Revisit of the equivalence of the displacement discontinuity method and boundary element method for solving crack problems

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ABSTRACT

In this note, it is shown explicitly that the displacement discontinuity method (DDM) is equivalent to the boundary element method (BEM) for solving crack problems. To show this, the direct traction boundary integral equation (BIE) in terms of the displacement jump across crack surfaces is applied to a crack in an infinite 2-D elastic domain. Then, the direct traction BIE is discretized with constant line elements. All the integrals are evaluated analytically. The yielded linear system of equations is found to be exactly the same as the original DDM system of equations in terms of the displacement discontinuities. This proof of the equivalence of the DDM and BEM suggests that the two methods are the same in nature and both are based on the same traction BIE for crack problems.

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1. Introduction

The displacement discontinuity method (DDM) was proposed by Crouch in 1976 [1] based on the earlier research results in the elasticity theory of dislocations (cracks or inclusions) in mining [2–6]. DDM has been applied widely in solving crack problems, especially in rock mechanics [7], such as modeling hydraulic fracture [8,9]. In the original DDM for two-dimensional (2-D) elasticity problems, the displacement discontinuity across the two surfaces of a crack are assumed to be constant on line segments representing the crack. Stresses in the cracked domain are related to the displacement discontinuities on the line segments by use of Papkovitch functions and superposition. A linear system of algebraic equations is formed directly that can be used to solve the unknown displacement discontinuities on the crack surfaces. The method is easy to implement and accurate when the number of line segments or displacement discontinuities is sufficiently large [8,9]. To improve the accuracy of the DDM, higher-order representations of the displacement discontinuities are also available [8–10].

The boundary element method (BEM) based on the boundary integral equation (BIE) for elasticity theory [11] has been applied to solve crack problems for more than three decades [12–20]. The multidomain BEM was introduced first to solve crack problems [14] using only the displacement (singular) BIE, in which a cracked body is divided into subdomains using artificial boundaries connecting the cracks. In the late 1980s and early 1990s, the traction (hypersingular) BIE [14–25] was introduced and the displacement discontinuity (or jump) across the crack surfaces is used as the primary unknown variable to solve the crack problems based on the one surface model of the crack. Since then, dual BIE formulations [19,26,27] using various combinations of the displacement and traction BIEs have been applied to solve crack problems in a more general setting, for example, by discretizing the two surfaces of a crack in the model and using the displacements on the two surfaces as separate variables. More comprehensive reviews of the BEM for crack problems can be found in Refs. [14,17,18].

What are the connections between the DDM and BEM for solving crack problems? Both methods use the solution due to a point force or dislocation, the method of superposition, and unknowns only on the boundary. Therefore, it is natural for researchers to link the two methods. In the literature, the DDM has been said to be a special indirect BEM, or similar to the direct BEM, or a third method based on physical arguments [7–10,28]. In 1988, Hong and Chen provided perhaps the earliest and closest linkage between the DDM and BEM in Refs. [15,16]. They started with the traction BIE for a crack in terms of the displacement discontinuity and discretized the BIE with constant elements. They obtained a set of equations containing derivatives of inverse tangent functions. Besides, the authors claim that the final equations are identical to the ones from the DDM [1] by using MACSYMA (a symbolic manipulator). In the 1990s, Linkov and Mogilevskaya also noted that the DDM is equivalent to the traction BIE for 2-D crack problems [29,30] and later in the context of 3-D crack problems [31]. However, to the authors' best knowledge, no

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equivalence has been shown explicitly in the literature demonstrating that the DDM and BEM systems of equations are identical, and therefore the two methods are the same for solving crack problems.

In this short note, it is shown explicitly that the discretization of the direct traction BIE for a crack in a 2-D infinite elastic domain using constant line elements will yield exactly the same system of equations as in the original DDM. This linkage will help advance the development of the DDM (or the BEM) as it is being increasingly used to solve large-scale crack problems, such as those in hydraulic fracture [8,9,32,33]. There are abundant knowledge and software tools in the field of the BEM research regarding solving large-scale BEM models, such as the fast multipole BEM, adaptive cross approximation method, and others [34], that can be applied to accelerate the solutions of the systems of equations obtained using the DDM (or the BEM) in solving large-scale crack problems.

2. Equivalence of the DDM and BEM

Consider a mode I crack in a 2-D infinite elastic medium under plane strain condition (Fig. 1). The displacement discontinuity method gives the following equations for determining the displacement discontinuities distributed over N line segments (Fig. 2) [1]:

$$p = \frac{G}{\pi(1-\nu)} \sum_{j=1}^{N} \frac{a_j}{(x_j - x_i)^2 - a_j^2} (\Delta u_2)_j, \quad \text{for } i = 1, 2, ..., N, \tag{1}$$

where x_j is x coordinate of the center of segment j, a_j is the half length of segment j, G is the shear modulus, ν is Poisson's ratio, $p = \sigma_{22}(x, 0^{\pm})$ with $|x| < a$ is the normal stress on the boundary in the y direction (negative pressure load on the entire crack surface), and

$$(\Delta u_2)_j = u_2(x_j, 0^+) - u_2(x_j, 0^-)$$

is the (constant) *displacement discontinuity* over line segment j.

To apply the BEM for the same crack problem (Fig. 1), we start with the direct displacement (singular) BIE for the configuration
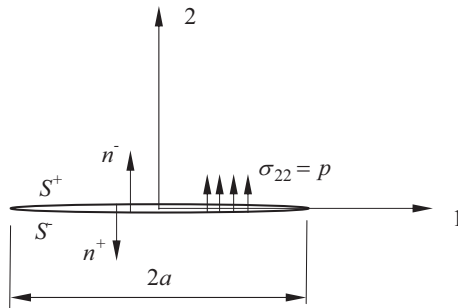

Fig. 1. A crack in an infinite 2-D elastic domain.
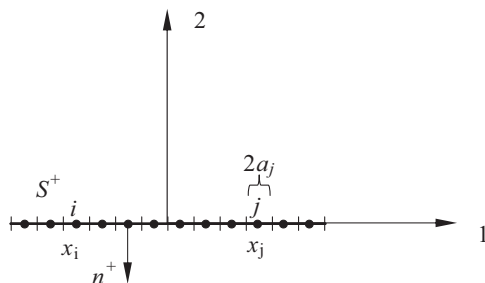

Fig. 2. One surface model of the crack and discretized with N constant line elements.

in Fig. 1 [11]:

$$(1/2)\mathbf{u}(\mathbf{x}) = \int_{S^+ \cup S^-} [\mathbf{U}(\mathbf{x}, \mathbf{y})\mathbf{t}(\mathbf{y}) - \mathbf{T}(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y})]dS(\mathbf{y}), \tag{2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the source point and field point, respectively, $\mathbf{u}$ and $\mathbf{t}$ are the displacement and traction vector, respectively, $\mathbf{U}$ and $\mathbf{T}$ are 2×2 matrices from the displacement and traction kernels in Kelvin's solution, respectively [20]. It is assumed that the crack surface is smooth at the source point $\mathbf{x}$.

The corresponding direct traction (hypersingular) BIE is [16,19,20]

$$(1/2)\mathbf{t}(\mathbf{x}) = \int_{S^+ \cup S^-} [\mathbf{K}(\mathbf{x}, \mathbf{y})\mathbf{t}(\mathbf{y}) - \mathbf{H}(\mathbf{x}, \mathbf{y})\mathbf{u}(\mathbf{y})]dS(\mathbf{y}), \tag{3}$$

where $\mathbf{K}$ and $\mathbf{H}$ are 2×2 matrices from the two new kernels based on Kelvin's solution [20]. For completeness, we list the expressions for the four kernels (U, T, K and H) in index notation for the case of plane strain in the following [20]:

$$U_{ij}(\mathbf{x}, \mathbf{y}) = (1/(8\pi G(1-\nu)))[(3-4\nu)\delta_{ij} \log(1/r) + r_{,i}r_{,j}], \tag{4}$$

$$T_{ij}(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi(1-\nu)r}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\}, \tag{5}$$

$$K_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi(1-\nu)r}[(1-2\nu)(\delta_{ij}r_{,k} + \delta_{jk}r_{,i} - \delta_{ik}r_{,j}) + 2r_{,i}r_{,j}r_{,k}] n_k(\mathbf{x}), \tag{6}$$

$$\begin{aligned} H_{ij}(\mathbf{x}, \mathbf{y}) = \frac{G}{2\pi(1-\nu)r^2}&\left\{2\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ik}r_{,j} + \nu(\delta_{ij}r_{,k} + \delta_{jk}r_{,i}) - 4r_{,i}r_{,j}r_{,k}]\right.\\ &+ 2\nu(n_i r_{,j}r_{,k} + n_k r_{,i}r_{,j}) - (1-4\nu)\delta_{ik}n_j \\ &\left.+ (1-2\nu)(2n_j r_{,i}r_{,k} + \delta_{ij}n_k + \delta_{jk}n_i)\right\}n_k(\mathbf{x}), \end{aligned} \tag{7}$$

in which r is the distance between the source point $\mathbf{x}$ and field point $\mathbf{y}$, $(\ )_{,i} = \partial(\ )/\partial y_i$, δ_{ij} is the Kronecker δ symbol, and n_i is the direction cosine of the normal.

When S^- is collapsed onto S^+ to form a one surface model for the crack (Fig. 2), the displacement BIE (2) is reduced to the following equation [26,35]:

$$(1/2)\Sigma\mathbf{u}(\mathbf{x}) = \int_{S^+} [\mathbf{U}(\mathbf{x}, \mathbf{y})\Sigma\mathbf{t}(\mathbf{y}) - \mathbf{T}(\mathbf{x}, \mathbf{y})\Delta\mathbf{u}(\mathbf{y})]dS(\mathbf{y}), \tag{8}$$

and the traction BIE (3) is reduced to the following equation [19, 22–27]:

$$(1/2)\Delta\mathbf{t}(\mathbf{x}) = \int_{S^+} [\mathbf{K}(\mathbf{x}, \mathbf{y})\Sigma\mathbf{t}(\mathbf{y}) - \mathbf{H}(\mathbf{x}, \mathbf{y})\Delta\mathbf{u}(\mathbf{y})]dS(\mathbf{y}), \tag{9}$$

where $\Delta\mathbf{u} = \mathbf{u}|_{S^+} - \mathbf{u}|_{S^-}$, $\Sigma\mathbf{u} = \mathbf{u}|_{S^+} + \mathbf{u}|_{S^-}$, $\Delta\mathbf{t} = \mathbf{t}|_{S^+} - \mathbf{t}|_{S^-}$, and $\Sigma\mathbf{t} = \mathbf{t}|_{S^+} + \mathbf{t}|_{S^-}$. Assuming that the load on the crack surface is in equilibrium (such as a pressure load), we have

$$\Sigma\mathbf{t} = \mathbf{t}|_{S^+} + \mathbf{t}|_{S^-} = \mathbf{0}. \tag{10}$$

Thus displacement BIE (8) is further reduced to

$$(1/2)\Sigma\mathbf{u}(\mathbf{x}) = -\int_{S^+} \mathbf{T}(\mathbf{x}, \mathbf{y})\Delta\mathbf{u}(\mathbf{y})dS(\mathbf{y}), \tag{11}$$

and the traction BIE (9) is further reduced to

$$(1/2)\Delta\mathbf{t}(\mathbf{x}) = -\int_{S^+} \mathbf{H}(\mathbf{x}, \mathbf{y})\Delta\mathbf{u}(\mathbf{y})dS(\mathbf{y}). \tag{12}$$

Note that an integral representation similar to BIE (11) and in terms of the displacement discontinuity was presented by Jaswon and Symm in Ref. [36] (Eq. (6.4.4), with the source point $\mathbf{x}$ inside the domain). If one let the source point approach the surface, the same BIE (11) can be derived from Jaswon and Symm's result in

Ref. [36]. However, the displacement BIE (11) is insufficient when it is applied alone to a crack problem, as it contains both the displacement sum and displacement discontinuity across the crack surfaces (two unknown functions). On the other hand, the traction BIE (12) is ideal for solving the displacement discontinuity (or jump) for crack problems (and then stress intensity factors) when the traction is specified on the crack surfaces or another load at a remote location.

In the following, we will show that the discretization of the traction BIE (12) leads directly to the DDM equation, that is, Eq. (1). First, the traction BIE (12) can be written in the following component form:

$$\frac{1}{2}\begin{Bmatrix} \Delta t_1 \\ \Delta t_2 \end{Bmatrix} = -\int_{S^+} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} dS, \qquad (13)$$

where H_{kl} are the coefficients of matrix $\mathbf{H}$. Now, apply the BIE in (12) or (13) to the crack problem shown in Fig. 1 using the discretization shown in Fig. 2. That is, the one surface crack model is discretized with N constant elements, where for element j, the element length is $2a_j$ and the x-coordinate of the center of the element is x_j, for $j=1,2,...,N$. Then, the discretized BIE when the source point $\mathbf{x}$ is placed at the center of element i is

$$\frac{1}{2}\begin{Bmatrix} \Delta t_1 \\ \Delta t_2 \end{Bmatrix}_i = -\sum_{j=1}^{N} \left(\int_{\Delta S_j} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}_i dS \right) \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_j, \qquad (14)$$

for $i=1,2,...,N$; in which ΔS_j indicates element j. Integration of the four coefficients can be done analytically using the results in Appendix A.2 of Ref. [20] for an arbitrary orientation of the element and location of the source point $\mathbf{x}$ (Fig. 3). The parameters shown in Fig. 3 have the relations $d=0$, $r_1^2 = T_1^2$, $r_2^2 = T_2^2$ for the special case of the given crack (Fig. 4). Applying the results in Eqs. (A.20), (A.22), and (A.25) of Appendix A.2 in Ref. [20], we have

$$\int_{\Delta S_j} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}_i dS = \begin{bmatrix} D_{111} & D_{121} \\ D_{211} & D_{221} \end{bmatrix} = \frac{G}{2\pi(1-\nu)} \begin{bmatrix} -\Gamma_{12} & 0 \\ 0 & -\Gamma_{12} \end{bmatrix}$$

$$= \frac{G}{2\pi(1-\nu)} \begin{bmatrix} \left(\frac{1}{T_1} - \frac{1}{T_2}\right)_{ij} & 0 \\ 0 & \left(\frac{1}{T_1} - \frac{1}{T_2}\right)_{ij} \end{bmatrix}, \quad \text{for } i,j = 1,2,...,N. \qquad (15)$$

From Fig. 4, we have

$$\left(\frac{1}{T_1} - \frac{1}{T_2}\right)_{ij} = \frac{1}{(x_j - x_i) - a_j} - \frac{1}{(x_j - x_i) + a_j} = \frac{2a_j}{(x_j - x_i)^2 - a_j^2}$$

Therefore, the integrals in (15) are found to be

$$\int_{\Delta S_j} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}_i dS = \frac{G}{\pi(1-\nu)} \begin{bmatrix} \frac{a_j}{(x_j - x_i)^2 - a_j^2} & 0 \\ 0 & \frac{a_j}{(x_j - x_i)^2 - a_j^2} \end{bmatrix},$$
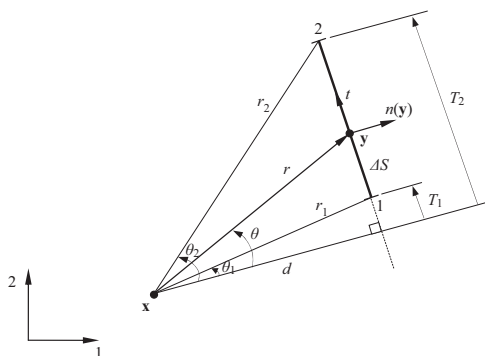
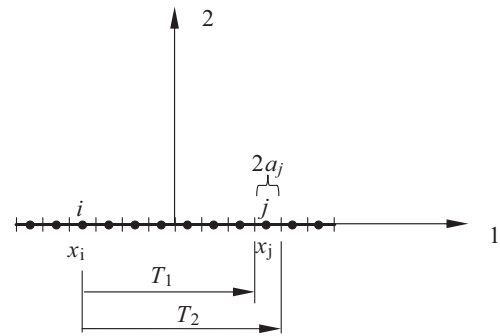**Fig. 3.** Analytical integration on a constant element.

Fig. 4. Parameters for the analytical integration on the crack.

for $i,j = 1,2,...,N$. $\qquad (16)$

For $i=j$, the integral is a Hadamard finite part (HFP) integral and the result given by the above equation is consistent with the one given in Eq. (A.29) of Ref. [20].

Substituting results of Eq. (16) in Eq. (14), we obtain the following BEM system of equations for the crack problem

$$\frac{1}{2}\begin{Bmatrix} \Delta t_1 \\ \Delta t_2 \end{Bmatrix}_i = -\frac{G}{\pi(1-\nu)}\sum_{j=1}^{N} \begin{bmatrix} \frac{a_j}{(x_j - x_i)^2 - a_j^2} & 0 \\ 0 & \frac{a_j}{(x_j - x_i)^2 - a_j^2} \end{bmatrix} \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_j, \quad \text{for } i=1,2,...,N. \qquad (17)$$

Note that the x-component and y-component of the equations are not coupled.

For the given load (negative pressure on the entire two crack surfaces) as shown in Fig. 1, we have

$$\Delta t_2 = t_2^+ - t_2^- = -p - p = -2p.$$

Thus, the second equation (y-component) of Eq. (17) becomes

$$-p = -\frac{G}{\pi(1-\nu)}\sum_{j=1}^{N} \frac{a_j}{(x_j - x_i)^2 - a_j^2}(\Delta u_2)_j, \quad \text{for } i=1,2,...,N, \qquad (18)$$

which is the same as Eq. (1) derived in the displacement discontinuity method [1].

Therefore, the equivalence of the DDM and BEM for 2-D crack problems is shown explicitly using the direct traction BIE which is discretized with constant elements.

3. Discussions

This note is limited to the case of a single crack in a 2-D infinite domain with the BEM discretized using constant line elements. However, the equivalence of the DDM and the BEM based on the traction BIE for modeling crack problems is believed to be general, not limited to a single crack in 2-D infinite domain, or straight cracks, or the use of constant elements. The 2-D case with constant elements is used in this note simply because the analytical integration results are readily available in Ref. [20]. For multiple cracks in infinite or finite domains, curved cracks in 2-D and 3-D, and BIEs discretized with linear, quadratic or other higher-order elements, similar equivalence results can be shown if analytical results for integrals of the kernels in the traction BIEs are available.

The proof of the equivalence of the DDM and BEM with constant elements for crack problems also indicates that the BEM with constant elements can be applied to solve crack problems successfully. In the BEM literature, the BEM has been reported in solving crack problems with almost exclusively higher-order elements, in the pursuit of higher accuracy and in dealing with the crack-tip singularity. However, the DDM shows that traction BIE and the BEM with constant elements can be applied,

and have been applied successfully for almost four decades, to solve crack problems if one uses the displacement discontinuities which are well defined near crack tips. In Ref. [37], the fast multipole BEM code in Ref. [20] that is based on a dual BIE formulation (a linear combination of the displacement BIE and traction BIE) and using constant elements, is applied to solve 2-D multiple crack interaction problems. Accurate BEM results of the stress intensity factors are obtained using the displacement discontinuity (crack-tip opening displacement) results.

The linkage between the DDM and BEM can help advance the development of the DDM, which has been increasingly used in solving large-scale crack problems, such as those in hydraulic fracture (fracking) [8,9,32,33]. The matrix for the system of equations from the DDM is dense and nonsymmetric in general, and therefore it requires $O(N^2)$ computational effort to obtain the solutions for large-scale problems with the conventional DDM approach. On the other hand, there are many fast solution methods for solving large-scale BEM models, such as the fast multipole BEM, adaptive cross approximation method, and others [34], that can be applied directly to accelerate the solutions of the DDM in solving large-scale crack problems. A couple of such applications are given in Refs. [38,39].

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