2)}^{(\varepsilon+s_2\tan\beta/s_2)}\frac{s^2-1}{(1+s^2)^4}ds \\ &= -\frac{2}{\pi}\frac{s}{1+s^2}\Big|_{(-\varepsilon+s_2\tan\beta/s_2)}^{(\varepsilon+s_2\tan\beta/s_2)}\end{aligned} \tag{I3}$$

where for $\gamma > 1$, in the limit $\varepsilon \to 0$, the right-hand side of Eq. (I3) tends to zero in agreement to Eq. (45).

Appendix II

This appendix is concerned the evaluation of the integrals that appear on the right-hand side of Eqs. (59), (62)–(64) using the point-limiting method. It is readily confirmed that Eqs. (59) and (63) are path dependent, so of particular interest here are Eqs. (62) or (64) as path dependence is uncertain.

Consider the point $\mathbf{x}$ approaching $\mathbf{x}_d$ as depicted in Fig. 9 where the approach is at an angle β to the surface nodal normal. The radial vector $\mathbf{r} = \mathbf{y} - \mathbf{x}_d$ is conveniently represented as $\mathbf{r} = s_1^\ell\mathbf{t}_1^\ell + s_2^\ell\mathbf{t}_2^\ell$ or $\mathbf{r} = s_1^b\mathbf{t}_1^b + s_2^b\mathbf{t}_2^b$, where $s_2^\ell = \varepsilon^\gamma\sin(\alpha-\beta)$, $s_2^b = \varepsilon^\gamma\sin(\alpha+\beta)$, $s_1^\ell = t_1^\ell - \varepsilon^\gamma\cos(\alpha-\beta) = t_1^\ell - s_2^\ell\cot(\alpha-\beta)$ and $s_1^b = t_1^b - \varepsilon^\gamma\sin(\alpha+\beta) = t_1^\ell - s_2^b\cot(\alpha+\beta)$, and where $r^2 = (s_1^\ell)^2 + (s_2^\ell)^2 = (s_1^b)^2 + (s_2^b)^2$.

The integrals of interest in Eqs. (62) and (64) are

$$\chi_{22}^{\ell} = \int_{\tilde{\Gamma}_{\varepsilon}^{p\ell}} \frac{\partial w}{\partial t_2^{\ell} \partial t_2^{\ell}} d\Gamma^p = \frac{1}{2\pi} \int_0^{\varepsilon} \frac{(s_1^{\ell})^2 - (s_2^{\ell})^2}{r^4} dt_1 \sim -\frac{\cos(\alpha-\beta)}{2\pi} \varepsilon^{-\gamma} \quad \text{(II1)}$$

$$\chi_{22}^{b} = \int_{\tilde{\Gamma}_{\varepsilon}^{pb}} \frac{\partial w}{\partial t_2^{b} \partial t_2^{b}} d\Gamma^p = \frac{1}{2\pi} \int_0^{\varepsilon} \frac{(s_1^{b})^2 - (s_2^{b})^2}{r^4} dt_1 \sim -\frac{\cos(\alpha+\beta)}{2\pi} \varepsilon^{-\gamma} \quad \text{(II2)}$$

$$\chi_{21}^{\ell} = \int_{\tilde{\Gamma}_{\varepsilon}^{p\ell}} \frac{\partial w}{\partial t_2^{\ell} \partial t_1^{\ell}} d\Gamma^p = -\frac{1}{\pi} \int_0^{\varepsilon} \frac{s_1^{\ell} s_2^{\ell}}{r^4} dt_1 \sim -\frac{\sin(\alpha-\beta)}{2\pi} \varepsilon^{-\gamma} \quad \text{(II3)}$$

$$\chi_{21}^{b} = \int_{\tilde{\Gamma}_{\varepsilon}^{pb}} \frac{\partial w}{\partial t_2^{b} \partial t_1^{b}} d\Gamma^p = -\frac{1}{\pi} \int_0^{\varepsilon} \frac{s_1^{b} s_2^{b}}{r^4} dt_1 \sim -\frac{\sin(\alpha+\beta)}{2\pi} \varepsilon^{-\gamma} \quad \text{(II4)}$$

The coefficient in Eq. (62) is $\alpha_{1j}^{\ell}(\chi_{2j}^{\ell} - \chi_{2j}^{d})$ and observe that

$$\alpha_{1j}^{\ell}(\chi_{2j}^{\ell} - \chi_{2j}^{d}) \sim \frac{\varepsilon^{-\gamma}}{2\pi}(\sin\alpha\sin(\alpha-\beta) - \sin\alpha\sin(\alpha+\beta) + \cos\alpha\cos(\alpha-\beta) - \cos\alpha\cos(\alpha+\beta)) = \frac{\varepsilon^{-\gamma}}{\pi}(\sin\alpha\cos\alpha\sin\beta - \cos\alpha\sin\alpha\sin\beta) = 0 \quad \text{(II8)}$$

which is independent of β.

Similarly the coefficient in Eq. (64) is $\alpha_{2j}^{\ell}(\chi_{2j}^{\ell} + \chi_{2j}^{d})$ and observe that

$$\alpha_{2j}^{\ell}(\chi_{2j}^{\ell} + \chi_{2j}^{d}) \sim \frac{\varepsilon^{-\gamma}}{2\pi}(\cos\alpha\sin(\alpha-\beta) + \cos\alpha\sin(\alpha+\beta) - \sin\alpha\cos(\alpha-\beta) - \sin\alpha\cos(\alpha+\beta)) = \frac{\varepsilon^{-\gamma}}{\pi}(\cos\alpha\sin\alpha\cos\beta - \sin\alpha\cos\alpha\cos\beta) = 0 \quad \text{(II11)}$$

which is again independent of β and hence confirming path independence.

References

[1] De Lacerda LA, Wrobel LC. Dual boundary element method for axisymmetric crack analysis. International Journal of Fracture 2002;113(3):267–84.

[2] Lu S, Huang Q. New type of dual BEM and Green's-function-library strategy for fracture analysis in complex structures. Acta Mechanica Solida Sinica 2000;13(4):363–73.

[3] Ariza MP, Dominguez J. Boundary element formulation for 3D transversely isotropic cracked bodies. International Journal for Numerical Methods in Engineering 2004;60(4):719–53.

[4] Hong H-K, Chen JT. Derivations of integral equations of elasticity. Journal of Engineering Mechanics, ASCE 1988;114(6):1028–44.

[5] Chen JT, Hong H-K. Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. Applied Mechanics Reviews, ASME 1999;52(1):17–33.

[6] Chen JT, Hong H-K. Dual boundary integral equations at a corner using contour approach around singularity. Advances in Engineering Software 1994;21:169–78.

[7] Guiggiani M. Hypersingular boundary integral equations have an additional free term. Computational Mechanics 1995;16(4):245–8.

[8] Chen JT, Liang MT, Yang SS. Dual boundary integral equations for exterior problems. Engineering Analysis with Boundary Elements 1995;16(4): 333–40.

[9] Toh K-C, Mukherjee S. Hypersingular and finite part integrals in the boundary element method. International Journal of Solids and Structures 1994;31(17): 2299–312.

[10] Mantic V, Paris F. Existence and evaluation of the two free terms in hypersingular boundary integral equation of potential theory. Engineering Analysis with Boundary Elements 1995;16:253–60.

[11] Gray LG. Evaluation of singular and hypersingular Galerkin integrals: direct limits and symbolic computation. In: Sladek V, Sladek J, editors. Singular Integrals in Boundary Element Methods (Advances in Boundary Elements Vol 3). WIT Press; 1998.

[12] Guigiiani M, Krishnasamy G, Rudolphi TJ, Rizzo FJ. A general algorithm for the numerical solution of hypersingular boundary integral equations. ASME Journal of Applied Mechanics 1992;59:604–14.

[13] Guigiiani M. Formulation and numerical treatment of boundary integral equations with hypersingular kernels. In: Sladek V, Sladek J, editors. Singular Integrals in Boundary Element Methods (Advances in Boundary Elements Vol 3). WIT Press; 1998.

[14] Mantic V. On computing boundary limiting values of boundary integrals with strongly singular and hypersingular kernels in 3D BEM for elastostatic. Engineering Analysis with Boundary Elements 1994;13:115–34.

[15] Sladek V, Sladek J, Tanaka M. Regularization of hypersingular and nearly singular integrals in the potential theory and Elasticity. International Journal of Numerical Methods in Engineering 1993;36:1609–28.

[16] Salvadori A. Analytical integrations of hypersingular kernel in 3D BEM problems. Computer Methods in Applied Mechanics and Engineering 2001;190(31):3957–75.

[17] Davey K, Alonso Rasgado MT. Integration over simplexes for accurate domain and boundary integral evaluation in boundary elements. Computers and Structures 2004:193–21182 2004:193–211.