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A necessary and sufficient BEM/BIEM for two-dimensional elasticity problems

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ABSTRACT

It is well known that the patch test is required for the finite element method (FEM). We may wonder whether we need any special test for the boundary element method (BEM). A sufficient and necessary boundary integral equation method (BIEM) to ensure a unique solution is our concern. In this paper, we revisit this issue for the interior two-dimensional (2-D) elasticity problem and investigate the equivalence of the solution space between the integral equation and the partial differential equation. Based on the degenerate kernel and the eigenfunction expansion, the range deficiency of the integral operator for the solution space in the degenerate-scale problem for the 2-D elasticity in the BIEM is analytically studied. According to the Fichera's idea, we enrich the conventional BIEM by adding constants and corresponding constraints. In addition, we introduce the concept of modal participation factor (MPF) to examine whether the adding term of rotation is required for interior simply-connected problems. Finally, two simple examples of degenerate-scale problems containing circular and elliptical boundaries subjected to various boundary conditions of the rigid body translation and rotation for 2-D elasticity problems are demonstrated by using the necessary and sufficient BIEM.

1. Introduction

In 1956, Kinoshita and Mura [1] derived the singular boundary integral equation for elasticity. Later, the boundary element method (BEM), or sometimes it was called the boundary integral equation method (BIEM), has been numerically implemented since Rizzo [2] discretized the integral equation for elastostatics problems in 1967. In recent decades, it is well known that the BEM and the BIEM are widely used to solve engineering problems. However, the mathematical models of the integral equation for engineering problems are not equivalent for the solution space to that of the partial differential equations (PDE) as pointed out by Feng [3] and Yu [4]. There are four classical ill-posed problems in the BEM/BIEM, degenerate-scale problems, degenerate-boundary problems, fictitious-frequency problems and spurious-eigenvalue problems. Not only the wedge problem [5], but also the degenerate scale in the BEM/BIEM due to the incomplete mathematical model [6–11] can be seen as paradox for 2-D elasticity problems. A degenerate scale [12] as a well-known problem in the BEM has been noticed for a long time. In the past experience of solving the Laplace problem [13–16], we found that the main key of the BEM was the fundamental solution containing the base of $\ln r$. Even though in the 2-D elasticity problem, the degenerate scale still exists in the BEM/BIEM [17–21]. The occurring mechanisms of two problems, Laplace and Navier, are similar due to the $\ln r$ term. However, there are two degenerate scales for elasticity problems instead of only one for Laplace problems. As we know, there are several regularized methods to deal with the degenerate scale in 2-D Laplace problems, e.g., rank promotion by adding the boundary flux equilibrium [22], the hypersingular formulation [12], the CHEEF method and the method of adding a rigid body mode [13]. Recently, Chen et al. [12,23] used the Fichera's method which offers the free constant term and the corresponding constraint to transform an ill-posed system (the conventional BEM) to a well-posed system through the analytical derivation by using the degenerate kernel and Fourier series. A necessary and sufficient BIE obtained from this approach was derived to avoid the pitfall for solving Laplace problems. The non-unique solution for the Neumann problem can be physically realized as a rigid body mode. The Fichera's method can be also employed to obtain reasonable solutions of these problems for the Laplace equation [24]. However, it is interesting that Hu et al. [25] and He et al. [20] proposed that both a translation term and a rotation term were required in the BIE for 2-D elasticity problems. It is different from the Fichera's method [26] which adds the free constant and the corresponding

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constraint. Therefore, we may wonder whether the rotation term is needed or not for the interior problem.

In this paper, we use the degenerate kernel instead of the closed-form Kelvin solution to derive the analytical degenerate scale and field response for the isotropic elasticity problem containing circular and ellipticalal boundaries. Through the analytical derivation, the range deficiency of the solution space is found and we find that a constant term could not appear in case of a degenerate scale. It means that the integral equation is not equivalent to the partial differential equation for the solution space. In the linear algebraic system, the singular value decomposition (SVD) and the modal participation factor (MPF) are used to study the contribution of the singular vector corresponding to the zero singular value for various boundary conditions. By way of the analytical derivation and numerical implementation, we find that the extra rotation term in [9,20] may not be needed for degenerate-scale problems. It is well known that FEM code should be examined by using the patch test to solve a simple example of constant strain case. Similarly, the BEM code should be checked by using a simpler solution of the rigid body translation and rotation. Since the conventional BEM/BIEM fails to solve the degenerate-scale problem, we propose the necessary and sufficient BEM/BIEM to deal with this problem. According to Fichera's idea, we enrich the conventional BEM/BIEM by adding constants and the corresponding constraints. Several examples of the degenerate-scale problem containing circular and elliptical boundaries for 2-D elasticity are demonstrated by using the necessary and sufficient BEM/BIEM.

2. Problem statement and formulation

For simplicity, the medium is considered to be linearly elastic, isotropic and homogenous. The governing equation is the Navier equation as follows:

$$(\lambda+G)\nabla(\nabla\cdot \underset{\sim}{u}(\mathbf{x}))+G\ \ \nabla^2\ \ \underset{\sim}{u}(\mathbf{x})=\underset{\sim}{0}\ \ ,\quad \mathbf{x}\in D, \tag{1}$$

where $\underset{\sim}{u}(\mathbf{x})$ is the displacement of the field point $\mathbf{x}$, D is the domain of interest, ∇^2 is the Laplace operator, λ and G are the Lamé constants for the isotropic elasticity. The integral representation of single-layer potential for the solution yields

$$u_i(\mathbf{x})=\int_B U_{ij}(\mathbf{x},\mathbf{s})\alpha_j(\mathbf{s})\ \ dB(\mathbf{s})\ ,\quad \mathbf{x}\in D\ , \tag{2}$$

where $U_{ij}(\mathbf{x},\mathbf{s})$ is the fundamental solution for the displacement response of the ith direction at the field point $\mathbf{x}$ due to a concentrated load of the jth direction at the source $\mathbf{s}$ and $\alpha_j(\mathbf{s})$ is the unknown boundary density. The explicit form of $U_{ij}(\mathbf{x},\mathbf{s})$, or the so-called Kelvin solution, is

$$U_{ij}(\mathbf{x},\mathbf{s})=-\frac{1}{8\pi G(1-\nu)}\left(\kappa\,\delta_{ij}\ln r-\frac{y_iy_j}{r^2}\right), \tag{3}$$

where δ_{ij} is the Kronecker delta, ν is the Poisson ratio ($\nu=\lambda/2(\lambda+G)$), $\kappa=3-4\nu$, $r\equiv|\mathbf{x}-\mathbf{s}|$, $y_i=x_i-s_i$, $i=1,\ \ 2$ and $j=1,\ \ 2$ for the plane elasticity [27]. Eq. (2) represents the displacement expression for plane elasticity in the form of single-layer potential. Because the fundamental solution in Eq. (3) is a weakly singular kernel, there is no jump when $\mathbf{x}\in D$ approaches the boundary.

To provide a test for the BEM, we design an exact solution including the constant term (rigid body translation) and the rotation term (rigid body rotation) as follows:

$$\bar{u}_1(\mathbf{x})=1-\gamma x_2,\quad \mathbf{x}\in D, \tag{4}$$

$$\bar{u}_2(\mathbf{x})=1+\gamma x_1,\quad \mathbf{x}\in D, \tag{5}$$

where $\mathbf{x}(x_1,x_2)$ is the field point and γ (an infinitesimal small value) is the rigid body rotation. The boundary condition is

$$u_i(\mathbf{x})=\bar{u}_i(\mathbf{x})\ \ ,\quad \mathbf{x}\in B,\quad i=1,2, \tag{6}$$

where B is the boundary of the domain.

3. Regularized methods for a degenerate-scale problem

3.1. Derivation of the degenerate scale of the BIEM in conjunction with the degenerate kernel

We examine simple cases with the rigid body translation and the rigid body rotation of circular and elliptical domains. The distributions of field points are shown in Fig. 1(a) and (b), respectively. For a circular domain, the source point $\mathbf{s}$ and the collocation point $\mathbf{x}$ are expressed as (R,θ) and (ρ,ϕ) in the polar

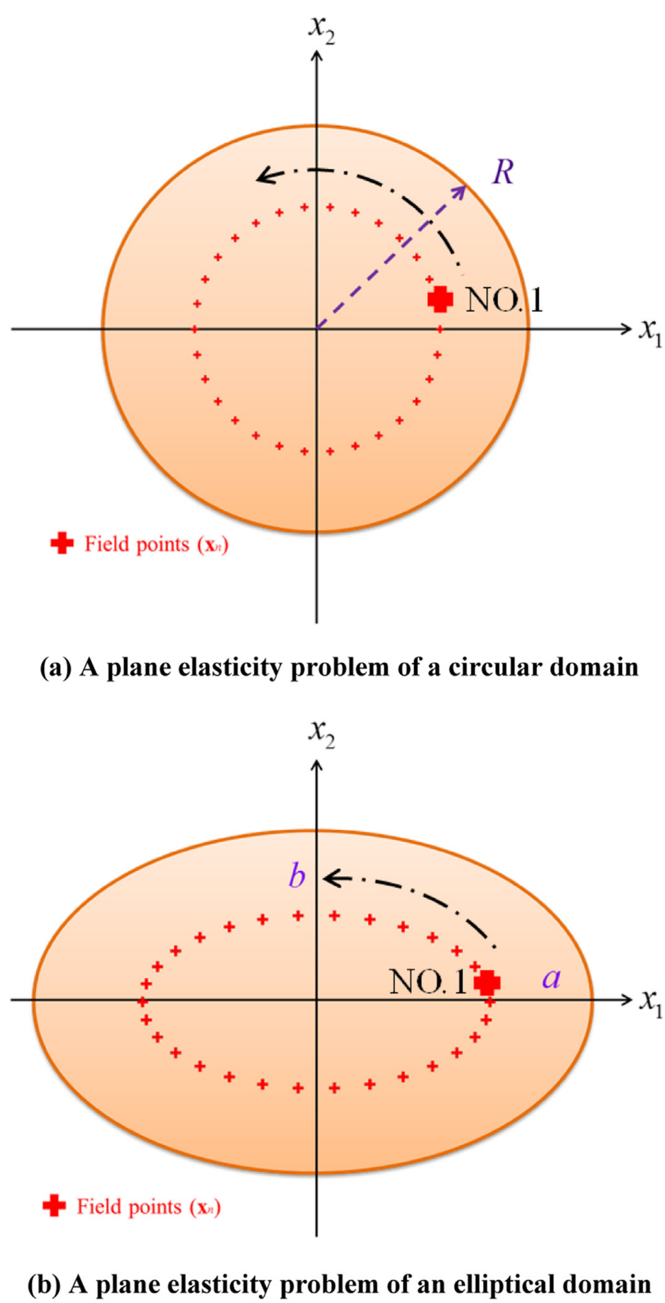


Fig. 1. The sketch of the problem.

coordinates [27,28], respectively. The unknown boundary densities $\alpha_j(\mathbf{s})$ and the given boundary conditions $f_i(\mathbf{x})$ along the boundary can be expressed by using the Fourier series expansion as shown below:

$$\alpha_j(\mathbf{s}) = a_0^{(j)} + \sum_{n=1}^{\infty} a_n^{(j)} \cos(n\theta) + b_n^{(j)} \sin(n\theta), \quad 0 \le \theta \le 2\pi, \quad j = 1, 2, \tag{7}$$

$$f_i(\mathbf{x}) = p_0^{(i)} + \sum_{n=1}^{\infty} p_n^{(i)} \cos(n\ \phi) + q_n^{(i)} \sin(n\phi), \quad 0 \le \phi \le 2\pi, \quad i = 1, 2, \tag{8}$$

where $a_0^{(j)}$, $a_n^{(j)}$ and $b_n^{(j)}$ are unknown coefficients, $p_0^{(i)}$, $p_n^{(i)}$ and $q_n^{(i)}$ are known coefficients from the specified boundary displacement. Similarly, the source point $\mathbf{s}$ and the collocation point $\mathbf{x}$ are expressed by $(\bar{\xi}, \bar{\eta})$ and (ξ, η) in the elliptical coordinates for an elliptical domain. The unknown boundary densities $\alpha_j(\mathbf{s})$ and the given boundary conditions $f_i(\mathbf{x})$ along the boundary can be expressed by using the eigenfunction expansion as shown below:

$$\alpha_j(\mathbf{s}) = \frac{1}{J_\mathbf{s}}\left(a_0^{(j)} + \sum_{n=1}^{\infty} a_n^{(j)} \cos(n\ \bar{\eta}) + b_n^{(j)} \sin(n\ \bar{\eta})\right), \quad 0 \le \bar{\eta} \le 2\pi, \quad j = 1, 2, \tag{9}$$

$$f_i(\mathbf{x}) = p_0^{(i)} + \sum_{n=1}^{\infty} p_n^{(i)} \cos(n\ \eta) + q_n^{(i)} \sin(n\ \eta), \quad 0 \le \eta \le 2\pi, \quad i = 1, 2, \tag{10}$$

where the Jacobian term for the source point is given by

$$J_\mathbf{s} = c\sqrt{\sinh^2\bar{\xi} + \sin^2\bar{\eta}}. \tag{11}$$

After matching the boundary conditions in the conventional BIEM, we can determine the unknown coefficients. It can be found that the coefficients of constant term ($a_0^{(1)}$ and $a_0^{(2)}$) could not be determined due to the zero denominator when a degenerate scale occurs [28]. The degenerate scales are $R = e^{(1/2\kappa)}$ and

$$2\kappa\left(\ln\frac{a+b}{2}\right) = \frac{2a}{a+b} \tag{12}$$

or

$$2\kappa\left(\ln\frac{a+b}{2}\right) = \frac{2b}{a+b} \tag{13}$$

for the circular domain and the elliptical domain, respectively [28], where a and b are the lengths of the semi-major axis and the semi-minor axis in the elliptical domain. However, the coefficients of the rotation term ($a_1^{(1)}$, $b_1^{(1)}$, $a_1^{(2)}$ and $b_1^{(2)}$) could be determined for any size. After derivation of the above two various domains, we found that the range of the integral operator is only short of the constant term instead of rotation term when the size of the domain is a degenerate scale. According to the Fredholm alternative theorem, we obtain the infinite solution when $\int_B f_i(\mathbf{x})\ dB(\mathbf{s}) = 0$, or no solution when $\int_B f_i(\mathbf{x})\ dB(\mathbf{s}) \neq 0$. In other words, the conventional BIEM of only single-layer potential is not sufficient (infinite solutions) and not necessary (no solution) to ensure a unique solution for a degenerate-scale problem.

3.2. Necessary and sufficient BEM/BIEM for 2-D elasticity problems

According to the above analytical result, it is found that only the coefficients of the constant term could not be determined due to the zero denominator when the size of the domain is a degenerate scale. It causes range deficiency to the solution space by a constant term. However, the coefficients of the rotation term and other higher order terms could be determined for any size of the domain.

Therefore, the range deficiency of the BIE only occurs in the constant term in case of a degenerate scale. Following the past experience of solving the Laplace problem, we introduce Fichera's idea to solve degenerate-scale problems. We enrich the range of the integral operator by adding constants c_i and the corresponding constraints to enforce the indeterminate constant term of the boundary densities to be zero. The necessary and sufficient BEM/BIEM could be written as follows:

$$u_i(\mathbf{x}) = \int_B U_{ij}(\mathbf{x}, \mathbf{s})\alpha_j(\mathbf{s})\ dB(\mathbf{s})\ + c_i\ , \quad \mathbf{x} \in D, \tag{14}$$

$$\int_B \alpha_i(\mathbf{s})\ dB(\mathbf{s}) = 0\ , \quad \mathbf{s} \in B. \tag{15}$$

This is different from Hu's necessary and sufficient BIEs [25] which contains a more rotation term and one more constraint as shown below:

$$u_i(\mathbf{x}) = \int_B U_{ij}(\mathbf{x}, \mathbf{s})\ \alpha_j(\mathbf{s})\ dB(\mathbf{s}) + c_i + \omega\ \ \varepsilon_{ijk}\ \underset{\sim}{x}_j\ \underset{\sim}{e}_k\ , \quad \mathbf{x} \in D, \tag{16}$$

$$\int_B \alpha_j(\mathbf{s})\ dB(\mathbf{s}) = 0\ , \quad \mathbf{s} \in B, \quad j = 1\ ,\ 2, \tag{17}$$

$$\int_B \underset{\sim}{\alpha}(\mathbf{s})\ \times\ \mathbf{s}\ \ dB(\mathbf{s}) = 0\ , \tag{18}$$

where ω is the unknown rotation term and $\underset{\sim}{e} = (0, 0, 1)$ for the 2-D case. We would employ the modal participation factor (MPF) in the numerical implementation to examine the content of rotation term in Eq. (16) and the role of constraint (Eq. (18)) that are necessary for the degenerate-scale problem or not.

3.3. Modal participation factor to examine the numerical instability

According to the analytical derivation, we clearly find that the adding rotation term in Hu's BIEs [25] may not be necessary for a degenerate-scale problem of 2-D elasticity of interior problems. In the linear algebraic system, we employed the SVD to find the MPF of the near-zero minimum singular vector due to a degenerate scale.

By using the SVD, the influence matrix $\mathbf{U}$ is decomposed as

$$\mathbf{U} = \Phi\ \Sigma\ \Psi^T\ = \sum_{n=1}^{N} \sigma_n\ \ \underset{\sim}{\phi}_n\ \ \underset{\sim}{\Psi}_n^T, \tag{19}$$

where $\underset{\sim}{\phi}_n$, $\underset{\sim}{\Psi}_n$ and σ_n are the nth left singular vector, the nth right singular vector and the nth singular value ($0 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_N$), respectively. When the rank deficiency exists in the influence matrix, the minimum singular value approaches zero. After the boundary element discretization, Eq. (2) could be expressed as follows:

$$\mathbf{U}\underset{\sim}{\alpha} = \underset{\sim}{f}, \tag{20}$$

where $\underset{\sim}{\alpha}$ and $\underset{\sim}{f}$ are the vector of the boundary densities and the boundary conditions, respectively. The boundary densities and boundary conditions could be expanded by

$$\underset{\sim}{\alpha} = \sum_n \gamma_n\ \ \underset{\sim}{\Psi}_n. \tag{21}$$

$$\underset{\sim}{f} = \sum_n \beta_n\ \ \underset{\sim}{\phi}_n. \tag{22}$$

Following the MPF in the structural dynamics, the MPF (β_n) with respect to $\underset{\sim}{\phi}_n$ is defined by

$$\beta_n = \underset{\sim}{\phi}_n^T\ \underset{\sim}{f}. \tag{23}$$

According to Eqs. (19), (21) and (22), (20) could be expressed as follows:

$$\sigma_n \ \gamma_n = \beta_n. \tag{24}$$

The coefficients of the boundary densities could be expressed as shown below:

$$\gamma_n = \frac{\beta_n}{\sigma_n}. \tag{25}$$

When the influence matrix **U** is a singular matrix, the minimum singular value is near zero in the numerical implementation. Therefore, the coefficients of the boundary densities could not be determined. It would cause the numerical instability. Once β_n is zero, it means that there is no contribution for the numerical instability of γ_n for the near-zero singular value (σ_n).

4. Illustrative examples and discussions

For a degenerate-scale problem of 2-D elasticity, two examples containing the circular and ellipticalal boundaries are considered. In addition, the MPF of the near-zero minimum singular vector in the influence matrix is also discussed. For the two examples, the exact solution of displacement field is given by Eqs. (4) and (5), subjected to the boundary conditions as Eq. (6), where the rotation (γ), the Lamé constants G and Poisson's ratio ν are given as 0.0001, 1.0 and 0.25, respectively.

Example 1. A circular domain with a degenerate scale

For a 2-D elasticity problem, the unique solution exists in the ordinary case (non-degenerate) by using the BEM/BIEM. However, the non-unique solution may occur for a degenerate-scale problem. According to the analytical derivation, the constant terms could not be determined when the size of the circular domain is a degenerate scale ($R = e^{1/2\kappa}$). It is the reason why a constant term of the solution space could not appear. In the numerical implementation, numerical results of the displacement solution by using the conventional BEM are not reasonable as shown in Fig. 2, where the number of boundary elements, the minimum singular value of the influence matrix and the radius (degenerate scale) are 100, 2.5×10^{-6} and 1.2845489395, respectively.

Since the reasonable displacement in the degenerate-scale problems could not be obtained by using the conventional BEM/BIEM, we employed the Fichera's idea and proposed an enriched formulation by adding constants and corresponding constraints to improve the conventional BEM/BIEM. This regularization approach, the necessary and sufficient BEM/BIEM, was also used to solve the problem. The results are more accurate than those obtained by using the conventional BEM/BIEM as shown in Fig. 2.

Example 2. An elliptical domain with a degenerate scale

For a degenerate-scale problem containing an elliptical boundary, two different degenerate scales are given in Eqs. (12) and (13). When the size of the elliptical domain is a degenerate scale, the constant term of the solution space is deficient. In Figs. 3 and 4, the numerical result by using the conventional BEM is also unreasonable, where the number of boundary elements is 100, the minimum singular values of the influence matrix are 2.9×10^{-6} and 3.6×10^{-6} corresponding to two degenerate scales, $b = 0.930785701$ and $b = 0.787891333$, respectively.

According to the necessary and sufficient BEM/BIEM, regularized results for the case of the degenerate scale are shown in Figs. 3 and 4. We can successfully solve the degenerate-scale problem even if the rotation term is not enriched in the regularized method.

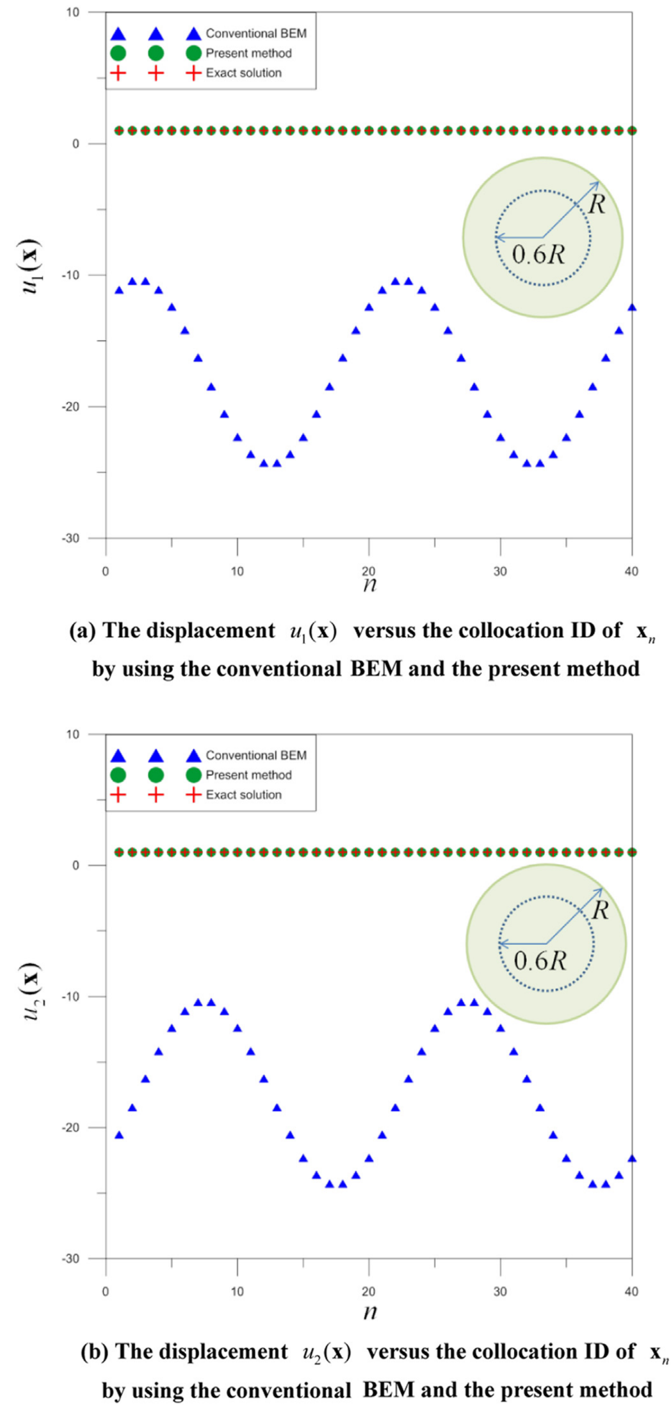


Fig. 2. The results of the degenerate-scale case (R=1.2845489395) for a circular domain.

In order to understand the MPF for various boundary conditions, we examine two cases of an ellipse. One is the rigid body translation as given by

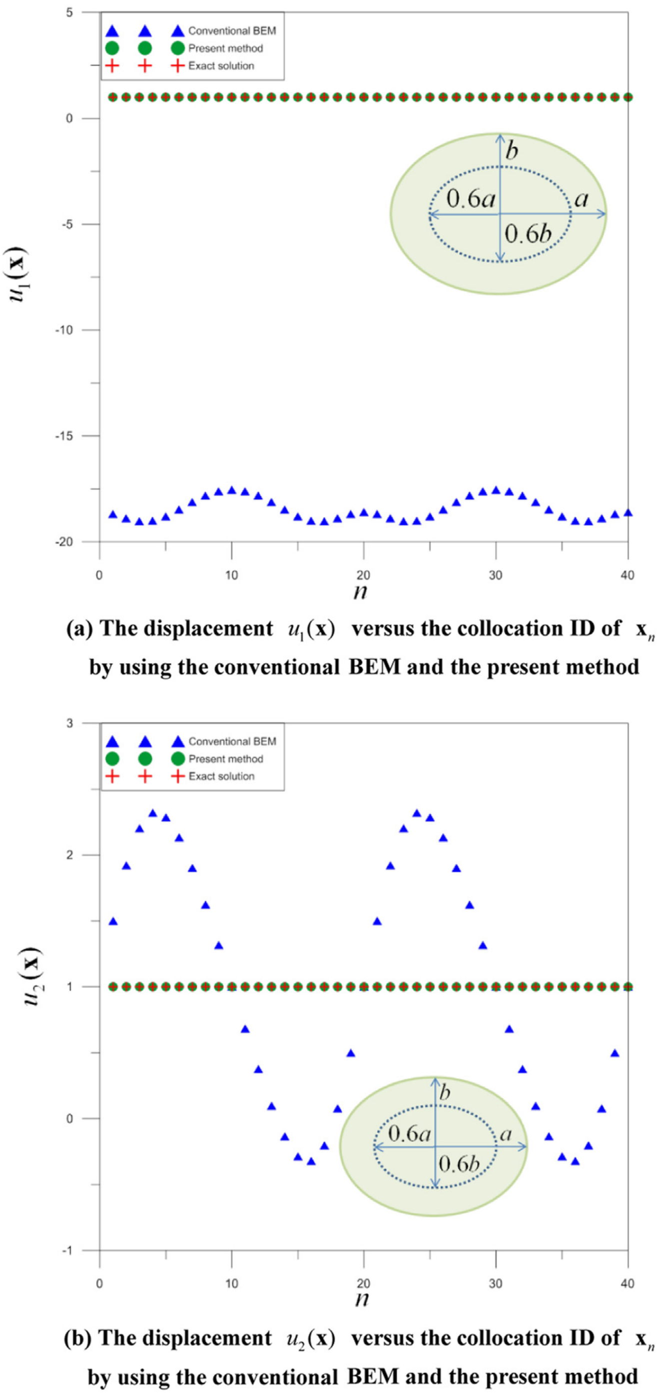
$$\bar{u}_i(\mathbf{x}) = 1, \quad i = 1, 2, \tag{26}$$

and the other is the rigid body rotation as given by

$$\bar{u}_1(\mathbf{x}) = -\gamma x_2, \tag{27}$$

$$\bar{u}_2(\mathbf{x}) = \gamma x_1. \tag{28}$$

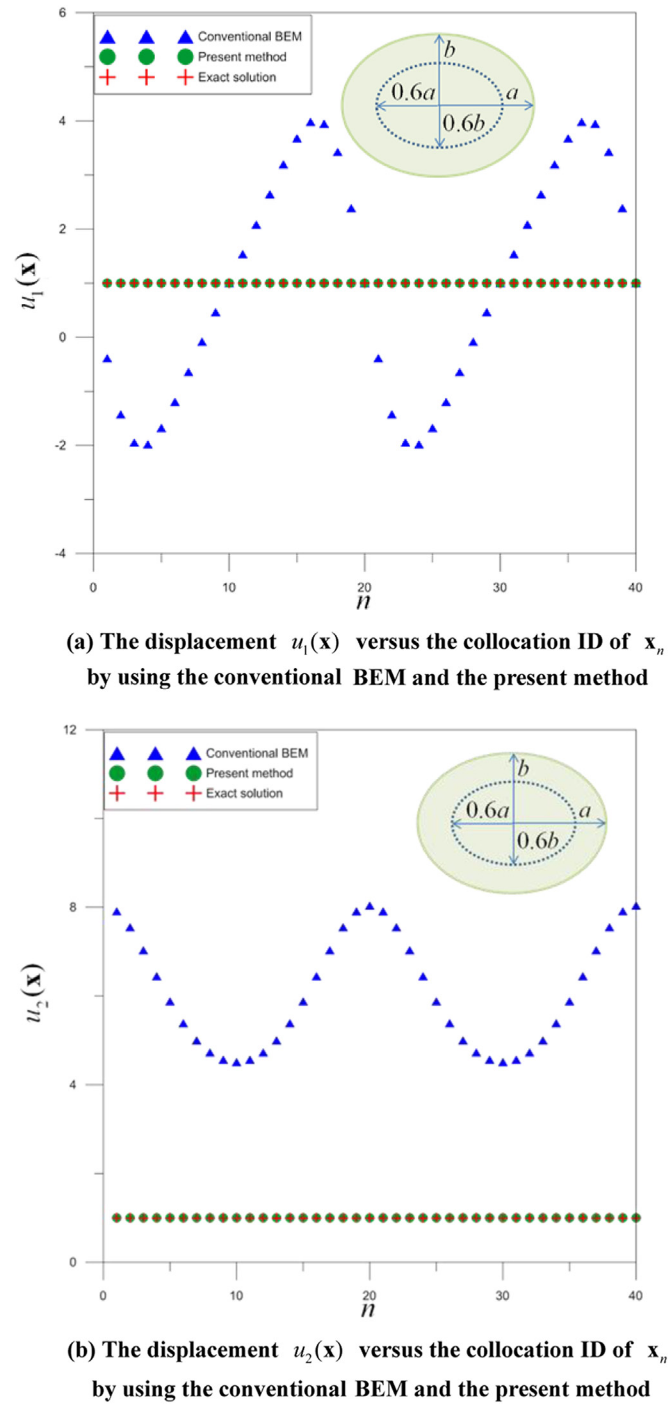
where the rotation (γ) is 0.0001, the size of domain is a degenerate scale ($b = 0.7878913330$).



(a) The displacement $u_1(\mathbf{x})$ versus the collocation ID of $\mathbf{x}_n$ by using the conventional BEM and the present method

(b) The displacement $u_2(\mathbf{x})$ versus the collocation ID of $\mathbf{x}_n$ by using the conventional BEM and the present method

Fig. 3. The results of the degenerate-scale case (b=0.9307857017, a=2b) for an elliptical domain.



(a) The displacement $u_1(\mathbf{x})$ versus the collocation ID of $\mathbf{x}_n$ by using the conventional BEM and the present method

(b) The displacement $u_2(\mathbf{x})$ versus the collocation ID of $\mathbf{x}_n$ by using the conventional BEM and the present method

Fig. 4. The results of the degenerate-scale case (b=0.7878913330, a=2b) for an elliptical domain.

The results for various boundary conditions by using the BEM are shown in Fig. 5. The numerical results are unreasonable for the case of the rigid body translation. In contrast, numerical results are acceptable for the case of the rigid body rotation. Therefore, we may wonder why the numerical results approach the exact solutions in the ill-posed system when the size of domain is a degenerate scale. We employed the SVD technique to find the MPF of the mode for the near-zero minimum singular value.

When the number of boundary elements is 20, the minimum singular value of the influence matrix is 6.3×10^{-6} in this case. The MPFs of the mode for the singular values are shown in Fig. 6, where $\beta_n^{(1)}$ and $\beta_n^{(2)}$ are the *nth* MPFs corresponding to the *n*th singular value for the case of the rigid body translation and the case of the rigid body rotation, respectively. When the size of the elliptical domain is the degenerate scale ($b = 0.787891333$), the MPFs corresponding to the near-zero minimum singular value ($\sigma_1 = 6.3 \times 10^{-6}$) are $\beta_1^{(1)} = -4.34859$ and $\beta_1^{(2)} = 0$. Two different MPFs are used to explain why the results of the displacement solution fail by using the conventional BEM to solve the elasticity problem with a degenerate scale if the solution contains the constant term. However, results of rotation are acceptable. In other

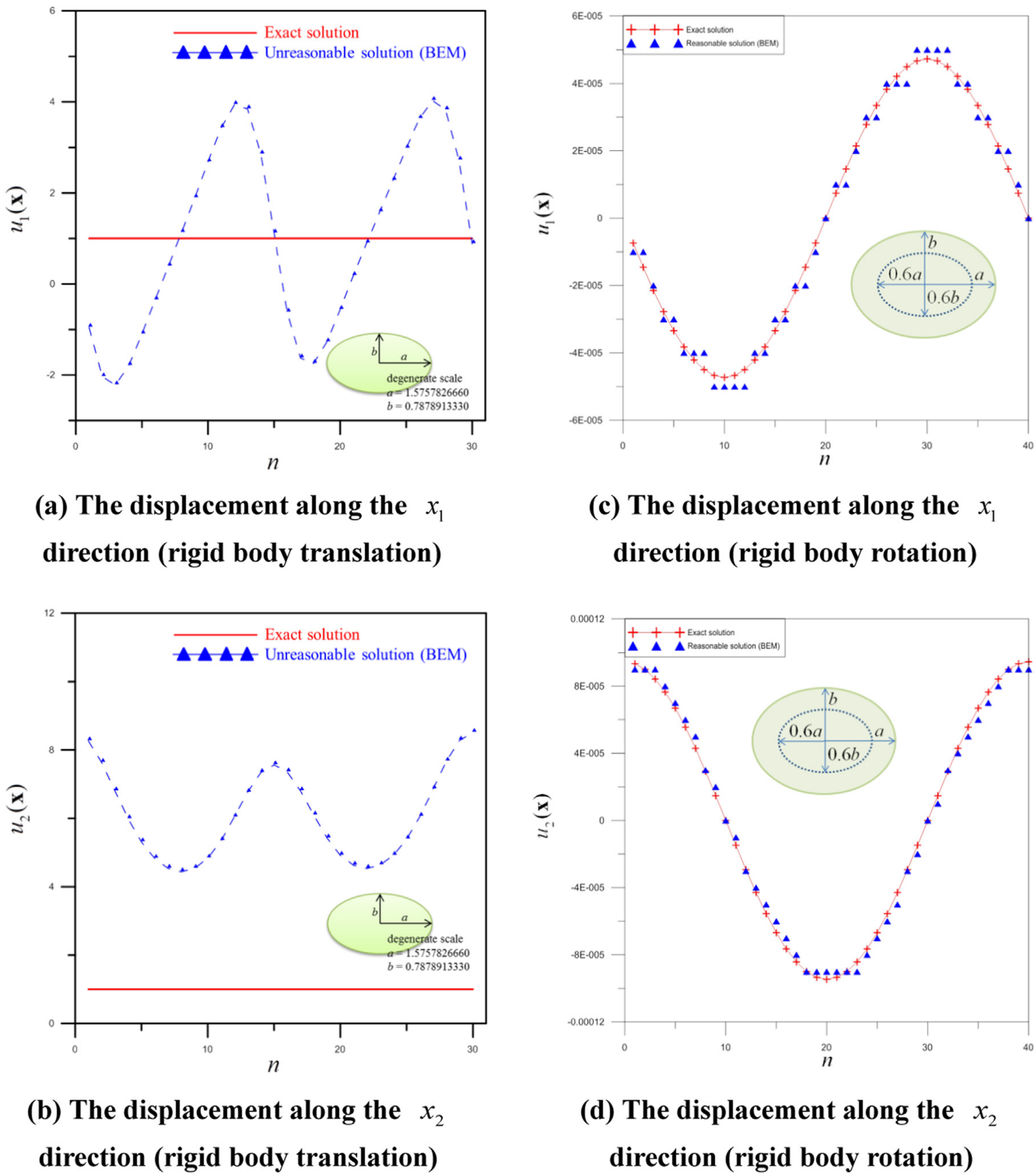


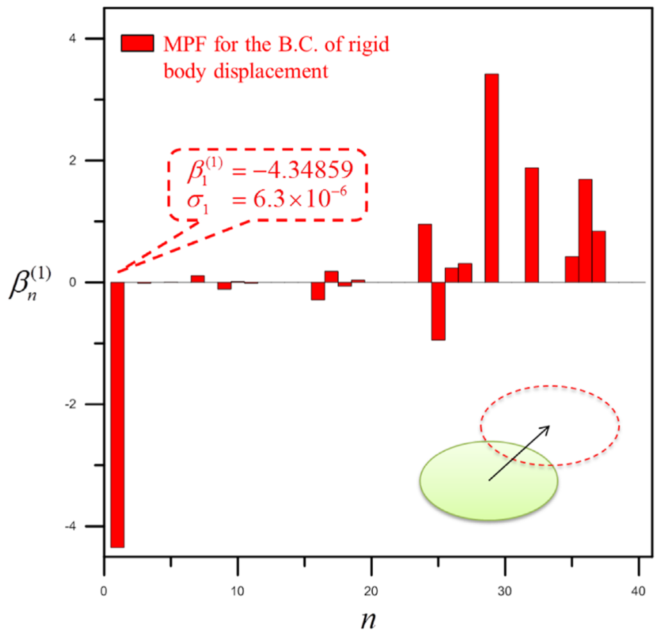
Fig. 5. The results of the degenerate-scale case (b=0.7878913330, a=2b) for an elliptical domain subjected to two different boundary conditions by using the conventional BEM.

words, the rotation term in Eq. (16) and the constraint (Eq. (18)) are not necessary for the degenerate-scale problem.

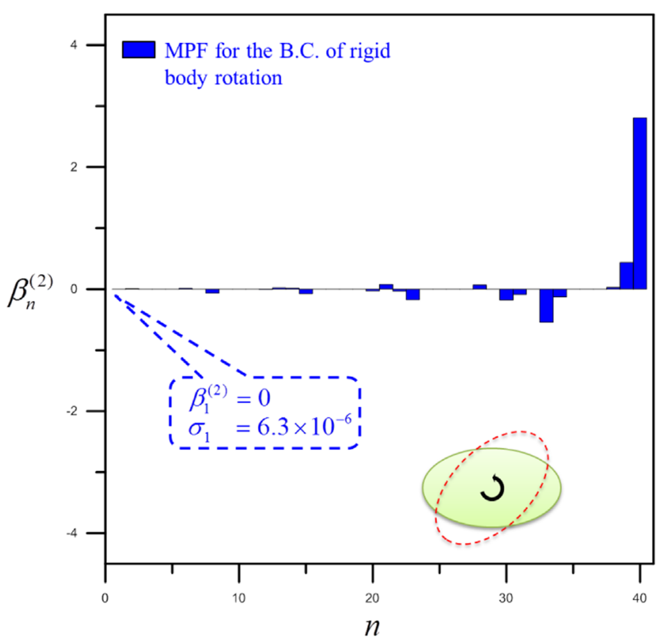
The aforementioned two examples including the boundary conditions of the rigid body translation or rotation are very simple for 2-D elasticity problems. However, the conventional BIEM fails to solve because the mathematical model is not well-posed and is incomplete. This simple test (rigid body translation) is used to examine the sufficient and necessary BIEM to ensure a unique solution as the patch test (constant strain) for the FEM.

5. Conclusions

A simple test of the BEM/BEIM for the rigid body translation and rotation problems was proposed in this paper. Two examples with the specified displacement boundary condition (rigid body translation and rigid body rotation) are benchmarks for the BIEM/BEM as the patch test for the FEM. Two tests are compared in Table 1. For the degenerate scale of the 2-D elasticity problem, we revisited the issue of the sufficient and necessary BIEs. We examined the role of the adding rotation term in Hu's sufficient and necessary BIEs for the degenerate-scale problem by using the degenerate kernel and the MPFs. Based on analytical results, the range deficiency only occurs in the constant term of the solution space instead of the rigid body rotation for the interior problem. In the numerical implementations, we investigated the MPF for various boundary conditions when the size of domain is a degenerate scale. It is found that the MPF (β_1) corresponding to the near-zero minimum singular value is zero when the boundary condition is a rigid body rotation without containing any translation. Therefore, the additional rotation term is not necessary to be included



(a) Boundary condition of a rigid body translation



(b) Boundary condition of a rigid body rotation

Fig. 6. The modal participation factors for various boundary conditions when the size of an ellipse is the degenerate scale (b=0.7878913330, a=2b).

Table 1
The comparison of the simple test for the BEM and the FEM.

	BEM/BIEM	FEM
Test	Rigid body test	Patch test
Boundary condition	Rigid body translation and rigid body rotation	Uniaxial stress to check the constant strain

in the regularized BIE for the interior degenerate-scale problem. We extended the Fichera's idea to add a constant and a corresponding constraint for the rank promotion. The necessary and sufficient BEM/BIEM to ensure a unique solution was examined. Finally, we successfully recovered the range deficiency by adding a constant term in the BIEM when the size of the domain is a degenerate scale.

Acknowledgments

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